Motion of a Dipole-Quadrupole System*

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Niels Bohr showed that the magnetic moment of an electron cannot be determined by a Stern-Gerlach experiment. His argument is extended to other moments, and general conditions are stated as to when a moment determination is feasible by such an experiment. In addition, the expectation-value equations are given for a stationary, gyrating, nonrelativistic, spin-1 particle in arbitrary, but classical, external electromagnetic fields. For such a dipole-quadrupole system there is no distinction between the classical and quantal descriptions. Of interest is the fact that $SU(3)$ is the invariance group of the motion. The theory is extended to the relativistic domain by Lorentz transformation. The group $SU(3,1)$ appears as the invariance group of the internal motion.

I. INTRODUCTION

IN recent times and in diverse circumstances, the intrinsic properties of elementary particles and their stable or relatively stable composites have come under increasing analysis. By intrinsic, we mean those properties of a system other than space-time trajectory, such as spin, isospin, quadrupole moment, etc.

Over the years, two differing views have arisen concerning these internal parameters and their relation to classical physics. In the late 1920's, Bohr[1] denied the validity of spin as a classically describable quality of the electron, and thus by implication nullified a classical view of any other intrinsic property of elementary particles. On the other hand, Bloch[2] showed, in his fundamental work on magnetic resonance, that if gyroscopic frequencies were of concern, then the spin motion could be analyzed by solving either the Schrödinger equation or a well-known classical gyroscopic equation which at a later date was derived[3] for the expection value of the spin operator. Without additional assumptions, the classical equation could not yield a discrete spin spectrum or the magnitude of the angular momentum, as does the quantum theory, but the predicted gyroscopic frequencies were the same in both theories. Moreover, a classical theory written in terms of spinor variables was indistinguishable from the quantal description of the internal motion, provided only that the external magnetic field was free of spatial dependence.

Bohr, in his early statement, reasoned that since no Stern-Gerlach experiment could yield the value of an electron's magnetic moment, the spin of the electron, which is proportional to the magnetic moment, falls outside the fold of classical description. While his denial represented an extreme empiricist view which was proved faulty in other circumstances by Bloch's analysis, there is nevertheless a significant truth contained in Bohr's original statement. It seems that Bohr has given the simplest example of a class of experiments in which the uncertainty principle prevents us from determining the values of the intrinsic moments of elementary systems. The experimental arrangement requires the simultaneous use of a confined beam of particles and a field-intensity measurement or particle-intensity measurement transverse to the beam. These experiments in no way rule out other procedures for the determination of the moment values, but they do extend the range of application of the uncertainty principle to an area of interplay of relativity, wave theory, and the dynamics of the system. Bohr's contention seems to have a wide applicability for deformations beyond the magnetic moment.

The measurement difficulties emphasized by Bohr do not limit the use of classical particle concepts and equations whenever the conditions of Bloch's theorem are satisfied. These conditions are certainly met by systems which can be considered both at rest and localized in space, and which couple linearly to classical internal and external fields. Such systems are describable by nonrelativistic classical dynamics in the Bloch sense,[4] irrespective of their internal complexity.

Furthermore, the conditions given above cannot be unique, since a system whose internal motions are always decoupled from its spatial trajectory is in conflict with the principles of relativity. One must then anticipate some minimal coupling between a system's internal dynamics and its spatial trajectory which will permit a reasonably precise classical description of a system's gyrations. Indeed, the relativistic extension of Bloch's theorem has been given by Bargmann, Michel, and Telegdi[5] (BMT) for leptons in constant, uniform fields, and the predicted motions have been confirmed by numerous experiments.

For higher-spin systems the BMT equations remain valid as long as the external fields are uniform and constant. However, for nonuniform fields the motion of higher multipole moments becomes significant. This may be seen in an explicit analysis of a spin-1 system

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which possesses both a magnetic-dipole moment and electric-quadrupole moment. The classical equations of such a system are derived from the density matrix of the Schrödinger equation and then immediately extended to relativistic motions by Lorentz transformation. The resulting equations are a generalization of the equations of BMT. The internal motions of the nonrelativistic system are best described by means of spinors associated with the group $SU(3)$, and the relativistic internal motions in terms of the group $SU(3,1)$.

