New Test of Quark-Model Predictions of Infinite-Energy Cross Sections Using Superconvergence Theory*

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The quark model predicts that certain combinations of total cross sections should be equal at infinite energy. One such prediction is that $\bar{\sigma}_\pi(\infty)=\frac{2}{3}\bar{\sigma}_N(\infty)$, where $\bar{\sigma}_\pi(\nu)=\frac{1}{2}[\sigma_{\pi^+p}(\nu)+\sigma_{\pi^-p}(\nu)]$ and $\bar{\sigma}_N(\nu)=\frac{1}{4}[\sigma_{pp}(\nu)+\sigma_{pn}(\nu)+\sigma_{\bar{p}n}(\nu)+\sigma_{\bar{p}n}(\nu)]$ are the average pion-nucleon and nucleon-nucleon cross sections, respectively, and ν is the laboratory beam energy. We have tested this prediction by using the implied superconvergence of the forward scattering amplitude $\bar{T}_\pi(\nu)-\frac{2}{3}\bar{T}_N(\nu)$, where $\bar{T}_\pi(\nu)$ and $\bar{T}_N(\nu)$ are defined in an analogous way. We find that the sum rule is badly violated in the laboratory frame with pions as the beam but satisfied in the "antilaboratory frame" with protons as the beam. We conclude that the quark-model prediction is correct so long as it is interpreted in the antilaboratory frame.

I. INTRODUCTION

DURING recent years there has been much discussion of the quark model. It has been applied in many calculations of high-energy hadron scattering with considerable success. The original idea is due to Frankfort and Levin,[1] who used the simple idea that nucleons are made of three quarks and mesons of two. They concluded that the ratio between the pion-nucleon and nucleon-nucleon infinite-energy total cross sections should be exactly $\frac{2}{3}$. This idea was later extended by Kokkedee and Van Hove[2] and independently by Lipkin and Scheck,[3] who deduced many other relations between infinite-energy total cross sections $\sigma_r(\infty)$ of the form

$$\sum_r n_r\sigma_r(\infty)=0\,, \qquad (1)$$

where the integers n_r are simply related to the number of quarks and antiquarks participating in the reactions.

Unfortunately, the straightforward experimental tests of such relations are inconclusive because it is not known at what energy the asymptotic region begins. However, if we define a function $T(\nu)$ as the weighted sum of forward scattering amplitudes T_r of reaction r,

$$T(\nu)=\sum_r n_rT_r(\nu)\,, \qquad (2)$$

and assume that its high-energy behavior is determined by (1), then we should be able to write down a superconvergent dispersion relation for $T(\nu)$.

A similar technique has been tried in photon-hadron scattering[4] but our application has the advantage that the strong-interaction cross sections are much better known than the corresponding electromagnetic ones.

Therefore, if our dispersion relation is well satisfied, it will serve as a test of the quark-model predictions. As an example we shall look at the following sum rule:

$$3\sigma_{\pi^+p}(\infty)+3\sigma_{\pi^-p}(\infty)-\sigma_{\bar{p}n}(\infty)-\sigma_{pn}(\infty)-\sigma_{\bar{p}p}(\infty)-\sigma_{pp}(\infty)=0\,. \qquad (3)$$

We choose this example for several reasons. (1) Its derivation does not depend on $SU(3)$ symmetry[5] but simply on the assumption that pions are made from two quarks and nucleons from three. (2) There are good experimental data available on all the cross sections. (3) We cannot only tell that this combination goes to zero but we can also predict how fast it goes to zero.

Besides the simple quark-model derivation of (3) there is also an equivalent Regge-theory derivation. The most sophisticated version of this is due to Cabibbo, Horwitz, and Ne'eman.[6] They show that under certain assumptions, by taking the combination in (3) we can cancel all the leading singularities for which $\alpha(0)\geq 0$, where $\alpha(0)$ is the intercept of the Regge trajectory at $t=0$. The possible Regge exchanges are the scalar type (P,P',f,f') and the vector type (φ,ω,ρ). The vector exchanges are cancelled in the pion-nucleon cross-section sum and the nucleon-nucleon sum separately and the scalar exchanges are cancelled when we combine both types of scattering.

There remains the question of which frame we should use to evaluate the sum rule. James and Watson[7] have shown that Eq. (3) actually becomes true for $\nu\geq 20$ BeV if we use the so-called antilaboratory system. In this frame the beam always consists of nucleons. The target is then either pions or nucleons, depending on which reaction is being considered. We have investigated both the laboratory-system and antilaboratory-system formulations of the sum rule and find it to be violated in the laboratory frame but satisfied in the antilaboratory frame.

