evaluating, one easily sees that it differs from the pole term by a factor

$$2^{2\alpha(0)}[(s_0-m^2)/\Lambda]^{\alpha(0)},$$

assuming $\alpha(t)$ to be linear in t. We cannot determine this factor, since we do not know Λ, but for the η meson this can be quite different from unity. Both the behavior of $\beta(t)$ near $t=0$ and an estimate of Λ can be obtained from high-energy Compton-scattering data (Regge region), for which we have no information so far.

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Use of Born Approximations in N/D Calculations

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The N/D equations have been solved with the first, second, and third Born approximations to the left-hand cut, for nonrelativistic, single-channel potential scattering, with potentials involving combinations of attraction and repulsion of different ranges, and the results are compared with the exact solution of the Schrödinger equation. It is found that for the sort of potential strengths which occur in strong-interaction dynamics, the third Born approximation is satisfactory. It is known that the first Born approximation, which is commonly used, suffers from several defects in that long-range repulsions can produce attractive effects, and "ghosts" appear on the physical sheet, and we explore the way in which the approximation breaks down. It is concluded that in dynamical calculations, such as those involving the strip approximation, much more satisfactory results are likely to be obtained if the left-hand cut is calculated from a few iterations of the potential.

I. INTRODUCTION

It has been known for some time that the forces which generate strongly interacting particles are likely to contain both attractive and repulsive components. In particular, it is known that the exchange of a Pomeranchon (P) trajectory gives rise to a long-range repulsion.[1]

This has created severe difficulties for the usual sort of N/D calculations which are used to solve dynamical problems.[2] In such calculations it is usual to impose unitarity on an amplitude whose left-hand cut is given by just the first Born approximation; that is, the left-hand cut of the amplitude is assumed to contain just the cut of the potential. This is not, of course, the same as taking the first Born approximation to the amplitude, in that exact unitarity (or at least exact within the framework of the possibility multichannel calculation which we wish to perform) is imposed on the right-hand cut, but the effects of the reaction to the potential on the left-hand cut are ignored. It has been found that if a repulsive force is combined with an attraction (the two having different ranges), the effect of the repulsion if often to give stronger binding, i.e., to act as an attraction.

This fact was commented on by Kayser,[3] and has been noted since by many authors,[4–7] particularly in the context of the Dashen-Frautschi type of perturbation calculation.[8,9]

What is worse, if the repulsion is really strong it is possible for "ghosts," by which is meant in this context resonances with negative residues, to appear.[2] These violate causality and so must be due to the inadequacy of our approximations.

In calculations involving the "new form of the strip approximation" it was found necessary to remove the P repulsion by normalizing the potential,[2] though it was realized at the time that the validity of this procedure was doubtful, and that the P repulsion probably represents the physically important effects of the presence of infinitely many channels with thresholds above the resonance region.[1]

It is to be expected that these defects of the strip approximation would be removed if we were able to use an exact expression for the left-hand cut, but in general, this is prohibitively difficult to calculate. The question thus arises as to the order of the Born approximation to the left-hand cut which is needed to give satisfactory accuracy in this sort of problem. The best way of trying to assess this is to examine the situation in single-channel potential scattering, where we can compare the solution of the N/D equations, for various types of potentials treated in various Born approximations, with the exact solution to the corresponding Schrödinger equation. We know, of course, that if we

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had used the exact left-hand cut the N/D equations would give the exact answer.

Luming[10] has examined the problem for a single attractive potential in the first and second Born approximations, and in this paper we extend his work to include the third Born approximation and also to consider combinations of potentials of different signs and ranges. It is anticipated that this will give us guidance as to the likelihood of obtaining reasonable results with similar approximations to the left-hand cut in strong interactions.

In Sec. II we briefly review the difficulties which have arisen in recent dynamical calculations and remind the reader of the P-repulsion effect. Succeeding sections briefly review the N/D equations and explain how the various approximations to the left-hand cut may be calculated using the Mandelstam iteration method. In Sec. V we present some numerical examples of our solutions which demonstrate how the various approximations break down. Section VI discusses the same phenomena in a simple model which can be solved analytically; finally, in Sec. VII we present some conclusions.

II. PROBLEMS IN RECENT DYNAMICAL CALCULATIONS

Since the pioneering work of Chew and Mandelstam,[11,12] many attempts have been made to perform calculations of the dynamics of strong interactions, bootstrap calculations, using methods which are closely analogous to those used in potential scattering.[13] Much recent work by various authors has been based on the strip approximation,[14] and particularly on the so-called "new form" of the strip approximation devised by Chew and Jones[15] on the basis of work by Chew.[16] But as a means of calculating the π-π scattering amplitude, for example, this approximation was found to be inadequate.[2,17,18] The ρ-exchange force was not strong enough to produce the desired trajectories in the direct channel, and the P repulsion gave rise to ghosts unless it was assumed that the potential could be normalized,[19] that is, that there were other isotopic spin $I=0$ contributions (lower-lying trajectories) which cancelled the repulsion, and turned the total $I=0$ exchange force into an attraction. The arguments for and against this are discussed in Sec. IV of Ref. 2.

The reason why the P gives a repulsion has been explained by Chew[1,20] on the basis of the Khuri-Jones representation[21] for a Regge pole. The force in the s channel due to the exchange of a t-channel Regge trajectory α can be approximated by a t-channel partial-wave series

$$V(s,t)=\sum_{l_t}(2l_t+1)V_{l_t}(t)P_{l_t}(1+s/2q_t^2)\,, \qquad (2.1)$$

with

$$V_{l_t}(t)=\gamma(t)(q_t^2)^{\alpha(t)}e^{-[l_t-\alpha(t)]\xi_1(t)}/[l_t-\alpha(t)]\,, \qquad (2.2)$$

where

$$\xi_1(t)=\ln\{z_1(t)+[z_1^2(t)-1]^{1/2}\} \qquad (2.3)$$

and

$$z_1(t)=1+s_1/2q_t^2\,,\quad s_1\gg m_\pi^2\,. \qquad (2.4)$$

It is evident from (2.2) that if $\alpha(t)$, which is constrained by the Froissart bound to be less than unity in the s-channel physical region, is greater than a given value of l_t, then we can expect a repulsion from that partial wave. The lowest allowed value of l_t is $l_t=0$ for an even-signature trajectory, and $l_t=1$ for odd signature, and the contribution of each succeeding (even or odd, respectively) partial wave will be much reduced over the preceding one by the exponential factor in (2.2). Thus an even-signature trajectory with $\alpha(t)$ above zero can be expected to give a repulsion, while an odd-signature trajectory will give an attraction. We refer the reader to Chew's paper for further details. The range of the P repulsion for a given elastic process, as measured by the inverse of its logarithmic derivative with respect to t at $t=0$, will be essentially the same as the width of the high-energy diffraction peak for that process.[1] This is a considerably longer range than that of most of the attractive particle-exchange forces.