II. MEASUREMENT OF MOMENTS

In his classic analysis, Bohr[1] discussed a beam of independent particles, each carrying a charge e, mass m, and magnetic moment $\mu \approx e\hbar/mc$. He showed that because of beam diffraction, no measurement could be made of μ by field-intensity measurements or particle-intensity measurements transverse to the beam.

Let us, for example, choose to measure the magnetic moment μ of a charged dipole in the beam. The magnitude of the magnetic field at the position r of the magnetometer is

$$B=\mu/r^3. \tag{2.1}$$

On the other hand, a moving charge has an accompanying magnetic field of magnitude

$$B'=ev/cr^2. \tag{2.2}$$

Since we are using a beam, the uncertainty in B' in the direction of B is

$$\Delta B'=e\Delta p/mcr^2 \approx e\hbar/mcr^2\Delta r. \tag{2.3}$$

A reasonable field measurement then requires that

$$\Delta B' \ll B, \tag{2.4}$$

which results in

$$\Delta r \gg r(e\hbar/mc)\mu^{-1}. \tag{2.5}$$

If the beam is not to overwhelm the apparatus, Eq. (2.5) gives

$$(e\hbar/cm) \ll \mu. \tag{2.6}$$

Equation (2.6) is true for systems with vanishing charge, but fails for charged systems, since, generally, $\mu \approx e\hbar/mc$. In this last case, no magnetometer measurement is feasible.

Bohr's ideas are easily extended to arbitrary moments. We assume a moment Q_n giving rise to a field (electric or magnetic) of magnitude

$$G_n=eQ_n/r^{n+2}, \quad n=0,1,2,\cdots, \tag{2.7}$$

We further assume that even values of n refer to electric moments and fields, while odd values of n refer to their magnetic counterparts. For convenience alone, our enumeration ignores the possible existence of magnetic charges, intrinsic electric-dipole moments, etc. If the system has a lower moment of opposite character (magnetic or electric), the indeterminacy in the motion will induce a field uncertainty (electric or magnetic) of magnitude

$$\Delta G_n'=eQ_{n-1}\Delta p/cmr^{n+1} \approx (e\hbar/mc)(Q_{n-1})/(\Delta r)r^{n+1}. \tag{2.8}$$

If the field G_n is to be measured, we must have

$$G_n \gg \Delta G_n'. \tag{2.9}$$

This last implies that

$$Q_n \gg (r/\Delta r)(\hbar/mc)Q_{n-1}. \tag{2.10}$$

On the other hand, if

$$(\hbar/mc)Q_{n-1} \approx Q_n, \tag{2.11}$$

the measurement must fail.

Little more of a general nature can be said concerning the possible validity of Eq. (2.11), since much depends on the moment in question and beyond that, the details of the dynamics of the system. For example, for a hypothetical massive spin-1 particle with anomalous electric-quadrupole moment and no anomalous magnetic-dipole moment, Eq. (2.10) is satisfied, so that both field measurements and Stern-Gerlach-type experiments are available for the determination of the quadrupole moment. A major contributing factor in making this particular measurement possible is the Thomas precession, an ever-present relativistic kinematic effect.

In a similar manner, consider the Bohr argument for the conditions for breakdown of a generalized Stern-Gerlach experiment. Assume that we have a beam of quadrupoles moving in the z direction and entering an inhomogeneous electric field. Further, to simplify the mathematical argument, arrange the external electric-quadrupole field so that its gradient has only terms

$$\partial E_x/\partial x=-\partial E_y/\partial y, \tag{2.12}$$

which satisfies the Maxwell equation, $\nabla\cdot\mathbf{E}=0$. The electric force on the quadrupoles in the beam is

$$\mathbf{F}=\tfrac{1}{2}\nabla(\mathbf{Q}:\nabla\mathbf{E}), \tag{2.13}$$

where Q is the diagonal quadrupole tensor $Q_{xx}=-Q_{yy}$. The magnitude of the force in the y direction is

$$F_y=Q_{yy}(\partial^2E_y/\partial y^2). \tag{2.14}$$

Assume that the particle moving with velocity $\mathbf{v}$ has, in addition, a magnetic-dipole moment $\boldsymbol{\mu}$, and thus a motion-induced electric-dipole moment coupling to the external electric field with potential $V=(\boldsymbol{\mu}\times\mathbf{v})\cdot(\mathbf{E}/c)$. If $\boldsymbol{\mu}$ has its major component in the y direction, the magnitude of the force in the same direction becomes