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II. FORMULATION IN THE LABORATORY FRAME

To derive the superconvergent dispersion relation we first normalize our forward scattering amplitude by writing the optical theorem in the form

$$\begin{aligned} \mathrm{Im}T_\pi(\nu) &= [(\nu^2-\mu^2)^{1/2}/8\pi][\sigma_{\pi^+p}(\nu)+\sigma_{\pi^-p}(\nu)], \\ \mathrm{Im}T_p(\nu) &= [M(\nu^2-M^2)^{1/2}/16\pi][\sigma_{pp}(\nu)+\sigma_{p\bar{p}}(\nu)], \\ \mathrm{Im}T_n(\nu) &= [M(\nu^2-M^2)^{1/2}/16\pi][\sigma_{np}(\nu)+\sigma_{n\bar{p}}(\nu)], \end{aligned} \tag{4}$$

where ν is the beam laboratory energy, M is the mass of the nucleon, μ is the mass of the pion, and the $\sigma(\nu)$ are total cross sections.

Next we define

$$F(\nu)=(3M\nu/2k_\pi^2)T_\pi(\nu)-(\nu/k_N^2)[T_p(\nu)+T_n(\nu)], \tag{5}$$

where $k_\pi^2=\nu^2-\mu^2$ and $k_N^2=\nu^2-M^2$, so that at large energies

$$\begin{aligned} \mathrm{Im}F(\nu) &\sim 3[\sigma_{\pi^+p}(\nu)+\sigma_{\pi^-p}(\nu)]-\sigma_{pp}(\nu) \\ &\quad -\sigma_{pn}(\nu)-\sigma_{p\bar{p}}(\nu)-\sigma_{\bar{p}n}(\nu) \\ &\sim \nu^{-1+\alpha(0)}, \end{aligned} \tag{6}$$

where the leading trajectory $\alpha(0)<0$. Hence $F(\nu)$ is superconvergent or

$$\int_{-\infty}^{\infty} \mathrm{Im}F(\nu)d\nu=0. \tag{7}$$

Note also that $\mathrm{Im}F(\nu)$ has the crossing property

$$\mathrm{Im}F(-\nu)=\mathrm{Im}F(\nu),$$

which yields

$$\tfrac{3}{2}M\int_{-\infty}^{\infty} d\nu\, \mathrm{Im}(\nu/k_\pi^2)T_\pi(\nu) = \int_{-\infty}^{\infty} d\nu\, \mathrm{Im}(\nu/k_N^2)[T_p(\nu)+T_n(\nu)]. \tag{8}$$

There are a number of pole terms in (8). The left-hand side has a pole at threshold $\nu=\mu$ and at the proton intermediate state $\nu=\nu_p=-\mu^2/2M$. The right-hand side has poles at threshold $\nu=M$, at the pion intermediate states (in $p\bar{p}$ and $\bar{p}n$) $\nu=\nu_\pi=(\mu^2-2M^2)/2M$, and at the deuteron intermediate state (in pn) $\nu=\nu_d$. Therefore (8) becomes

$$\begin{aligned} &\tfrac{3}{2}M\,\mathrm{Re}T_\pi(\mu)+\frac{3M\nu_pR_p}{\nu_p^2-\mu^2}-\frac{3M}{8\pi^2}\int_\mu^\infty \frac{\nu d\nu}{k_\pi}[\sigma_{\pi^+p}(\nu)+\sigma_{\pi^-p}(\nu)] \\ &= \mathrm{Re}T_p(M)+\mathrm{Re}T_n(M)+\frac{3\kappa}{4B(1-\kappa r_t)}-\frac{3M\nu_\pi R_\pi}{\nu_\pi^2-M^2} \\ &\quad -\frac{2}{\pi}\int_{\nu_0}^{M}\frac{\nu d\nu}{k_N^2}[\mathrm{Im}T_p(\nu)+\mathrm{Im}T_n(\nu)] \\ &\quad -\frac{M}{8\pi^2}\int_M^\infty \frac{\nu d\nu}{k_N}[\sigma_{pp}(\nu)+\sigma_{pn}(\nu)+\sigma_{p\bar{p}}(\nu)+\sigma_{\bar{p}n}(\nu)], \end{aligned} \tag{9}$$

where

$$R_p=4R_\pi=(g^2/4\pi)(\mu^2/4M^2).$$

In (9) the deuteron pole parameters are $\kappa^2=MB$, B=binding energy, and r_t=triplet effective range. The unphysical threshold ν_0 corresponds to the reaction $N\bar{N}\to 2\pi$. Equation (9) is exact under the assumption of superconvergence. Details of the pole-term derivations can be found elsewhere.[8]