This repulsion, which corresponds to the effect of the many high-threshold channels which open up above the resonance region, is expected[1] (by analogy with the situation in nuclear physics) to be important in producing narrow resonances, and also in keeping the trajectories rising to higher values of angular momentum. It was found that the trajectories calculated in the new form of the strip approximation always turn over almost as soon as the threshold is reached,[2,17] making it very hard to produce a P-wave resonance corresponding to the ρ, let alone a D-wave resonance on the P trajectory. When resonances were produced they were much too wide.

The inability of the new form of the strip approximation to cope with a strong repulsion is due simply

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to the fact that it employs only the Born approximation to the left-hand cuts of the partial-wave amplitude. The presence of the repulsion has caused a renewal of interest in the old form of the strip approximation,[22,23] in which elastic unitarity is used to calculate the complete double spectral function in the strip region by iterating the potential out to asymptotic values of t, and then identifying the s-channel poles from the asymptotic behavior of the amplitude, $\sim t^{\alpha(s)}$. The numerical accuracy required to calculate the trajectories in this way is very great, however, particularly in situations where several trajectories occur one below the other.

Our alternative proposal is to use the iterative method simply to calculate the first few Born approximations to the left-hand cut, and then to impose unitarity by means of the N/D equations in the usual way. This will be advantageous if, in the expected situation that the forces are combinations of attractions and repulsions with different ranges, only a few iterations are needed to get a satisfactory approximation to the potential. In the following sections we shall try to obtain guidance in this matter by examining the same problem in nonrelativistic potential scattering.

III. N/D EQUATIONS

The numerical calculations are made for the nonrelativistic scattering of equal-mass scalar particles due to a superposition of N simple Yukawa potentials. We choose units such that $\hbar=c=1$, and the external mass $m=1$. Then in the c.m. system the radial Schrödinger equation is

$$\psi''(r)+\{q_s^2-V(r)-l(l+1)r^{-2}\}\psi(r)=0\,, \qquad (3.1)$$

where q_s is the magnitude of the momentum of the particles. The potential is taken to be

$$V(r)=\sum_{i=1}^{N} V_i(r)\,, \qquad (3.2)$$

where

$$V_i(r)=g_i e^{-m_i r}/r\,. \qquad (3.3)$$

Defining the cms scattering angle to be θ, we introduce the variables (corresponding to the usual relativistic invariants)

$$s=4(1+q_s^2)\,, \qquad (3.4)$$

$$t=-(\text{c.m. momentum transfer})^2$$
$$=2q_s^2(1-\cos\theta)\,. \qquad (3.5)$$

The scattering amplitude has a Mandelstam representation,[24] and it has Regge asymptotic behavior in t.[25] We can thus write a fixed-energy dispersion relation[26]

$$A(s,t)=\frac{1}{\pi}\int^{\infty}\frac{D_t(s,t')}{t'-t}dt'\,, \qquad (3.6)$$

where the necessary limiting process to approach the cut is understood. Separating out the poles, we may express the cut discontinuity $D_t(s,t)$ in terms of the Mandelstam double spectral function $\rho(s,t)$;

$$D_t(s,t)=\pi\sum_{i=1}^{N} g_i\delta(m_i^2-t)+\frac{1}{\pi}\int_{s_0(t)}^{\infty}\frac{\rho(s',t)}{s'-s}ds'\,, \quad (3.7)$$

where $s_0(t)$ is the boundary of $\rho(s,t)$ in the s-t plane. In Sec. IV we describe how $\rho(s,t)$ can be calculated from the pole parameters g_i and m_i, i.e., from $V(r)$.

The partial-wave series for the scattering amplitude may be written

$$A(s,t)=\sum_{l=0}^{\infty}(2l+1)A_l(s)P_l(\cos\theta)\,, \qquad (3.8)$$

and we define the "reduced" partial-wave amplitudes

$$B_l(s)=q_s^{-2l}A_l(s) \qquad (3.9)$$

so as to enforce the correct threshold behavior upon $A_l(s)$ and remove the kinematical branch point at threshold for nonintegral l.

Physical partial-wave amplitudes are uniquely interpolated in l by the Froissart-Gribov projection[27]

$$B_l(s)=\frac{1}{2\pi q_s^{2l+2}}\int_0^{\infty}Q_l(\cos\theta)D_t(s,t)dt\,, \quad (3.10)$$

and no problem of signature arises because we do not consider exchange forces.

In the s plane, $B_l(s)$ is real analytic, with possible bound-state poles, with a right-hand branch point determined by unitarity at $s=4$, and with a series of left-hand branch points stemming from the t singularities of $A(s,t)$. As usual, we write

$$B_l(s)=N_l(s)/D_l(s)\,, \qquad (3.11)$$

where the N and D functions are real analytic in s with only left-hand, and only right-hand, cuts, respectively. We can normalize these functions so that

$$N_l(s)\xrightarrow[s\to\infty]{}0\,, \qquad (3.12)$$

$$D_l(s)\xrightarrow[s\to\infty]{}1\,. \qquad (3.13)$$

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[27] M. Froisart, Report to the La Jolla Conference on High-Energy Physics, 1961 (unpublished); V. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 667 (1961); **41**, 1962 (1961) [English transls.: Soviet Phys.—JETP **14**, 478 (1962); **14**, 1395 (1962)].