$$F_y'=\mu_y(\partial E_x/\partial y)(v_z/c). \tag{2.15}$$

For a particle slightly off the center of the beam and orthogonal to the force direction, we find

$$\Delta F_y'=\mu_y(\partial^2E_x/\partial x\partial y)\Delta x(v_z/c) \approx \mu_y(\partial^2E_y/\partial y^2)(\hbar/mc)(v_z/\Delta v_x). \tag{2.16}$$

We have used the uncertainty principle and (2.12) to arrive at the last term in (2.16).

If the Stern-Gerlach experiment is to be feasible,

$$F_y \gg \Delta F_y' \tag{2.17}$$

or

$$\Delta v_x \gg v_z(\hbar/mc)\mu_y/Q_{yy}. \tag{2.18}$$

Should

$$(\hbar/mc)\mu_y \approx Q_{yy}, \tag{2.19}$$

no Stern-Gerlach experiment is possible, since, instead of a beam, we have a cylindrical wave.

Note that the condition (2.19) is a particular example of (2.11) which prevented us from making a transverse (to the beam) field measurement. Similar but lengthier arguments are clearly possible for Stern-Gerlach-type experiments involving higher moments. In general, the pattern of failure of this type of experiment is clear: We must have a moment Q_n and a "dual" moment of lower order Q_{n-1}, with

$$Q_n \approx (\hbar/mc)Q_{n-1}. \tag{2.11}$$

In addition, failure may occur for lower moments with

$$Q_n \approx (\hbar/mc)Q_{n-a}r^{a-1} \quad a=1, 2, 3, \cdots. \tag{2.20}$$

III. NONRELATIVISTIC QUADRUPOLE MOTION

Assume that we have a system with intrinsic angular momentum j and magnetic moment represented by the vector matrix

$$\boldsymbol{\mu} = \alpha\hbar\boldsymbol{\sigma}, \tag{3.1}$$

with $\boldsymbol{\sigma}$ the spin matrices for spin j. If the system is exposed to a time-dependent magnetic field $\mathbf{B}$, then in the nonrelativistic limit the Schrödinger equation for the internal motion is

$$\partial\Psi/\partial t = i\alpha\mathbf{B}\cdot\boldsymbol{\sigma}\Psi. \tag{3.2}$$

Since the equation is independent of Planck's constant, it must describe a classical system. In classical terms, we have a charged spherical top with angular momentum $\mathbf{S}$, gyrating according to

$$d\mathbf{S}/dt = \alpha\mathbf{S}\times\mathbf{B}. \tag{3.3}$$

Equation (3.3) is the classical spinor form of (3.2). In fact, if we identify $\mathbf{S} \propto \psi^\dagger\boldsymbol{\sigma}\Psi$ and use (3.2), the expectation value of the spin operator $\boldsymbol{\sigma}$ obeys (3.3).

The spin describes the orientation of the system. If the spin is ≥ 1, higher moments will be needed to complete the description of the system's internal dynamics. If more complex external fields are introduced, the description and the motion become correspondingly more complex. Nevertheless, if the internal motions can be decoupled from the system's spatial motion, as occurs, for example, for a nucleus in a crystal, then no matter how complex the internal dynamics, the resulting Schrödinger equation describes a classical system.

This nonrelativistic classical Schrödinger equation, first given by Fano, provides a valid, although approximate, quantal description of an arbitrary system subjected to arbitrary classical fields. Fano's equation[6] is of the form (3.3), but is expressed in the language of irreducible tensorial sets, a natural generalization of the density matrix for systems with arbitrary spin. As such, it simultaneously includes a wave-function description, (3.2), and an expectation-value description, (3.3).

In this paper, we are interested, in particular, in Fano's results for massive spin-1 particles in external gradient fields. The analysis of this relatively primitive system is readily extended to relativistic particle motions, and the resulting equations for such systems form the natural generalization of the equations of Bargmann, Michel, and Telegdi. In addition, the theory is related to recent speculations in elementary-particle physics and nuclear physics.