We now approximate the unphysical cut by a series of poles corresponding to η, σ, ω^0, and ρ. Then we may write

$$\begin{aligned} \frac{2}{\pi}\int_{\nu_0}^{M}\frac{\nu d\nu}{k_N^2}\mathrm{Im}T_p(\nu) &= \sum_i \frac{MR_i\nu_i}{\nu_i^2-M^2}, \quad i=\eta,\sigma,\omega^0,\rho^0 \\ \frac{2}{\pi}\int_{\nu_0}^{M}\frac{\nu d\nu}{k_N^2}\mathrm{Im}T_n(\nu) &= \frac{2M\nu_\rho R_\rho}{\nu_\rho^2-M^2}, \end{aligned} \tag{10}$$

where for particle i with mass m_i

$$\nu_i \equiv (m_i^2-2M^2)/2M. \tag{11}$$

The residues are given by[8]

$$\begin{aligned} R_\eta &= \frac{g_\eta^2}{4\pi}\frac{M_\eta^2}{16M^2}, \quad R_\sigma=\frac{g_\sigma^2}{4\pi}\left(\frac{m_\sigma^2-4M^2}{16M^2}\right), \\ R_{\rho,\omega} &= \frac{1}{4\pi}\frac{1}{4M}\left\{g_1^2M+[g_1g_2+(g_1+g_2)^2]\frac{m^2}{2M}\right\}, \end{aligned} \tag{12}$$

where g_1 and g_2 are the vector and tensor couplings, respectively. Equation (9) becomes

$$\begin{aligned} &\frac{M}{24\pi^2}\int_M^\infty \frac{\nu d\nu}{k_\pi}[\sigma_{pp}(\nu)+\sigma_{pn}(\nu)+\sigma_{p\bar{p}}(\nu)+\sigma_{\bar{p}n}(\nu)]-\frac{M}{8\pi^2}\int_\mu^\infty\frac{\nu d\nu}{k_N}[\sigma_{\pi^+p}(\nu)+\sigma_{\pi^-p}(\nu)]=\tfrac{1}{3}[\mathrm{Re}T_p(M)+\mathrm{Re}T_n(M)] \\ &\quad -\tfrac{1}{2}M\,\mathrm{Re}T_\pi(\mu)+\frac{\kappa}{4B(1-\kappa r_t)}-M\left[\frac{\nu_pR_p}{\nu_p^2-\mu^2}+\frac{\nu_\pi R_\pi}{\nu_\pi^2-M^2}+\frac{\nu_\rho R_\rho}{\nu_\rho-M^2}+\frac{\nu_\eta R_\eta}{3(\nu_\eta^2-M^2)}+\frac{\nu_\sigma R_\sigma}{3(\nu_\sigma^2-M^2)}+\frac{\nu_\omega R_\omega}{3(\nu_\omega^2-M^2)}\right]. \end{aligned} \tag{13}$$

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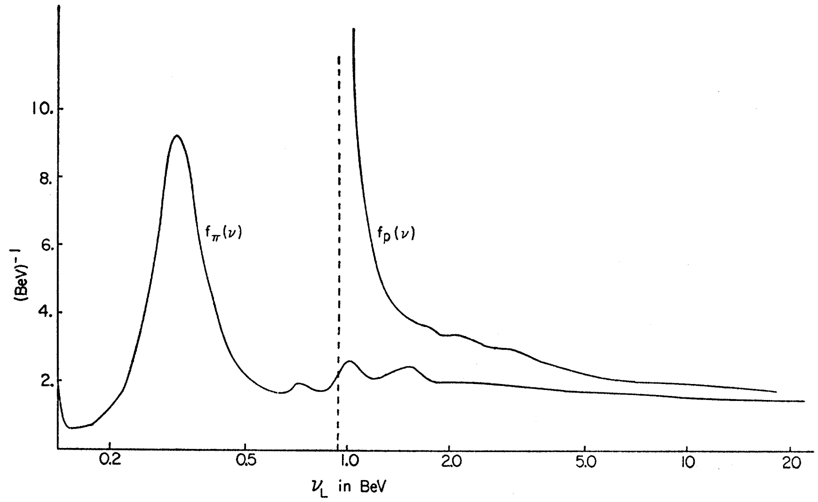

FIG. 1. The values of $f_p(\nu)$ and $f_\pi(\nu)$ defined in Eq. (18) as a function of the laboratory beam energy ν. Note that the ν scale is logarithmic.

