Relations among the Superconvergence Conditions for Forward Elastic Scattering. III. General Method to Obtain All Independent Superconvergence Conditions for Forward Amplitudes*

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The superconvergence conditions for forward elastic scattering of two spinning particles are discussed. The crossing relation implies that the t-channel forward amplitude is related to those s-channel forward amplitudes which do not flip helicities by more than 1. Therefore, in general, the superconvergence conditions for the t-channel forward amplitudes do not exhaust all possible superconvergence conditions for forward amplitudes. A general method to obtain all independent superconvergence conditions for the s-channel forward amplitudes is described. The linear relations among the derivatives (with respect to t) of the t-channel helicity amplitudes at $t=0$ are derived and discussed.

I. INTRODUCTION

It was suggested by de Alfaro, Fubini, Rossetti, and Furlan[1] that, because of the kinematic structure of the scattering amplitudes for particles with spin, the high-energy behavior of certain amplitudes leads to superconvergence conditions. A general way to obtain superconvergence conditions for particles with spin is to construct amplitudes free from the s-kinematic singularities directly from the t-channel helicity amplitudes.[2] However, the superconvergence conditions for the forward-elastic-scattering amplitudes need special treatment for the following reasons: (a) At $t=0$, the t-channel helicity amplitudes are no longer kinematically independent of each other; therefore the corresponding superconvergence conditions are related to each other; (b) the s-channel forward amplitudes which flip helicities by more than 1 do not contribute to the t-channel forward amplitudes. Therefore, in general, the superconvergence conditions for the t-channel forward amplitudes do not exhaust all possible superconvergence conditions for the s-channel forward amplitudes.[3] The superconvergence conditions for forward amplitudes are investigated in a series of papers. In the previous papers,[3,4] we discussed the linear relations among the t-channel forward amplitudes and showed that there exists a set of t-channel helicity amplitudes with the following properties (at $t=0$): (a) The corresponding superconvergence conditions are independent; (b) the superconvergence conditions for any other t-channel helicity amplitude are linearly related to the superconvergence conditions for this set of amplitudes. In this paper, we shall discuss the linear relations among the derivatives (with respect to t) of the t-channel helicity amplitudes at $t=0$ and show how to obtain all independent superconvergence conditions for the s-channel forward amplitudes.

Using only the Trueman-Wick crossing relations,[5] Wang developed a general method to identify the kinematic singularities and zeros of the helicity amplitudes.[6] It was pointed out that the t-kinematic zeros of the s-channel helicity amplitudes imply kinematic constraints among the t-channel helicity amplitudes.[7] We show in Sec. II that these kinematic constraints introduce certain linear relations among the derivatives (with respect to t at $t=0$) of the t-channel amplitudes. The algebraic structure of these linear relations and a general method to obtain all independent superconvergence conditions for forward amplitudes are discussed in Sec. III for the special case where one of the scattering particles has no spin. The general case is discussed in Sec. IV. Several mathematical details are given in the Appendices.

II. LINEAR RELATIONS AMONG THE DERIVATIVES OF t-CHANNEL HELICITY AMPLITUDES AT $t=0$

We consider the elastic scattering of the particles a (spin J, mass M_a) and b (spin J', mass M_b): $a+b \to a+b$. Their helicities are denoted by α, β, α^*, and β^*, respectively. We use the convention $J \geqslant J'$. The crossed channel is defined to be $\bar{b}+b \to a+\bar{a}$, where $\bar{a}$ means the antiparticle of a, and the corresponding helicities are β', β'', α', and α'', respectively. The square of the center-of-mass energy in the direct (crossed) channel is s (t). The s-channel helicity amplitude $f_{\alpha^*\beta^*,\alpha\beta}{}^s(s,t)$ has a kinematic factor $t^{|\lambda^*-\mu^*|/2}$, where $\lambda^* \equiv \alpha-\beta$ and $\mu^* \equiv \alpha^*-\beta^*$.[6,8] The t-channel helicity amplitude

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[1] V. de Alfaro, S. Fubini, G. Rossetti, and G. Furlan, Phys. Letters 21, 576 (1966). Their idea was generalized by R. Gatto, Nuovo Cimento 51A, 212 (1967).
[2] An excellent and general discussion was given by T. L. Trueman, Phys. Rev. Letters 17, 1198 (1966). This subject was briefly discussed, within the framework of the Regge-pole model, by M. Gell-Mann, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley* (University of California Press, Berkeley, Calif., 1967). The author would like to thank Dr. J. M. Wang for calling his attention to Gell-Mann's article.
[3] K. Y. Lin, Phys. Rev. 163, 1568 (1967).
[4] K. Y. Lin, Phys. Rev. 167, 1499 (1968).
[5] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) 26, 322 (1964).
[6] L. L. C. Wang, Phys. Rev. 142, 1187 (1966). We follow Wang's notations in this paper.
[7] E. Abers and V. L. Teplitz, Phys. Rev. 158, 1365 (1967).
[8] Y. Hara, Phys. Rev. 136, B507 (1964).