Since a spin-1 system is comparatively simple and another form of Fano's equation is desirable, we shall derive the classical equations in usual tensor form directly from the ordinary density matrix.

The density matrix ρ for a spin-1 particle is a 3×3 matrix. In common with all density matrices describing pure states, ρ is a constant of the motion,

$$\partial\rho/\partial t + (i\hbar)^{-1}(\rho,H) = 0. \tag{3.4}$$

The expectation value of a given matrix operator A_{mn} is

$$\langle A\rangle = \rho_{mn}A_{nm} \equiv \mathrm{Tr}(\rho A). \tag{3.5}$$

ρ may be decomposed in terms of spin matrices as follows:

$$\rho_{mn} = \tfrac{1}{3}\delta_{mn} + \tfrac{1}{2}S_a\sigma_{mn}{}^a + \tfrac{1}{4}Q_{ab}(\sigma_{me}{}^a\sigma_{en}{}^b + \sigma_{me}{}^b\sigma_{en}{}^a - \tfrac{4}{3}\delta_{ab}\delta_{mn}), \tag{3.6}$$

where

$$S_a = \langle\sigma^a\rangle, \quad Q_{ab} = Q_{ba} = \langle\sigma^a\sigma^b + \sigma^b\sigma^a - \tfrac{4}{3}\delta_{ab}\rangle,$$

and $Q_{aa}=0$.

If we choose as a representation for the spin matrices $\sigma_{mn}{}^a = -i\epsilon_{amn}$, we find

$$\rho_{mn} = \tfrac{1}{3}\delta_{mn} - \tfrac{1}{2}iS_{mn} - \tfrac{1}{2}Q_{mn}, \tag{3.7}$$

with $S_{mn} = S_i\epsilon_{imn}$.

In addition to (3.4), the density matrix satisfies

$$\rho_{mn}\rho_{nm} = 1, \ \rho_{mn}\rho_{nl}\rho_{lm} = 1, \cdots, \tag{3.8}$$

for pure states.

As a result of (3.8), the following are the only two independent invariants for the expectation values:

$$I_1 = S_{ik}S_{ik} + Q_{ik}Q_{ik}, \tag{3.9a}$$

$$I_2 = 3S_{ik}S_{kl}Q_{il} - Q_{ik}Q_{kl}Q_{il}. \tag{3.9b}$$

I_1 and I_2 must be dynamical constants of the motion for a spin-1 system for arbitrary external fields. They

[6] Reference 4, Eq. (7).

are the two invariants associated with the group $SU(3)$. For systems with more complex structure, there are the additional invariants of $SU(N)$. While I_1 and its generalizations to $SU(N)$ have been noted by Fano,[4] he apparently has overlooked these other invariants, although he has stressed the really significant fact that, in general, terms of the form $S_{ik}S_{ik}$, $Q_{ik}Q_{ik}$, etc., are not constants of the motion.

This fact distinguishes general spin systems from spin-$\frac{1}{2}$ particles. In the case of spin-$\frac{1}{2}$ systems, the expectation values of the square of the spin operator, $\psi^\dagger\sigma^2\psi$, and the square of the expectation value of the spin operator, $(\psi^\dagger\sigma\psi)^2$, are both constants of the motion for unit spinor wave functions, $\psi^\dagger\psi=1$. For higher spins, $(\psi^\dagger\sigma\psi)^2$ is no longer constant, so that in discussing the average properties of such systems, the rotation group and its generators no longer play the same role as in the case spin $\frac{1}{2}$.

The classical dynamics of a spin-1 system is given by the differential equations for the expectation values of relevant operators. These are found directly from (3.4). Choose as Hamiltonian

$$H=-\alpha h\sigma_m B_m-\tfrac{1}{2}h\beta E_{mn}(\sigma_m\sigma_n+\sigma_n\sigma_m-\tfrac{4}{3}\delta_{mn}), \quad (3.10)$$

with B_m the magnetic field and $E_{mn}=\partial E_m/\partial x_n+\partial E_n/\partial x_m$ the gradient tensor of the electric field. Again, if we choose $\sigma_{ik}{}^m=-i\epsilon_{ijm}$, the matrix H above becomes

$$H_{ik}=ih\alpha B_{ik}+h\beta E_{ik}, \quad (3.11)$$

and $B_{ik}=\epsilon_{ike}B_e$.