On the right-hand cut, unitarity requires

$$\mathrm{Im}D_l(s) = -\rho_l(s)N_l(s)\,, \qquad (3.14)$$

where the phase-space factor is

$$\rho_l(s) = q_s^{2l+1}. \qquad (3.15)$$

We introduce the potential function $B^V(s)$, defined to have the same left-hand cut as $B_l(s)$, viz.,

$$B_l^V(s) = B_l(s) - \frac{1}{\pi}\int_4^\infty \frac{\mathrm{Im}B_l(s')}{s'-s}ds'. \qquad (3.16)$$

We can then deduce the usual integral equation[28] for $N_l(s)$:

$$N_l(s) = B_l^V(s) + \frac{1}{\pi}\int_4^\infty \frac{B_l^V(s') - B_l^V(s)}{s'-s}\rho_l(s')N_l(s')ds'\,, \qquad (3.17)$$

and $D_l(s)$ is determined by

$$D_l(s) = 1 - \frac{1}{\pi}\int_4^\infty \frac{\rho_l(s')N_l(s')}{s'-s}ds'. \qquad (3.18)$$

To derive an expression for $B_l^V(s)$ more useful than (3.16) we substitute (3.7) into (3.10). The imaginary part of $B_l(s)$ on the right-hand cut (RHC) is

$$\mathrm{Im}B_l(s)\,|_{\mathrm{RHC}} = \frac{1}{\pi}\int \rho(s,t)Q_l\left(1+\frac{t}{2q_s^2}\right)\frac{dt}{2q_s^{2l+2}}\,, \qquad (3.19)$$

and so (3.16) gives[29]

$$B_l^V(s) = \frac{1}{2q_s^{2l+2}}\sum_{i=1}^N g_iQ_l\left(1+\frac{m_i^2}{2q_s^2}\right) + \frac{1}{\pi^2}\int\int \frac{\rho(s',t)}{s'-s} \times \left\{\frac{Q_l(1+t/q2_s^2)}{2q_s^{2l+2}} - \frac{Q_l(1+t/2q_{s'}^2)}{2q_{s'}^{2l+2}}\right\}ds'dt. \qquad (3.20)$$

Provided that all the infinite integrals converge, it is easy to solve the N/D equations (3.17) and (3.18); if $B_l^V(s)$ were known exactly, the results would be just those obtained by solving the Schrödinger equation (3.1).[30]

We shall see in Sec. IV that for our approximations to $\rho(s,t)$ the integrals of (3.20) present no difficulty, and

$$B_l^V(s) \underset{s\to\infty}{\sim} \ln s/s^{l+1}\,, \qquad (3.21)$$

that is, the potential approaches its first Born approximation.

From (3.17)

$$N_l(s) \underset{s\to\infty}{\sim} 1/s\,, \qquad (3.22)$$

where we now neglect possible logarithmic factors [cf. (3.12)]. Therefore the integral equation for $N_l(s)$ is well defined for l in the range $-1<l<\frac{3}{2}$.

In practice, we shall replace all the infinite upper limits by a cutoff s_1 for the purpose of numerical calculation. This introduces a logarithmic singularity in $B_l^V(s)$ at $s=s_1$, and correspondingly renders (3.17) non-Fredholm; but it has been shown[31] that despite this, the equation can be uniquely solved for $N_l(s)$. In fact, the solution is essentially independent of s_1 if it is taken sufficiently large.

In Sec. IV we describe the calculation of $B_l^V(s)$ from $V(r)$.

IV. POTENTIAL FUNCTION

When we refer to the potential function as determined in the nth Born approximation, we mean that it is derived from g_i, m_i, and $\rho(s,t)$ correct to order g_i^n for all $i=1,\cdots,N$. If we were to use this $\rho(s,t)$ in (3.7) to calculate $D_t(s,t)$, and then apply (3.6) to find $A(s,t)$, this would be equivalent to summing the first n terms of the Born series.[30] The amplitude would not, of course, be unitary. What we shall in fact do is to solve the N/D equation with $B_l^V(s)$ known to the nth Born approximation. This will not give the same answer because the N/D equations enforce unitarity, starting from the approximate left-hand cut, and for low n there may be a considerable difference.

The potential in the first Born approximation is found by setting $\rho(s,t)=0$ and keeping only the pole terms. The potential in the nth Born approximation is obtained by iteration using the Mandelstam elastic unitarity equation,[24,32] which we write in the form

$$\rho(s,t) = \frac{1}{2\pi q_s}\int\int \frac{D_{t*}(s,t')D_t(s,t'')\theta(K)dt'dt''}{K^{1/2}(q_s^2,t,t',t'')}\,, \qquad (4.1)$$

where

$$K = t^2 + t'^2 + t''^2 - 2(tt' + t't'' + t''t) - (tt't''/q_s^2). \qquad (4.2)$$

The θ function in (4.1) defines a region of integration in the $t'-t''$ plane which (4.2) gives as

$$t^{1/2} \geqslant t'^{1/2} + t''^{1/2} \qquad (4.3)$$

at $s=\infty$, and as a smaller region with a curved boundary for finite s.[32] Therefore, an iterative procedure for calculating $\rho(s,t)$ using (4.1) and (3.7), emerges. For (4.3) tells us that to find $\rho(s,t)$ exactly at $t=\bar{t}$, we need to know $D_t(s,t)$ only for $t \leqslant \bar{t}$. It is thus possible to calculate the nth Born term from the previous $n-1$.

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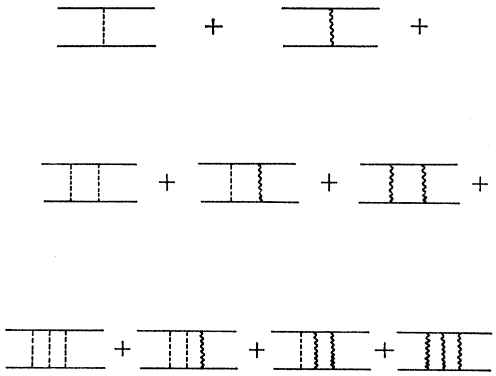


FIG. 1. The unitarity diagrams for two different Yukawa potentials up to the third Born approximation.