We now turn to the numerical evaluation of the right-hand side of Eq. (13). We found that the only large contributions come from the nucleon-nucleon scattering lengths and the deuteron pole. Fortunately these quantities are well determined.

The scattering length contributions are

$$\tfrac{1}{2}M\ \mathrm{Re}T_\pi(\mu)=\tfrac{1}{6}M(a_{1/2}{}^{\pi p}+2a_{3/2}{}^{\pi p})\times(1+\mu/M)=0.067\pm0.27\,, \quad (14a)$$

$$\tfrac{1}{3}\ \mathrm{Re}T_n(M)=(1/24)M(a_s{}^{pn}+a_t{}^{pn})=1.49\pm0.5\,, \quad (14b)$$

$$\tfrac{1}{3}\ \mathrm{Re}T_p(M)=\tfrac{1}{12}Ma_t{}^{pp}=7.00\pm1.6\,. \quad (14c)$$

In (14b) and (14c) we have enlarged the experimental errors to include the $\bar{p}p$ and $\bar{p}n$ scattering lengths. These are not known but can be shown to be less than 1 fm, yielding the uncertainty quoted.

The deuteron pole parameters are[8]

$$B=2.22\text{ MeV}\,,\quad \kappa=45.7\text{ MeV}\,,\quad r_t=1.7\text{ fm}\,, \quad (15)$$

and the coupling constants are

$$g_\pi{}^2/4\pi=14.5\,,\quad g_\eta{}^2/4\pi=10\,,\quad g_{\rho1}{}^2/4\pi=0.6\,,$$
$$g_{\rho2}{}^2/4\pi=2.82\,,\quad g_{\omega1}{}^2/4\pi=1.5\,,\text{ and } g_{\omega2}{}^2/4\pi=0\,. \quad (16)$$

Although these coupling constants are not very well known, their contributions to the sum rule are not large. The paremeters of the σ resonance, if one exists, are not known. If we take the usual estimates $m_\sigma\approx\frac{1}{2}M$ and $g_\sigma{}^2\approx2$, we get a contribution of

$$M\nu_\sigma R_\sigma/3(\nu_\sigma{}^2-M^2)\approx-1.3\,. \quad (17)$$

With these values we have calculated the contributions of the various terms and the results are listed in Table I. In the terms where the coupling constants are not well known, we have included a 100% uncertainty. The errors are then combined as standard deviations.

We now define

$$\Delta(\Lambda)=\int_M^\Lambda f_p(\nu)d\nu-\int_\mu^\Lambda f_\pi(\nu)d\nu\,, \quad (18)$$

where

$$f_p(\nu)=(M/24\pi^2)(\nu/k_N)\times[\sigma_{pp}(\nu)+\sigma_{pn}(\nu)+\sigma_{p\bar{p}}(\nu)+\sigma_{\bar{p}n}(\nu)]\,,$$

and

$$f_\pi(\nu)=(M/8\pi^2)(\nu/k_\pi)[\sigma_{\pi^+p}(\nu)+\sigma_{\pi^-p}(\nu)]\,,$$

so that we can express Eq. (13) as

$$\Delta(\infty)=16.5\pm2.2\,. \quad (19)$$

We have calculated $\Delta(\Lambda)$ for various cutoffs Λ by using the experimental data.[9–11] The functions $f_p(\nu)$ and $f_\pi(\nu)$ are plotted in Fig. 1. We find that for values of

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TABLE I. The contributions of the various terms to the right-hand side of Eq. (13).

Term	Contribution
$\frac{1}{3}\,\mathrm{Re}T_n(M)$	1.49 ±0.5
$\frac{1}{3}\,\mathrm{Re}T_p(M)$	7.00 ±1.6
$-\frac{1}{2}M\,\mathrm{Re}T_\pi(\mu)$	−0.067 ±0.27
$\kappa/4B(1-\kappa r_t)$	8.78 ±0.008
$-M\nu_pR_p/(\nu_p^2-\mu^2)$	−0.0407±0.011
$-M\nu_\pi R_\pi/(\nu_\pi^2-M^2)$	−0.901 ±0.25
$-M\nu_\rho R_\rho/(\nu_\rho^2-M^2)$	−0.645 ±0.65
$-M\nu_\eta R_\eta/3(\nu_\eta^2-M^2)$	−0.189 ±0.19
$-M\nu_\omega R_\omega/3(\nu_\omega^2-M^2)$	−0.191 ±0.19
$-M\nu_\sigma R_\sigma/3(\nu_\sigma^2-M^2)$	1.3 ±1.3
Total	16.5 ±2.2