The dynamical equations are then easily found from (3.4):

$$\dot{S}_{ik}=-\alpha(S_{ie}B_{ek}-B_{ie}S_{ek})+\beta(Q_{ie}E_{ek}-Q_{ke}E_{ei}), \quad (3.12a)$$

$$\dot{Q}_{ik}=-\alpha(Q_{ie}B_{ek}+Q_{ke}B_{ei})-\beta(S_{ie}E_{ek}+S_{ke}E_{ei}). \quad (3.12b)$$

These classical expectation-value equations maintain in time the constancy of the invariants I_1 and I_2 of (3.9).

The same Eqs. (3.12) may be derived in another fashion from classical dynamics. Let us assume that the spin and quadrupole tensors may be represented in terms of coordinates and conjugate momenta associated with the internal state of rotation and deformation of the system. Further, choose the Poisson-bracket relations among the dynamical variables to be

$$[S_{ik},S_{mn}]=\delta_{im}S_{kn}-\delta_{in}S_{km}+\delta_{mk}S_{ni}-\delta_{nk}S_{mi}, \quad (3.13a)$$

$$[Q_{ik},S_{mn}]=\delta_{in}Q_{mk}-\delta_{im}Q_{nk}+\delta_{kn}Q_{mi}-\delta_{km}Q_{in}, \quad (3.13b)$$

$$[Q_{mn},Q_{ik}]=S_{ni}\delta_{km}+S_{nk}\delta_{im}+S_{mk}\delta_{in}+S_{mi}\delta_{kn}. \quad (3.13c)$$

This choice follows the usual rule that in the transition from the quantum theory to classical mechanics, commutators are to be replaced by Poisson brackets. The corresponding spin and quadrupole operators $\Sigma_{ik}\equiv\epsilon_{ika}\sigma^a$ and $K_{mn}\equiv\sigma^m\sigma^n+\sigma^n\sigma^m-\frac{4}{3}\delta_{mn}$ have for their commutator values the same right-hand sides as appear for S_{mn} and Q_{mn} in (3.13). Thus, for example, we have

$$(K_{ik},K_{em})=i(\delta_{im}\Sigma_{ke}+\delta_{km}\Sigma_{ie}+\delta_{ek}\Sigma_{im}+\delta_{ei}\Sigma_{km}). \quad (3.14)$$

If we make use of the Hamiltonian

$$H=-\tfrac{1}{2}\alpha S_{ik}B_{ik}-\tfrac{1}{2}\beta Q_{mn}E_{mn} \quad (3.15)$$

and the Poisson-bracket relation (3.13), we recover the dynamical equations (3.12).

We have assumed the existence of canonically conjugate dynamical variables describing the internal state of a system. Such variables may be introduced as follows:

The classical Hermitian tensor representation of $SU(3)$ is given by the form $A_{mn}=A_{nm}{}^\dagger$, $A_{mm}=0$, with Poisson-bracket relations

$$[A_{mn},A_{ik}]=-i(\delta_{mk}A_{in}-\delta_{ni}A_{mk}). \quad (3.16)$$

The group $SU(3)$ enters the analysis, since the Poisson-bracket relations (3.13) define the classical three-dimensional unitary group, i.e., the Poisson-bracket analog of the group $SU(3)$.

In terms of the A_{mn}, the spin and quadrupole tensors may be written

$$S_{ik}=i(A_{ik}-A_{ki}), \quad (3.17a)$$

$$Q_{ik}=-(A_{ik}+A_{ki}). \quad (3.17b)$$

From (3.16) and (3.17) we easily recover (3.13).