If we regard the Yukawa potential as representing the exchange of a particle, this is analogous to calculating a given graph from a knowledge of the lower-order ones, starting with the single-particle-exchange diagram. For example, if we calculate to order $n=3$, for $N=2$ different Yukawa potentials, we sum the graphs depicted in Fig. 1.

We can find $\rho(s,t)$ explicitly for $n=2$ and $n=3$ as follows: Using only the pole term of (3.7), we derive from (4.1)

$$\rho^{(2)}(s,t)=\sum_{i,j=1}^{N}\rho_{ij}^{(2)}(s,t)\,, \tag{4.4}$$

where

$$\rho_{ij}^{(2)}(s,t)=\pi a_{ij}(t)[s-s_0^{ij}(t)]^{-1/2} \tag{4.5}$$

and

$$a_{ij}(t)=g_ig_j\{t[t+(m_i^2-m_j^2)^2/t-2(m_i^2+m_j^2)]\}^{-1/2}. \tag{4.6}$$

There are $\frac{1}{2}N(N+1)$ distinct terms in the double sum of (4.4) because of the i-j symmetry. (This is evident from the graphs of Fig. 1.)

Each distinct piece of double spectral function $\rho_{ij}^{(2)}(s,t)$ has a boundary in the s-t plane given by

$$s=s_0^{ij}(t)\equiv 4\{1+(m_im_j)^2[t+(m_i^2-m_j^2)^2/t-2(m_i^2+m_j^2)]^{-1}\}. \tag{4.7}$$

Using (3.7), we then derive

$$D_t^{(2)}(s,t)=\text{pole terms}+\sum_{i,j=1}^{N}D_t^{(2)ij}(s,t)\,, \tag{4.8}$$

where

$$\mathrm{Re}D_t^{(2)ij}(s,t)=\pi a_{ij}(t)\{s_0^{ij}(t)-s\}^{-1/2}\,,\quad \mathrm{Im}D_t^{(2)ij}(s,t)=0 \tag{4.9}$$

for $s\leqslant s_0^{ij}(t)$, and

$$\mathrm{Re}D_t^{(2)ij}(s,t)=0\,,\quad \mathrm{Im}D_t^{(2)ij}(s,t)=\rho_{ij}^{(2)}(s,t) \tag{4.10}$$

for $s\geqslant s_0^{ij}(t)$.

Using (4.1) again, we find

$$\rho^{(3)}(s,t)=\rho^{(2)}(s,t)+\sum_{i,j,k=1}^{N}\rho_{ijk}^{(3)}(s,t)\,, \tag{4.11}$$

where the terms of order g^3 are given by

$$\rho_{ijk}^{(3)}(s,t)=\frac{g_i}{2q_s}\int\int\frac{\delta(m_i^2-t'')\,\mathrm{Re}D_t^{(2)jk}(s,t')\vartheta(K)}{K^{1/2}(q_s^2,t,t',t'')}dt'dt''. \tag{4.12}$$

Using (4.9), this becomes

$$\rho_{ijk}^{(3)}(s,t)=\pi g_ig_jg_k\int_{(m_j+m_k)^2}^{L(s,t)}\{(-1)(ax^2-2b_1x+c_1)\times(ax^2-2b_2x+c_2)\}^{-1/2}dx\,, \tag{4.13}$$

where we write

$$\begin{aligned}a&=s-4\,,\\ b_1&=a(m_i^2+t)+2tm_i^2\,,\\ c_1&=a(m_i^2-t)^2\,,\\ b_2&=a(m_j^2+m_k^2)+2m_j^2m_k^2\,,\\ c_2&=a(m_j^2-m_k^2)^2.\end{aligned} \tag{4.14}$$

We see that symmetry in j and k leaves only $N(N+1)$ distinct terms in the triple sum of (4.11), as might be expected from the graphs of Fig. 1.

The upper limit of the integral in (4.13) is given by the lowest zero of the denominator $(ax^2-2b_1x+c_1)$, namely,

$$x=L(s,t)=a^{-1}(s)\{[b_1^2(t)-a(s)c_1(t)]^{1/2}-b_1(t)\}. \tag{4.15}$$

The boundary of $\rho_{ijk}^{(3)}(s,t)$ in the s-t plane is given by

$$L(s,t)=(m_j+m_k)^2, \tag{4.16}$$

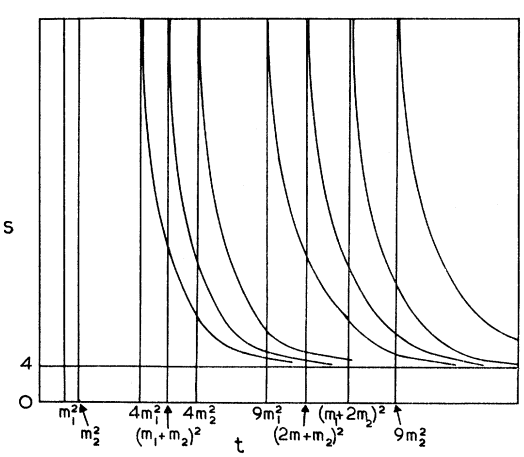


FIG. 2. A sketch of the s-t plane singularities for two Yukawa potentials, showing the positions of the poles at $t=m_1^2$ and $t=m_2^2$, and the curved boundaries of the double spectral function for the second and third Born approximations. Our units throughout are chosen such that the external masses are equal to one.

and we write the solution of the equation as

$$s = s_0^{ijk}(t). \tag{4.17}$$

In Fig. 2 we depict the s-t plane, and show the boundaries of $\rho_{ij}^{(2)}(s,t)$ and $\rho_{ijk}^{(3)}(s,t)$ for the case $N=2$.

The general features of $\rho_{ijk}^{(3)}(s,t)$ are easily found from (4.13). We find that

$$\rho_{ijk}^{(3)}[s_0^{ijk}(t),t] = 0, \tag{4.18}$$

and that all derivatives of $\rho_{ijk}^{(3)}(s,t)$ at $s=s_0^{ijk}(t)$ are infinite. Therefore, from its boundary, the function rises sharply to a peak, and then falls away, being eventually proportional to s^{-1} at fixed t, and to $t^{-3/2}$ at fixed s. Its main features are sketched in Fig. 3.

