may have attractive effects, and ghosts may appear. Both of these unpleasant features are removed for a wide range of potential strengths, including those which are likely to be encountered in particle physics, if the third Born approximation is used.

It is expected that this demonstration of how the solutions go awry will enable us to recognize the breakdown of these approximation schemes more readily in the future.

Also, it is believed that there is good reason to hope that in calculations of strong-interaction dynamics, such as, for example, those based on the strip approximation, it will also be the case that an approximation to the left-hand cut involving just a few iterations of the "potential" provides a satisfactory input to the N/D equations. It may be, of course, that the sort of "equivalent potential" obtained from the Mandelstam representation can never be made to give a satisfactory account of strong-interaction dynamics, and that arbitrary parameters specifying the more important features of trajectories are required as input.[40] However, we shall not be able to arrive at a clear decision on such matters until the current-calculational schemes have been explored fully, without the debilitating effects of unnecessarily poor approximations. Calculations are in hand to use this same sort of iteration of the potential for π-π scattering in the strip approximation, and we hope to present results shortly.

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[40] P. D. B. Collins, R. C. Johnson, and E. J. Squires, Phys. Letters **26B**, 223 (1968).

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Tests of Saturation in Strong-Coupling Theory*

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Saturation of sum rules is tested in several strong-coupling static models, and the results are compared with the known properties of the models. A meson current commutation relation is very poorly saturated by low-lying states. Superconvergence relations are discussed in both s- and p-wave models. There are no such relations in s-wave models, but there are two in the p-wave model, one of which is saturated, the other not. It is also shown that certain truncated Chew-Low equations have no solutions.

INTRODUCTION

THE purpose of this paper is to carry out several tests of the saturation hypothesis[1] in soluble field-theoretical models. Obviously, caution must be exercised in generalizing from these, but they have the advantage of being soluble. Usually[2] the only test of saturation is the experimental data; this often leads to little insight into the situation, and therefore in contrast we shall ask our questions of a toy (but understood) world.

All our examples will be in the strong-coupling limit of static models.[3,4] Properties of these models are well known, and this makes it easy to test a few sum rules within them. We shall first try saturating a meson current commutator and find that it fails miserably. We next consider some superconvergence relations, which have been derived in the model. We find contradictions in the sum rules (some do not saturate) and prove a related theorem on the existence of solutions to a class of cutoff static models. The p-wave model is seen to have one derivable sum rule which is valid and another which fails.

SATURATION OF A CURRENT COMMUTATOR

We shall deal first with the question of saturation in charge-symmetric scalar theory, which may be defined by the interaction Hamiltonian

$$H_I = g_0 \int d^3r\, \tau_\alpha u(r) \phi_\alpha(\mathbf{r})\,, \tag{1}$$

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[1] Recent applications of the saturation scheme include: V. de Alfaro, S. Fubini, C. Rossetti, and G. Furlan, Phys. Letters **21**, 576 (1966); B. Sakita and K. C. Wali, Phys. Rev. Letters **18**, 319 (1967); K. Bardakci and G. Segrè, *ibid.* **159**, 1263 (1967); S. Weinberg, *ibid.* **18**, 507 (1967).

[2] J. J. Sakurai, Phys. Rev. Letters **19**, 893 (1967); T. Das, V. S. Mathur, and S. Okubo, *ibid.* **19**, 470 (1967).

[3] G. Wentzel, Helv. Phys. Acta **13**, 269 (1940); S. Tomonaga, Progr. Theoret. Phys. (Kyoto) **2**, 6 (1947).

[4] C. Goebel, Midwest Research Conference, 1965 (unpublished); T. Cook, C. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965).

where

τ_α = Pauli matrices, $\alpha = 1, 2, 3$;

$u(r)$ = shape of extended source,

$\phi_\alpha(\mathbf{r})$ = meson field operator,

g_0 = bare Yukawa coupling.

We shall make use of the coupling operator $\mathcal{G}_\alpha$ which is defined on the discrete isobar states

$$\mathcal{G}_\alpha{}^{s's} \equiv \langle s'|g_0\tau_\alpha|s\rangle, \tag{2}$$

where s and s' are discrete physical states. In the strong-coupling limit ($g_0 \to \infty$), the $\mathcal{G}_\alpha$ satisfy the coupling condition

$$[\mathcal{G}_\alpha,\mathcal{G}_\beta]=0. \tag{3}$$

We shall test saturation on a sum rule[5] which is an identity satisfied by Pauli matrices

$$[\tau_\alpha,\tau_\beta]=i\epsilon_{\alpha\beta\gamma}\tau_\gamma. \tag{4}$$

Now if we multiply through by g_0^2 and insert a complete set of physical intermediate states we obtain

$$[\mathcal{G}_\alpha,\mathcal{G}_\beta]^{s's}+\sum_n \langle s'|g_0\tau_\alpha|n\rangle\langle n|g_0\tau_\beta|s\rangle - \sum_n \langle s'|g_0\tau_\beta|n\rangle\langle n|g_0\tau_\alpha|s\rangle = g_0 i\epsilon_{\alpha\beta\gamma}\mathcal{G}_\gamma{}^{s's}, \tag{5}$$