A particular representation of the classical tensor A_{mn} in terms of canonical variables is

$$\begin{aligned} &A_{11}=\tau+\xi-\tfrac{2}{3}C,\quad A_{22}=\tau-\xi-\tfrac{2}{3}C,\quad A_{33}=\tfrac{4}{3}C-2\tau,\\ &A_{12}=e^{-i\eta}(\tau^2-\xi^2)^{1/2},\quad A_{13}=[(2C-2\tau)(\tau+\xi)]^{-1/2}\\ &\qquad\times e^{-i(\phi+\eta)/2}, \quad (3.18)\\ &A_{23}=-i[(2C-2\tau)(\tau-\xi)]^{1/2}e^{-i(\phi-\eta)/2}, \end{aligned}$$

where τ and ξ are momenta conjugate to the coordinates ϕ and η. They satisfy the usual Poisson-bracket relation

$$[\phi,\tau]=1,\quad [\eta,\xi]=1, \quad (3.19)$$

with all other relations vanishing.

τ and ξ are classical variables: ξ is the analog of the z component of the isotopic spin, while τ corresponds to the hypercharge $Y=2\tau-\frac{4}{3}C$. C is an arbitrary constant related to the invariant I_1 of (3.9a); $\frac{1}{4}I_1\equiv A_{ik}A_{ki}=(8/9)C^2$. This classical representation of the group $SU(3)$ is closely related to the usual representation in its identification of moment components. The appearance of half-angles in (3.18) is a peculiarity of the theory attributable to the character of the internal configuration space. This configuration space may be characterized by five generalized Euler angles for rotations and deformations of our particle. If we introduce three complex spinor variables ψ_1, ψ_2, and ψ_3, with

$$\begin{aligned} &\psi_1=(\tau+\xi)^{1/2}e^{i(\phi+\eta+\psi)/2},\\ &\psi_2=(\tau-\xi)^{1/2}e^{i(\phi-\eta+\psi)/2}, \quad (3.20)\\ &\psi_3=[2(C-\tau)]^{1/2}e^{i\psi/2}, \end{aligned}$$

where $\xi = C\cos\theta_1\cos\theta_2$ and $\tau = C\cos\theta_2$, then the ψ_a provide a representation of "quarks" in terms of these five angles.

The nonrelativistic gyroscopic motion of (3.12) may be described in $SU(3)$ terms by a simple Schrödinger equation of the form (3.2). We first define the generalized field $C_{mn} = \beta E_{mn} - i\alpha B_{mn}$, with space components m and n. Introduce the $SU(3)$ spinor ψ_a of (3.20) which satisfies the equation

$$i(\partial\psi_a/\partial t) = C_{mn} A_{ab}{}^{mn}\psi_b. \qquad (3.21)$$

Here the nine $SU(3)$ matrices are

$$(A^{mn})_{ab} = -\tfrac{1}{2}(K^{mn} + i\Sigma^{mn})_{ab}.$$

K and Σ are matrices given in the text above (3.14). For our present limited purposes, raised indices are for convenience in writing and are indistinguishable from lowered ones. All indices run from 1 to 3.

If we define the spin and the quadrupole moments in terms of expectation values,

$$S^{mn} = \psi_a{}^\dagger \Sigma_{ab}{}^{mn}\psi_b, \qquad (3.22a)$$

$$Q^{mn} = \psi_a{}^\dagger K_{ab}{}^{mn}\psi_b, \qquad (3.22b)$$

then (3.21) and (3.22) lead to (3.12), while (3.20) and (3.22) lead to (3.18).

The gyration frequencies of (3.21) are the same as those of (3.12). For stationary motions the quantal eigenstates will be given by the equation

$$i\hbar(\partial\psi_a/\partial t) = E\psi_a. \qquad (3.23)$$

The exact eigenvalues associated with $SU(3)$ spinors are given by (3.23). Alternatively, other $SU(3)$ representatives (tensors) might have been chosen as geometrical candidates satisfying (3.21) and (3.23). In the classical theory the choice is arbitrary, since (3.23) is invalid, and only expectation values of the form (3.22) have significance.

The set of dynamical equations derived from the classical Hamiltonian

$$H = C_{mn}A_{mn}, \qquad (3.24)$$

with $C_{mn} = \beta E_{mn} - i\alpha B_{mn}$, $A_{mn} = -\frac{1}{2}(Q_{mn} + iS_{mn})$, is identical with (3.12) and equal in content to (3.21). In (3.24) the C_{mn} give the external fields, while the A_{mn} are classical dynamical variables. If we make use of H and the Poisson-bracket relations (3.16), we may rederive the equations of motion (3.12).