This behavior is to be contrasted with that of $\rho_{ij}^{(2)}(s,t)$, which has an inverse square-root singularity at its boundary, and falls monotonically in both s and t, being eventually proportional to $s^{-1/2}$ at fixed t, and to t^{-1} at fixed s.

The double spectral function cannot readily be calculated to higher order, but the general features are clear. Successively higher terms would have less pronounced peaking close to their s-t plane boundaries. This agrees with the numerical findings of Bali,[23] and of Bransden *et al.*,[33] whose double spectral functions were calculated from the exchange of a Breit-Wigner shape for the ρ meson in π-π scattering. They found oscillations corresponding to the boundary peaks, which died away quickly with increasing t. We obtain more severe oscillations since we are iterating a δ function. The singularity at the boundary disappears, however, by the time the third Born approximation is reached. The oscillations make it difficult to calculate higher Born terms for more than one Yukawa potential.

The improved asymptotic behavior of successive terms in the iteration, each by a factor $\sim(st)^{-1/2}$, follows by inspection of (4.1), but we note that the total double spectral function must have an asymptotic behavior proportional to $t^{\alpha(s)}$ if calculated exactly. It is clear that in this case, where we take only terms up to those in g^3, the infinite integrals of (3.20) will converge. The divergent parts correspond to the s-channel strip of the new form of the strip approximation, and give an unimportant short-range component to the force.[2]

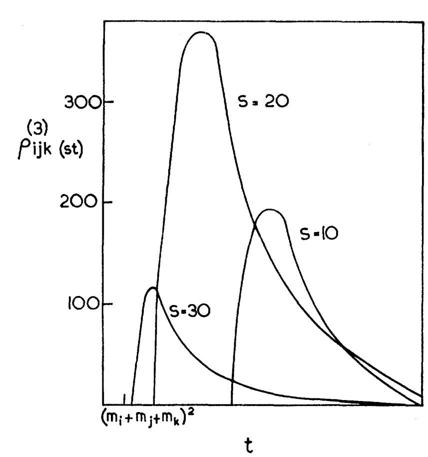


FIG. 3. A piece of double spectral function $\rho_{ijk}^{(3)}(s,t)$, plotted against t for three values of s.

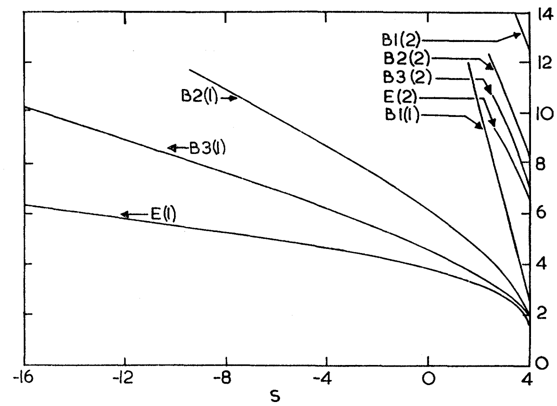


FIG. 4. S-wave bound-state positions plotted against the coupling constant for a single attractive Yukawa potential. $E(1)$, $E(2)$ denote the exact solution, primary and secondary. $B3(1)$, $B3(2)$ denotes the 3rd Born approximation primary and secondary. $B2(1)$, $B2(2)$ denotes the 2nd Born approximation primary and secondary. $B1(1)$, $B1(2)$ denotes the 1st Born approximation primary and secondary. The symbols E, $B3$, $B2$, and $B1$ throughout this paper denote, respectively, the exact solution and the N/D solution with the third, second, and first Born approximations to the left-hand cut.

V. NUMERICAL EXAMPLES

In this section we present, mainly in graphical form, the results of solving the N/D equations with potentials calculated in the manner just described. We are mainly interested in potential strengths which are similar to those found in strong-interaction physics. The relevant parameter is g/m, where m is the mass of the exchanged particle, in units of the reduced mass of the scattering system. According to the calculations of Finkelstein,[34] the equivalent energy-dependent potential due to the exchange of the ρ meson in π-π scattering will correspond to $g/m \simeq 3$, over the range of energies between threshold and $100 m_\pi^2$, and this is a fairly typical order of magnitude for such forces.

Firstly, in Fig. 4 we plot the position of an S-wave bound state resulting from a single Yukawa potential of the form (3.3) as a function of the strength of the coupling. Corresponding curves (except for the inclusion of the third Born approximation) are to be found in Refs. 10 and 29. We see that the first Born approximation is not really satisfactory if the potential is strong enough to produce a bound state, but that the third Born approximation is quite good even for large

[33] B. H. Bransden, P. G. Burke, J. W. Moffat, R. G. Moorhouse, and D. Morgan, Nuovo Cimento **30**, 297 (1963).

[34] J. Finkelstein, Phys. Rev. **145**, 1185 (1966).

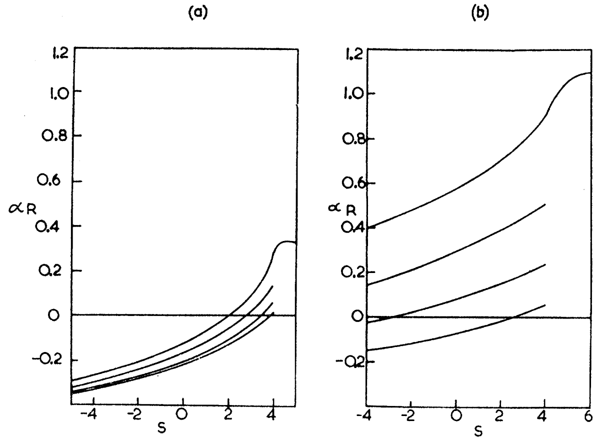


FIG. 5. Regge trajectories for a single attractive Yukawa potential (a) $V=-3e^{-r}/r$; (b) $V=-8e^{-r}/r$ using E, $B3$, $B2$, and $B1$ (top to bottom). The notation is the same as in Fig. 4.

couplings, and also gives quite a satisfactory account of the secondary trajectory which appears for $g>6.5$.

The corresponding trajectories for two different couplings are given in Fig. 5, and again for the weaker force the third Born approximation is very good.