where the $|n\rangle$ are physical (in or out) states of more than one particle. The matrix elements in the sum are related to scattering and multiple meson-production amplitudes. For example, the scattering amplitude $f_{\beta\alpha}{}^{s's}(\omega)$ is given by

$$f_{\beta\alpha}{}^{s's}(\omega)=\frac{u(k)(2\omega)^{1/2}}{2\pi}\langle s',k,\beta|g_0\tau_\alpha|s\rangle, \tag{6}$$

where $u(k)$ is the Fourier transform of $u(r)$. In a strong-coupling model, $f_{\beta\alpha}{}^{s's}(\omega)$ is given by[4]

$$f_{\beta\alpha}{}^{s's}(\omega)=\frac{\Lambda_{\beta\alpha}{}^{s's}}{-\mu-ik} \tag{7}$$

for ω much less than the cutoff energy which is defined by $u(k)$, and $\Lambda_{\beta\alpha}{}^{s's}$ is a certain matrix satisfying[4]

$$\sum_\gamma \Lambda_{\beta\gamma}\Lambda_{\gamma\alpha}=\Lambda_{\beta\alpha}. \tag{8}$$

Equation (8) is called the "mass condition." Finally, it is known[6] that below the cutoff all multiple meson-production cross sections vanish for large enough coupling. Now Eqs. (3) and (6)–(8), together with the vanishing production amplitudes, give the left-hand side of Eq. (5) to be zero when the sum over n is truncated at energies below the cutoff; whereas the right-hand side of Eq. (5) is of order g^2, where g is the renormalized coupling. (Note that in this model[6] $g^2 \approx \frac{1}{3}g_0^2$.) We have found, then, that not only saturation by the discrete states fails, but even inclusion of the multiple-particle states fails if we again cut off the energy sum over intermediate states. The reason is clearly that the very top of the continuum conspires to give a contribution which will match the order g^2 quantity on the right-hand side. This last is sensible if, for example, at very large ω the scattering approaches the "bare" Born approximation.

SUPERCONVERGENCE

We now turn to the question of superconvergence in s-wave static models. It has been claimed[7] that Eq. (3) is the result of a superconvergence relation imposed by the cutoff in the Chew-Low equation. We shall first find a contradiction in that derivation and in the next section produce a theorem which helps to explain the paradox.

The derivation consists of writing the Chew-Low equation for the function $h_{\beta\alpha}{}^{s's}(\omega) \equiv f(\omega)/u(k)^2$:

$$h_{\beta\alpha}{}^{s's}(\omega)=\frac{1}{\pi}\int\frac{u(k')^2\rho_{\beta\alpha}{}^{s's}(\omega')d\omega'}{\omega'-\omega}, \tag{9}$$

where the integral runs along the upper edge of the left- and right-hand cuts and includes poles lying between $-\mu$ and μ, the meson mass. As usual, $\rho_{\beta\alpha}{}^{s's}$ is given by unitarity and crossing. Now Pande[7] pointed out that if $u(k)\equiv 0$ for $|\omega|>\Omega$, it is easy to derive an infinite number of sum rules, viz.,

$$\int_{-\Omega}^{\Omega} d\omega\, \omega^n\, \mathrm{Im}h_{\beta\alpha}{}^{s's}(\omega)=0, \quad n=0, 1, 2, \cdots. \tag{10}$$

For $n=0$, the saturation by discrete states leads to the coupling condition Eq. (3). Pande dismisses the cases for $n\geq 1$ because the isobars become degenerate and each of the terms on the left-hand side would vanish separately. Our point is that this is not so for $n=1$, because the masses become degenerate only as $1/g^2$ and this leads to a contribution from the pole terms to the left-hand side equal to

$$[\mathcal{G}_\beta, [\mathbf{M}, \mathcal{G}_\alpha]]^{s's} \equiv 2\mu\Lambda_{\beta\alpha}{}^{s's}, \tag{11}$$

where $\mathbf{M}^{s's}\equiv \mathbf{M}_s\delta_{s's}$ is the mass operator for the isobar states defined relative to the ground state. In Eq. (11), $\Lambda_{\beta\alpha}{}^{s's}$ is the same as in Eqs. (7) and (8), and is of order 1, not zero. Equation (10) then is false for $n=1$, when we use the saturation hypothesis.

One might now try to explain the failure by pointing out that, after all, with $n=1$ we are weighting the higher-

[5] Sum rules like this date back at least to Ref. 11 and M. Cini and S. Fubini, Nuovo Cimento **3**, 764 (1956). For a recent treatment but in a slightly different model, cf. H. Miyazawa, Progr. Theoret. Phys. (Kyoto) Suppl. **37** and **38**, 315 (1966).

[6] This result is probably in the literature, but I was made aware of it by Professor C. Goebel (private communication).