The theory contained in (3.21) is for massive boson particles. It is a classical theory of a dipole-quadrupole moment and is the logical extension of the ideas of Klein,[7] who made use of spinors in his analysis of the rotating top. If we permit the simplest deformation of a spherical top, we are led to the group $SU(3)$.

[7] One of the earliest and one of the most incisive discussions of the use of spinors in the classical top problem appears in F. Klein, *The Mathematical Theory of the Top* (Scribner and Sons, New York, 1897).

IV. RELATIVISTIC MOTION

The relativistic generalization of Eqs. (3.12) may be achieved in several ways. The dynamical equations corresponding to (3.12) may be found as the WKB limit of a specific one-particle relativistic field theory, or from the density matrix of that same field theory. A third possibility is to postulate the dynamical equations in the instantaneous rest frame of the particle and then Lorentz-transform the equations so that they are valid for any observer moving with arbitrary but constant velocity. This latter method was adopted by Bargmann, Michel, and Telegdi in their work on uniform fields, and I shall follow the same approach. In addition to the equations of motion in the rest frame, some further assumptions are needed concerning the moments themselves. We represent the spin moment by an antisymmetric tensor $S_{\mu\nu} = -S_{\nu\mu}$ and the quadrupole moment by a symmetric tensor $Q_{\mu\nu} = Q_{\nu\mu}$, $\mu, \nu = 1, 2, 3, 4$. Since experiment and theory indicate the presence of only motion-induced electric-dipole moments and magnetic-quadrupole moments, we restrict $S_{\mu\nu}$ and $Q_{\mu\nu}$ as follows:

$$\begin{aligned} &S_{\mu\nu}V_\nu = Q_{\mu\nu}V_\nu = 0, \quad V_\nu V_\nu = -1, \\ &Q_{\mu\mu} = 0, \quad S_{\mu\nu}S_{\mu\nu} + Q_{\mu\nu}Q_{\mu\nu} = \text{const}, \\ &3S_{\mu\rho}S_{\rho\nu}Q_{\mu\nu} - Q_{\mu\rho}Q_{\rho\nu}Q_{\mu\nu} = \text{const}, \end{aligned} \qquad (4.1)$$

with V_μ the usual four-velocity of the particle. The above relations must be maintained in the course of time.

We assume the following equations to hold in the instantaneous rest frame of the particle:

$$m\dot{v}_i = eE_i + \tfrac{1}{2}aS_{mn}B_{mn,i}, \qquad (4.2a)$$

$$\dot{S}_{ik} = -\alpha(S_{ie}B_{ek} - S_{ke}B_{ei}) + \beta(Q_{ie}E_{ek} - Q_{ke}E_{ei}), \qquad (4.2b)$$

$$\dot{Q}_{ik} = -\alpha(Q_{ie}B_{ek} + Q_{ke}B_{ei}) - \beta(S_{ie}E_{ek} + S_{ke}E_{ei}). \qquad (4.2c)$$

In postulating (4.2), we have deliberately neglected terms involving second derivatives in the external electric field. We consider such terms small in comparison to the terms included.

The relativistic theory of (4.2) is easily found by Lorentz transformation:

$$m\,dV_\mu/d\tau = eF_{\mu\nu}V_\nu + \tfrac{1}{2}a\Delta_{\tau\mu}\lambda_\tau, \qquad (4.3a)$$

$$\begin{aligned} dS_{\mu\nu}/d\tau = &-\alpha F_{\rho[\nu}S_{\mu]\rho} + (e/m - \alpha)V_{[\nu}S_{\mu]\rho}F_{\rho\tau}V_\tau \\ &+ (a/2m)V_{[\nu}S_{\mu]\rho}\lambda_\rho + \beta\Delta_{\sigma[\nu}Q_{\mu]\rho}E_{\rho\sigma}, \end{aligned} \qquad (4.3b)$$

$$\begin{aligned} dQ_{\mu\nu}/d\tau = &-\alpha F_{\rho(\nu}Q_{\mu)\rho} + (e/m - \alpha)V_{(\nu}Q_{\mu)\rho}F_{\rho\tau}V_\tau \\ &+ (a/2m)V_{(\nu}Q_{\mu)\alpha}\lambda_\alpha - \beta\Delta_{\sigma(\nu}S_{\mu)\rho}E_{\rho\sigma}, \end{aligned} \qquad (4.3c)$$