Λ of 16 and 18 BeV

$$\Delta(16\text{ BeV})=22.3\pm3\,,\qquad \Delta(18\text{ BeV})=23.9\pm3.5\,. \tag{20}$$

Referring to Fig. 1, we can see that $f_p(\nu)-f_\pi(\nu)>0$ and so $\Delta(\Lambda)$ is increasing with Λ away from the sum-rule prediction. If $f_p(\nu)-f_\pi(\nu)$ remains at about 0.5 BeV^{-1} for only a further 10 BeV, we can estimate $\Delta(30\text{ BeV})\approx 28$. It can be seen that the sum rule is badly violated unless there are negative contributions to $\Delta(\Lambda)$. This would imply a crossing of $f_p(\nu)$ and $f_\pi(\nu)$ at some energy. This, however, we do not consider likely, as no other theory predicts such a crossing.

III. FORMULATION IN THE ANTILABORATORY FRAME

The nucleon-nucleon part of the sum rule is, of course, unchanged by transforming to the antilaboratory frame. In the pion-nucleon part the variables k and ν become[7]

$$\bar{\nu}=(M/\mu)\nu\,,$$
$$\bar{k}=(\bar{\nu}^2-M^2)^{1/2}=(M/\mu)k\,.$$

The pole term $3M\nu_pR_p/(\nu_p^2-\mu^2)$ becomes

$$3M\bar{\nu}_p\bar{R}_p/(\bar{\nu}_p^2-M^2)\,,$$

where

$$\bar{\nu}_p=-\tfrac{1}{2}\mu\,,\quad \bar{R}_p=\tfrac{1}{48}g^2/4\pi\,.$$

The new value of the term is

$$3M\bar{\nu}_p\bar{R}_p/(\bar{\nu}_p^2-M^2)=-0.269\pm0.093\,,$$

so that the sum rule becomes

$$\bar{\Delta}(\infty)=16.3\pm2.2\,,$$

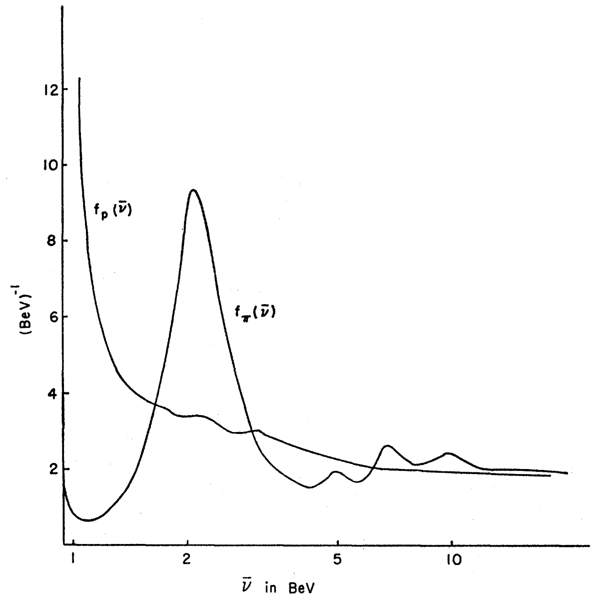

FIG. 2. The values of $f_p(\bar{\nu})$ and $f_\pi(\bar{\nu})$ as a function of the antilaboratory beam energy $\bar{\nu}$.

where now

$$\bar{\Delta}(\Lambda)=\int_M^\Lambda[f_p(\bar{\nu})-f_\pi(\bar{\nu})]d\bar{\nu}$$

and $f_p(\bar{\nu})$ and $f_\pi(\bar{\nu})$ are defined as in Eq. (18). These functions are plotted in Fig. 2.

It can be seen from Fig. 2 that $f_p(\bar{\nu})=f_\pi(\bar{\nu})$ for $\bar{\nu}\geq 18$ BeV, so that $\bar{\Delta}(18\text{ BeV})$ may correspond to $\bar{\Delta}(\infty)$. This is not certain, of course, because they may diverge at higher energies yet unstudied. However, we find

$$\bar{\Delta}(18\text{ BeV})=14.2\pm2.9\,,$$

which agrees with the sum-rule prediction within experimental limits.

IV. CONCLUSION

The quark model and dispersion theory give us a sum rule for a certain cross-section combination of total cross sections. This sum rule is found to be violated in the laboratory system at presently available energies and shows no sign of being satisfied at higher energies. In the antilaboratory frame, however, it is satisfied within experimental errors and appears to remain satisfied as the energy increases.