$f_{\alpha'\alpha'',\beta'\beta''}{}^t(s,t)$ has a kinematic factor $t^{1/2}$ if and only if the number $\lambda-\mu$ ($\lambda\equiv\alpha'-\alpha''$, $\mu\equiv\beta'-\beta''$) is odd.[6,8] We define

$$\bar{f}_{\alpha^*\beta^*,\alpha\beta}{}^s(s,t)\equiv t^{-|\lambda^*-\mu^*|/2} f_{\alpha^*\beta^*,\alpha\beta}{}^s(s,t)\,. \qquad (1)$$

Under parity symmetry,[9] we have

$$\begin{aligned} f_{\alpha^*\beta^*,\alpha\beta}{}^s(s,t)&=(-1)^{\lambda^*-\mu^*}f_{-\alpha^*-\beta^*,-\alpha-\beta}{}^s(s,t)\,,\\ f_{\alpha'\alpha'',\beta'\beta''}{}^t(s,t)&=(-1)^{\lambda-\mu}f_{-\alpha'-\alpha'',-\beta'-\beta''}{}^t(s,t)\,. \end{aligned} \qquad (2)$$

Time-reversal invariance implies

$$\begin{aligned} f_{\alpha^*\beta^*,\alpha\beta}{}^s(s,t)&=(-1)^{\lambda^*-\mu^*}f_{\alpha\beta,\alpha^*\beta^*}{}^s(s,t)\,,\\ f_{\alpha'\alpha'',\beta'\beta''}{}^t(s,t)&=(-1)^{\lambda-\mu}f_{\alpha''\alpha',\beta''\beta'}{}^t(s,t)\,. \end{aligned} \qquad (3)$$

It was argued by Hara[8] and Wang[6] that all s-kinematic singularities and zeros of the amplitude $f_{\alpha'\alpha'',\beta'\beta''}{}^t(s,t)$ are included in the kinematic factor $(\cos\frac{1}{2}\theta_t)^{|\lambda+\mu|}(\sin\frac{1}{2}\theta_t)^{|\lambda-\mu|}$, where the angle θ_t is the scattering angle in the t channel. We have

$$\sin\theta_t=2(-K-st)^{1/2}[(t-4M_a{}^2)(t-4M_b{}^2)]^{-1/2}, \qquad (4)$$

where

$$K\equiv[s-(M_a+M_b)^2][s-(M_a-M_b)^2].$$

The s-channel helicity amplitudes are related to the t-channel helicity amplitudes by the following crossing relations[5,10]:

$$f_{\alpha^*\beta^*,\alpha\beta}{}^s(s,t)=\sum_{\alpha',\alpha'',\beta',\beta''} d_{\alpha'\alpha^*}{}^J(\tfrac{1}{2}\pi+\phi)d_{\alpha''\alpha}{}^J(\tfrac{1}{2}\pi-\phi)$$
$$\times d_{\beta'\beta^*}{}^{J'}(\tfrac{1}{2}\pi-\phi')d_{\beta''\beta}{}^{J'}(\tfrac{1}{2}\pi+\phi')f_{\alpha'\alpha'',\beta'\beta''}{}^t(s,t)\,, \qquad (5)$$

where

$$\begin{aligned} \sin\phi&=(s+M_a{}^2-M_b{}^2)t^{1/2}[(t-4M_a{}^2)K]^{-1/2},\\ \sin\phi'&=(s+M_b{}^2-M_a{}^2)t^{1/2}[(t-4M_b{}^2)K]^{-1/2}. \end{aligned}$$

At $t=0$, we have the following kinematic constraints[7]:

$$[\partial^n/\partial(t^{1/2})^n][\sum_{\alpha',\alpha'',\beta',\beta''} d_{\alpha'\alpha^*}{}^J(\tfrac{1}{2}\pi+\phi)d_{\alpha''\alpha}{}^J(\tfrac{1}{2}\pi-\phi)$$
$$\times d_{\beta'\beta^*}{}^{J'}(\tfrac{1}{2}\pi-\phi')d_{\beta''\beta}{}^{J'}(\tfrac{1}{2}\pi+\phi')f_{\alpha'\alpha'',\beta'\beta''}{}^t(s,t)]$$
$$\begin{aligned} &=0 &&\text{if}\quad |\lambda^*-\mu^*|>n\,,\\ &=n!\bar{f}_{\alpha^*\beta^*,\alpha\beta}{}^s(s,0) &&\text{if}\quad |\lambda^*-\mu^*|=n\,. \end{aligned} \qquad (6)$$