where

$$\Delta_{\mu\nu} = \delta_{\mu\nu} + V_\mu V_\nu, \quad \Delta_{\mu\nu}V_\nu = 0,$$
$$E_{\mu\nu} = (F_{\mu\alpha,\nu} + F_{\nu\alpha,\mu})V_\alpha, \quad \lambda_\alpha = S_{\rho\sigma}F_{\rho\sigma,\alpha},$$

and the symbols $F_{\rho[\nu}S_{\mu]\alpha} \equiv F_{\rho\nu}S_{\mu\alpha} - F_{\rho\mu}S_{\nu\alpha}$ and $F_{\rho(\nu}S_{\mu)\alpha} \equiv F_{\rho\nu}S_{\mu\alpha} + F_{\rho\mu}S_{\nu\alpha}$. τ is the proper time. The constants

e, a, α, and β characterize the charge distribution. m is the rest mass of the particle.

Equations (4.3) describe the four-dimensional gyrations of a relativistic dipole quadrupole. These classical relativistic equations differ from others proposed for such a system,[8,9] in that the quadrupole moment is treated as a true deformation, and not as a compounded rotation. The quadrupole moment is independent of the dipole moment, and as a consequence, the homogeneous Lorentz group no longer plays its former crucial role in describing the intrinsic electrodynamic properties of the particle. In other treatments of the problem the assumption has generally been that a classical quadrupole moment is of the form

$$Q_{\mu\nu}'=S_{\mu\alpha}S_{\nu\alpha}-\tfrac{1}{3}(\delta_{\mu\nu}+V_\mu V_\nu)S_{\rho\sigma}S_{\rho\sigma}, \qquad (4.4)$$

so that deformations *per se* are not a measurable attribute of the classical system and presumably fail to appear in the correspondence-principle limit of its quantal counterpart. As is clear from Fano's nonrelativistic analysis, the assumption (4.4) as the unique quadrupole moment is too restrictive. On the other hand, an induced quadrupole moment of the form $Q_{\mu\nu}'$ may always be added to $Q_{\mu\nu}$ in (4.3) without further change in the equations. If, however, the quadrupole moment arises solely as an induced effect, then (4.3c) is unnecessary, and the Casimir invariant of the theory becomes $S_{\rho\sigma}S_{\rho\sigma}$. Under these conditions our equations would become identical with Good's.[9]

The classical set of Eqs. (4.3) should provide us with respectable gyrofrequencies for a moving spin-1 system or higher-spin systems where octupole-moment coupling and still greater moment coupling are insignificant. These same frequencies must be derivable from the quantum theory, but I defer this question to a future work. On the other hand, space-time trajectories deduced from (4.3) are as significant as they usually are in the quantum theory: Calculated orbits may have approximate validity if in our experimental arrangement we have not trifled with the uncertainty principle. However, such calculated trajectories, even when they do violence to the uncertainty principle, may still be useful in constructing solutions to the Schrödinger equation.

Equations (4.3) transform covariantly under Lorentz transformation. However, the universal constants of the motion for the internal variables $S_{\mu\nu}$ and $Q_{\mu\nu}$ are the invariants of the group $SU(3,1)$. The same group[10] has appeared in an approach to the problems of elementary particles via an analysis of Maxwell's matter-free equations. The group $SU(3,1)$ appearing in our paper and in Ref. 10 is closely tied to the Lorentz group. This lack of independence does not conform to the general views concerning the relation of the Lorentz group to the internal symmetry groups of elementary particles. The theory presented here is not intended to serve as a theory of elementary particles. However, beyond its utility in calculating gyroscopic frequencies, the theory may prove helpful in a phenomenological analysis of nuclear structure. In addition, the techniques employed in this paper seem readily applicable to higher-spin systems and other forms of interaction.

[8] P. Havas, Phys. Rev. **116**, 202 (1959). This paper contains references to earlier work and includes a detailed analysis of self-interactions for higher multipoles generated by the spin tensor.

[9] R. H. Good, Phys. Rev. **125**, 2112 (1962).

[10] B. Kurşunoğlu, Phys. Rev. **135**, B761 (1964).