Evidently, a very strong force is needed to produce a P-wave resonance. If we arrange combinations of attractive and repulsive potentials to produce the same S- or P-wave states, as in Fig. 6, then we get a somewhat steeper, and certainly higher-rising trajectory the larger the repulsion.[35] It is also found that the width of the P-wave resonance is smaller with a larger repulsion. This is exactly the effect which we hope for from the P repulsion in dynamical calculations.

The next question we want to ask is: How good are the various Born approximations for producing trajectories when both attractive and repulsive forces are present? In Fig. 7 we show the results for a comparatively weak attractive force and various longer-range repulsions. Evidently, the lower Born approximations are much less accurate than they are when there is only an attractive force. Indeed, we see in Fig. 8 that the lower Born approximations even give trajectories which are in the wrong order, i.e., the trajectory is more highly bound the stronger the repulsion. This effect is shown with greater clarity in Fig. 9, where we plot the change in the position of the S-wave bound state due to the introduction of a fixed repulsion against

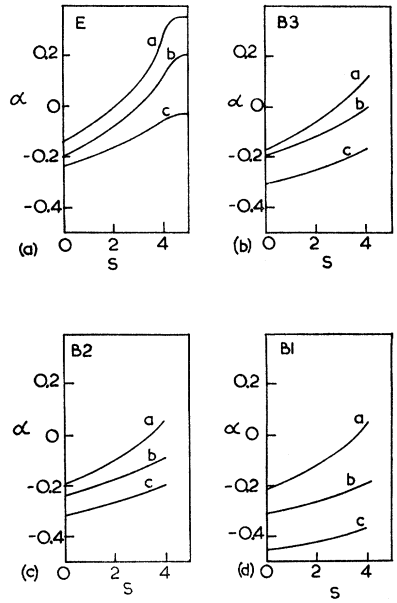


FIG. 7. Regge trajectories for an attractive force combined with various longer-range repulsions. The potentials are (a) $V=-3e^{-r}/r$; (b) $V=-3e^{-r}/r+0.5e^{-0.3r}/r$; and (c) $V=-3e^{-r}/r+1.0e^{-0.3r}/r$. The notation is as for Fig. 4.

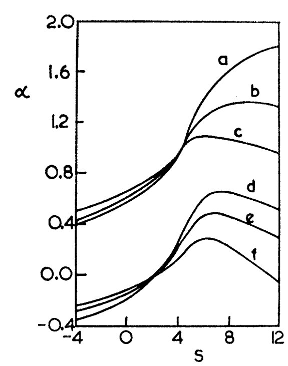


FIG. 6. An S-wave bound state and a P-wave resonance produced by various forces. The potentials for the P-wave resonance are (a) $V=-24.3e^{-r}/r+7.5e^{-0.3r}/r$; (b) $V=-18e^{-r}/r+5e^{-0.3r}/r$; and (c) $V=-8e^{-r}/r$. For the S-wave bound state (d) $V=-14e^{-r}/r+3.8e^{-0.3r}/r$; (e) $V=-7e^{-r}/r+2e^{-0.3r}/r$; and (f) $V=-3e^{-r}/r$.

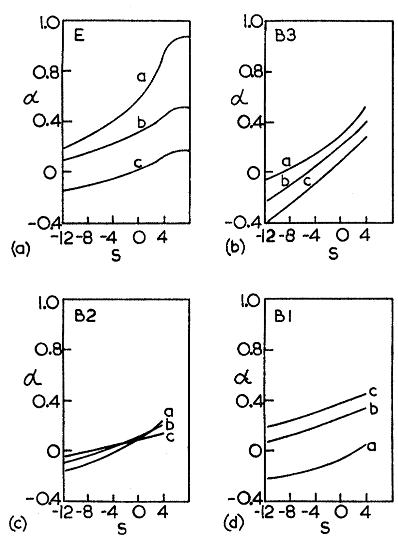


FIG. 8. Regge trajectories as in Fig. 7, but with a stronger attraction. The three cases are (a) $V=-8e^{-r}/r$; (b) $V=-8e^{r}/r+1.5e^{-0.3r}/r$; and (c) $V=-8e^{r}/r+3.0e^{-0.3r}/r$.

[35] A. Ahmadzadeh, Ph.D. thesis, University of California Radiation Laboratory Report No. UCRL-11096, 1963 (unpublished).

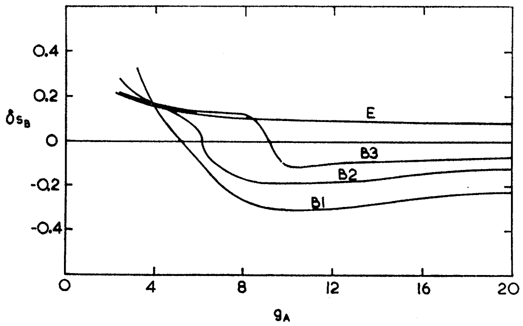


FIG. 9. The shift δs_B of the S-wave bound-state position when the potential $V=-g_Ae^{-r}/r$ is changed to $V=-g_Ae^{-r}/r+0.1e^{-0.3r}/r$. The notation is as for Fig. 4.

the coupling strength of the original attractive force. The response of the exact solution to the repulsive perturbation is almost independent of how deeply the state is bound, but this is certainly not true of the lower approximations. Only the third Born approximation is able to produce reasonable results for a wide range of couplings.

In Fig. 10 we plot the same effect the other way round, that is, we fix the strength of the attraction and vary the repulsion. For clarity, we have chosen a case in which none of the Born approximations gives a satisfactory result in that the response in each case is in the wrong direction. The important thing is that as the repulsion is increased there comes a point at which the position of the bound state has moved off to $s=-\infty$. If the repulsion is increased beyond this, the ghost phenomenon, mentioned above, appears.