[7] L. K. Pande, Phys. Letters **24B**, 243 (1967).

energy states and therefore we might not expect saturation by low-lying states to work as well in that case. One actually has even more help in the $n=0$ case since $\mathrm{Im}h_{\beta\alpha}{}^{s's}(\omega)$ is nearly an odd function along the continuum of the integration path [cf. Eq. (7)]. This last leads to cancellation of the continuum contributions in the $n=0$ case, but constructive adding in the $n=1$ case. In the next section we shall prove that the *derivation* of equations like Eq. (10) is probably suspect in the static models.

EXISTENCE OF SOLUTIONS

We may yet understand the failure of Eq. (10) by consideration of the following theorem:

The truncated Chew-Low equation

$$f(\omega)=\frac{-g^2}{\omega}+\frac{1}{\pi}\int_1^{\Omega}\frac{\mathrm{Im}f(\omega')}{\omega'-\omega}d\omega', \qquad (12)$$

with $\mathrm{Im}f=k|f|^2$, has no solutions for $g^2>0$.

The proof follows from examining the inverse function $\beta(\omega)\equiv g^2/\omega f$ and observing that for sufficiently small g^2 it must have a zero to the right of $\omega=\Omega$. Therefore, Eq. (12) must be amended by at least a pole to the right of $\omega=\Omega$ if it is to have a solution at all.

The physical (or unphysical) significance of the theorem will now be explained. Equation (12) defines the model which has a direct channel pole, a right-hand cut, and no left-hand cut. This is the ordinary N–θ sector of the Lee model[8] except that here we have truncated the continuum integration. The spurious state (pole) is just the ghost[9] which appears here to the right of the continuum for small coupling and moves to $\omega=+\infty$ as the coupling is raised to a critical value. After this, the state moves in from $\omega=-\infty$ toward the origin and is the *usual* ghost.

Now since the higher models do not have ghosts, we must be careful that we are not just studying a peculiarity of the Lee model. Fortunately, the analog of our theorem may be checked in charged scalar theory where crossing symmetry is still tractable. We do not give the argument here,[10] but the result is a spurious pole either to the right of $\omega=\Omega$, or the left of $\omega=-\Omega$. This pole never comes near the ordinary bound-state region, and so we are never troubled by a ghostlike state in the theory.

The meaning of our theorem is that truncating the Chew-Low equation does not correspond to a conventional extended-source model. Equation (10) then is probably false even without saturation. Equation (10) is also false in the case of the conventional extended-source (s-wave) model because the functions $f_{\alpha\beta}{}^{s's}(\omega)$ which go to zero rapidly at infinity (they are proportional to the cutoff function) do not lead to superconvergence relations because these functions have singularities on the imaginary ω axis. One may eliminate these singularities by dividing them out (this is usually done[11]), but the resulting quotient does not go to zero rapidly enough at infinity to give superconvergence.

[8] T. D. Lee, Phys. Rev. **95**, 1329 (1954).

[9] G. Källén, in *Brandeis Lectures* (W. A. Benjamin, Inc., New York, 1962), Vol. 2, p. 188.

[10] The relevant (but not cutoff) inverse Chew-Low equation is given in C. J. Goebel, Phys. Rev. **109**, 1846 (1958).

p-WAVE MODELS

Before concluding, we should mention the p-wave static models[11] which are of phenomenological interest. Here, the scattering amplitude is normalized such that

$$T_{IJ}(\omega)=\frac{\eta_{IJ}e^{2i\delta_{IJ}}-1}{2ip^3}, \qquad (13)$$

where δ_{IJ} is the phase shift in the partial wave (IJ) and p is the meson momentum. Now $T_{IJ}(\omega)$ goes to zero as $1/\omega^3$ and $\omega\to\infty$ aside from any damping due to a cutoff function. This *does* lead to a superconvergent sum rule and the saturation hypothesis leads to Eq. (3), the correct answer[12,13] according to strong-coupling theory.

From here the discussion is the same as in the s-wave case. The only difference being that the superconvergence relations for $n=0$, 1, $(\Omega=\infty)$ in Eq. (10) are definitely valid for the p-wave case. In the p-wave case then we have a clear-cut test of saturation. It works [i.e., gives Eq. (3)] for $n=0$, and as in the s-wave case it fails (i.e., leads to $\Lambda_{\beta\alpha}{}^{s's}=0$) for $n=1$.

The saturation hypothesis has now been tested in several strong-coupling models and the results are seen to coincide rarely with the known answers. We conclude that it is extremely difficult to determine *a priori* whether a given sum rule can be saturated by discrete states or even continuum states which are cut off in energy.

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It is a pleasure to thank Dr. M. P. Khanna and Professor G. Segrè for helpful conversations on this subject.

[11] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[12] This observation is not new and has been variously credited to many investigators. I have not seen it contrasted with the s-wave case, however, where the result obtains without superconvergence.

[13] H. Goldberg, Phys. Letters **24B**, 71 (1967). Note that we differ in point of view from this paper, which uses experimental information in the static model. We are concerned with the fixed-source theories as theories consistent within themselves rather than with their usefulness as a phenomenological aid.