This is readily explained if we examine the behavior of the corresponding N and D functions obtained with, for example, the first Born approximation (see also Ref. 2). Figure 11(a) shows the form of the N and D functions when a normal bound state is produced. In Fig. 11(b) more repulsion has been added and the state

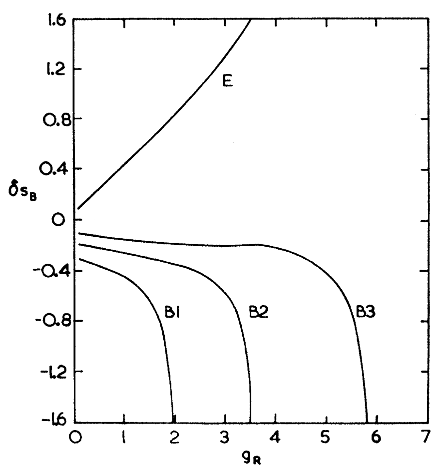


FIG. 10. The shift δs_B of the S-wave bound-state position when the potential $V=-11e^{-r}/r$ is changed to $V=-11e^{-r}/r+g_Re^{-0.3r}/r$. The notation is as for Fig. 4.

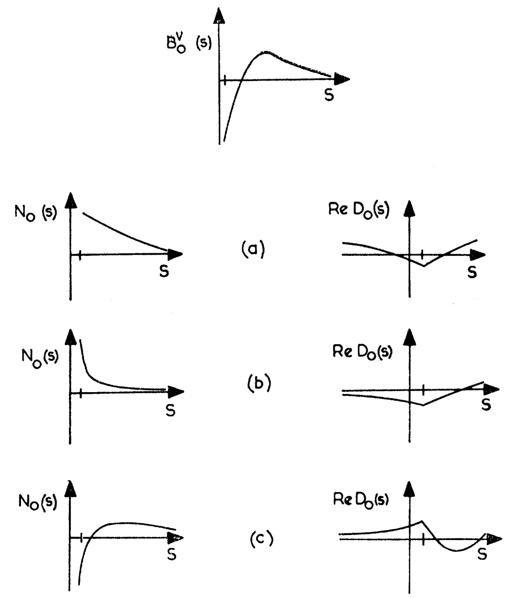


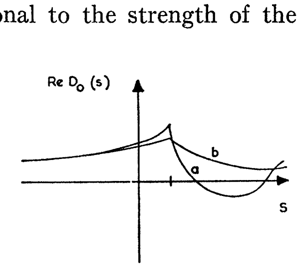
FIG. 11. A sketch of the potential function $B_0^V(s)$ and the N and D functions for the potential $V=-11e^{-r}/r+g_Re^{-0.3r}/r$ in the first Born approximation. The three cases have (a) $g_R=2.7$; (b) $g_R=3.0$; and (c) $g_R=3.3$. Over this range of g_R, $B_0^V(s)$ changes only slightly.

becomes more tightly bound. A further increase in the repulsion results in the development of a pole of the N function at threshold, as the position of the bound state moves to $-\infty$, and the N and D functions flip their signs near threshold. This happens when we reach a solution of the homogeneous equation obtained from (3.17) (see Ref. 36). The first zero of the D function, shown in Fig. 11(c), corresponds not to a resonance but to a ghost.

An examination of (3.17) shows why this happens. The sign change occurs at the point where the integral vanishes at threshold, and we get $N\simeq B^V$. Increasing the repulsion further results in the ghost moving first somewhat nearer to the threshold and then farther away, finally to vanish as the dip in the D function fails to reach zero (as in Fig. 12). Ghosts thus arise only after the bound state has moved off to $-\infty$.

It is evident from Fig. 9 that the way to check that the Born approximation used is adequate is to ensure that when a repulsion is added the bound state is really repelled, and (Fig. 10) that the amount by which it moves is roughly proportional to the strength of the

FIG. 12. A sketch of the S-wave D functions for the potential $V=-11e^{-r}/r+g_Re^{-0.3r}/r$ in first Born approximation. The two cases have (a) $g_R=3.3$; (b) $g_R=8.0$.



[36] J. S. Ball, Phys. Rev. **137**, B1573 (1965).

repulsion. If it is not, a higher approximation is needed. We can anticipate from these figures that whereas the first Born approximation, which has usually been used, will nearly always be unsatisfactory unless the coupling is very weak, the third Born approximation is likely to be good for most of the types of forces encountered in strong interactions, and indeed for rather stronger ones.

VI. SOLUBLE MODEL

It is interesting to look at some of the anomalous properties of repulsive forces in a simple soluble model.

Following Kayser,[6] we consider S-wave scattering with nonrelativistic kinematics, replacing the left-hand cut by simple poles. With one pole, the potential function is

$$B_0{}^V(s)=\lambda/(q_s{}^2+a^2) \qquad (6.1)$$

(a real and positive), which is well known[37] to be the first Born approximation to the potential

$$V(r)=-4\lambda a e^{-2ar}. \qquad (6.2)$$

For $\lambda>2a$, the potential function (6.1) gives rise to a bound state on the physical sheet. If a small long-range perturbation is added, in the form of a second pole at $q_s{}^2=-b^2$ $(b<a)$, the small shift in the bound-state position is easily calculated by the Dashen-Frautschi method.[8] Kayser[6] has shown that for weak binding such that the bound state lies to the right of $q_s{}^2=-b^2$, a repulsive perturbation moves the bound state towards threshold in the correct manner. For a stronger binding force, however, where the bound state lies between the two force poles, a repulsive perturbation appears to act like an extra binding force, moving the bound state to the left.

The reason for this is easily found. The potential function (6.1) is the full left-hand cut for the potential[38]

$$V(r)=\frac{-4\lambda a e^{-2ar}}{[(\lambda/2a)e^{-2ar}+1]^2}. \qquad (6.3)$$

The addition of a second pole at $q_s{}^2=-b^2$, of residue λ', gives the first Born approximation to the potential

$$V(r)=-4\lambda a e^{-2ar}-4\lambda' b e^{-2br}, \qquad (6.4)$$

or the full left-hand cut for the potential[39]

$$V(r)=2\frac{[dF(r)/dr]^2-F(r)d^2F(r)/dr^2}{[F(r)]^2}, \qquad (6.5)$$

where

$$F(r)=1+\frac{\lambda}{2a}e^{-2ar}+\frac{\lambda'}{2b}e^{-2br}-\frac{\lambda\lambda'}{a+b}e^{-(a+b)r}. \qquad (6.6)$$

For small λ, λ' the potentials (6.4) and (6.5) are quite similar, and as λ, λ' approach zero, (6.5) approaches (6.4).

For large λ, i.e., a strong attraction, expressions (6.4) and (6.5) are very different, and it is evident that $\lambda'<0$ can no longer be interpreted as corresponding to a simple repulsive force.

The conclusion to be drawn from this example is that a given potential function $B_l{}^V(s)$, although it can be interpreted as some Born approximation to a given potential, also can often be regarded as the exact potential function for an almost completely different force which coincides with that potential only in the limit of weak coupling. This gives us some insight into the qualitative features of the numerical results presented in Sec. V, although, unlike this example, our results are not dependent on the bound state's being inside the left-hand cut.

Kayser[6] has shown that these considerations enable us to understand a similar problem presented by Sawyer[5] in connection with the Dashen-Frautschi method. If the force producing a bound state is approximated by a simple pole, and if the pole is moved slightly to the left, leading, according to (6.1) and (6.2), to a weakening of the binding force because of the decrease in its range, then a Dashen-Frautschi calculation predicts that the bound state becomes more tightly bound. By inspection of (6.2) and (6.3), however, it is clear that an increase of a does not correspond simply to a decrease in the range of $V(r)$. The change is more complicated, and as Kayser[6] has demonstrated, leads not to a weakening, but to a strengthening of the binding force. Therefore, the result of the perturbation calculation of Sawyer is in no way anomalous (cf. Ref. 7).

Unfortunately, this simple model is unable to encompass the ghosts which our numerical calculations have produced. The poles correspond to potentials $V(r)$ which satisfy the conditions for the Mandelstam representation to be obeyed by the amplitude.[30] Therefore, causality will not be violated, and ghosts cannot be produced. The potentials $V(r)$ corresponding to the functions $B_l{}^V(s)$ used in the numerical examples must violate the Mandelstam representation. Our approximations have mutilated the analytic properties of the potentials to such an extent that they cease to bear any relation to the potentials they are supposed to approximate, and therefore nonsensical results occur.

VII. CONCLUSIONS

We have solved the N/D equations for potential scattering with various Born approximations to the left-hand cut. It turns out that the approximation which is most commonly employed, the first Born approximation, is often quite inadequate when we have a combination of attractive and repulsive potentials, and leads to anomalous behavior in that repulsions

[37] H. A. Bethe and R. Bacher, Rev. Mod. Phys. **8**, 111 (1936).
[38] V. Bargmann, Rev. Mod. Phys. **21**, 488 (1949); C. Eckart, Phys. Rev. **35**, 1303 (1930).
[39] P. J. S. Watson, Ph.D. thesis, University of Durham, 1967 (unpublished).

may have attractive effects, and ghosts may appear. Both of these unpleasant features are removed for a wide range of potential strengths, including those which are likely to be encountered in particle physics, if the third Born approximation is used.

It is expected that this demonstration of how the solutions go awry will enable us to recognize the breakdown of these approximation schemes more readily in the future.

Also, it is believed that there is good reason to hope that in calculations of strong-interaction dynamics, such as, for example, those based on the strip approximation, it will also be the case that an approximation to the left-hand cut involving just a few iterations of the "potential" provides a satisfactory input to the N/D equations. It may be, of course, that the sort of "equivalent potential" obtained from the Mandelstam representation can never be made to give a satisfactory account of strong-interaction dynamics, and that arbitrary parameters specifying the more important features of trajectories are required as input.[40] However, we shall not be able to arrive at a clear decision on such matters until the current-calculational schemes have been explored fully, without the debilitating effects of unnecessarily poor approximations. Calculations are in hand to use this same sort of iteration of the potential for π-π scattering in the strip approximation, and we hope to present results shortly.

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[40] P. D. B. Collins, R. C. Johnson, and E. J. Squires, Phys. Letters **26B**, 223 (1968).

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Tests of Saturation in Strong-Coupling Theory*

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Saturation of sum rules is tested in several strong-coupling static models, and the results are compared with the known properties of the models. A meson current commutation relation is very poorly saturated by low-lying states. Superconvergence relations are discussed in both s- and p-wave models. There are no such relations in s-wave models, but there are two in the p-wave model, one of which is saturated, the other not. It is also shown that certain truncated Chew-Low equations have no solutions.

INTRODUCTION

THE purpose of this paper is to carry out several tests of the saturation hypothesis[1] in soluble field-theoretical models. Obviously, caution must be exercised in generalizing from these, but they have the advantage of being soluble. Usually[2] the only test of saturation is the experimental data; this often leads to little insight into the situation, and therefore in contrast we shall ask our questions of a toy (but understood) world.

All our examples will be in the strong-coupling limit of static models.[3,4] Properties of these models are well known, and this makes it easy to test a few sum rules within them. We shall first try saturating a meson current commutator and find that it fails miserably. We next consider some superconvergence relations, which have been derived in the model. We find contradictions in the sum rules (some do not saturate) and prove a related theorem on the existence of solutions to a class of cutoff static models. The p-wave model is seen to have one derivable sum rule which is valid and another which fails.

SATURATION OF A CURRENT COMMUTATOR

We shall deal first with the question of saturation in charge-symmetric scalar theory, which may be defined by the interaction Hamiltonian

$$H_I = g_0 \int d^3r\, \tau_\alpha u(r) \phi_\alpha(\mathbf{r})\,, \tag{1}$$

* Supported by U. S. Atomic Energy Commission Contract No. AT-30-1-2171.

[1] Recent applications of the saturation scheme include: V. de Alfaro, S. Fubini, C. Rossetti, and G. Furlan, Phys. Letters **21**, 576 (1966); B. Sakita and K. C. Wali, Phys. Rev. Letters **18**, 319 (1967); K. Bardakci and G. Segrè, *ibid.* **159**, 1263 (1967); S. Weinberg, *ibid.* **18**, 507 (1967).

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[3] G. Wentzel, Helv. Phys. Acta **13**, 269 (1940); S. Tomonaga, Progr. Theoret. Phys. (Kyoto) **2**, 6 (1947).

[4] C. Goebel, Midwest Research Conference, 1965 (unpublished); T. Cook, C. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965).