The case of $n=1$ has been discussed in Ref. 3. Carrying out the differentiation, we have ($t=0$)

$$\sum_{\substack{\alpha',\alpha'',\beta',\beta''\\ \lambda-\mu=\text{even}}} d(\alpha''\alpha)d(\alpha'\alpha^*)d(\beta''\beta)d(\beta'\beta^*)$$
$$\times(\partial^m/\partial t^m)f_{\alpha'\alpha'',\beta'\beta''}{}^t(s,t)$$
$$\begin{aligned} &=0 &&\text{if}\quad |\lambda^*-\mu^*|>2m\,,\\ &\equiv 0 &&\text{if}\quad \lambda^*-\mu^*=\text{odd}\,,\ (\text{Ref. 11})\\ &=m!F_{\alpha^*\beta^*,\alpha\beta}(s) &&\text{if}\quad |\lambda^*-\mu^*|=2m\,, \end{aligned} \qquad (7)$$

$$\sum_{\substack{\alpha',\alpha'',\beta',\beta''\\ \lambda-\mu=\text{odd}}} d(\alpha''\alpha)d(\alpha'\alpha^*)d(\beta''\beta)d(\beta'\beta^*)(\partial^m/\partial t^m)$$
$$\times[t^{-1/2}(K+st)^{-1/2}f_{\alpha'\alpha'',\beta'\beta''}{}^t(s,t)]$$
$$\begin{aligned} &=0 &&\text{if}\quad |\lambda^*-\mu^*|>2m+1\,,\\ &\equiv 0 &&\text{if}\quad \lambda^*-\mu^*=\text{even}\,,\ (\text{Ref. 11})\\ &=m!K^{-1/2}F_{\alpha^*\beta^*,\alpha\beta}(s) &&\text{if}\quad |\lambda^*-\mu^*|=2m+1\,, \end{aligned} \qquad (8)$$

where

$$d(\alpha''\alpha)\equiv d_{\alpha''\alpha}{}^J(\tfrac{1}{2}\pi)\,,$$
$$d(\beta''\beta)\equiv d_{\beta''\beta}{}^{J'}(\tfrac{1}{2}\pi)\,,$$
$$F_{\alpha^*\beta^*,\alpha\beta}(s)\equiv\sum_{r=0}^{|\lambda^*-\mu^*|}(r!)^{-1}H^r\bar{f}_{\alpha^*\beta^*,\alpha\beta}{}^s(s,0)\,,$$
$$\begin{aligned} H&\equiv C^a(\alpha\downarrow+\alpha^*\uparrow)+C^b(\beta\uparrow+\beta^*\downarrow)\quad\text{if}\quad \lambda^*>\mu^*\,,\\ &\equiv -C^a(\alpha\uparrow+\alpha^*\downarrow)-C^b(\beta\downarrow+\beta^*\uparrow)\quad\text{if}\quad \lambda^*<\mu^*\,, \end{aligned}$$
$$\begin{aligned} C^a&\equiv 2^{-1}(\partial/\partial t^{1/2})\phi\\ &=i(s+M_a{}^2-M_b{}^2)(4M_b)^{-1}K^{-1/2}\,,\\ C^b&\equiv 2^{-1}(\partial/\partial t^{1/2})\phi'\\ &=i(s+M_b{}^2-M_a{}^2)(4M_a)^{-1}K^{-1/2}\,, \end{aligned}$$

the operators $(\alpha\downarrow)^r$, $(\alpha^*\uparrow)^p$, $(\beta\uparrow)^{r'}$, and $(\beta^*\downarrow)^{p'}$ commute with each other and are defined by

$$\begin{aligned} (\alpha\downarrow)^r\bar{f}_{\alpha^*\beta^*,\alpha\beta}{}^s(s,0)&\equiv g(\alpha)g(\alpha-1)\cdots\\ &\quad\times g(\alpha-r+1)\bar{f}_{\alpha^*\beta^*,(\alpha-r)\beta}{}^s(s,0)\,,\\ (\alpha^*\uparrow)^r\bar{f}_{\alpha^*\beta^*,\alpha\beta}{}^s(s,0)&\equiv g(-\alpha^*)g(-\alpha^*-1)\cdots\\ &\quad\times g(-\alpha^*-r+1)\bar{f}_{(\alpha^*+r)\beta^*,\alpha\beta}{}^s(s,0)\,,\\ (\beta^*\downarrow)^r\bar{f}_{\alpha^*\beta^*,\alpha\beta}{}^s(s,0)&\equiv \bar{g}(\beta^*)\bar{g}(\beta^*-1)\cdots\\ &\quad\times \bar{g}(\beta^*-r+1)\bar{f}_{\alpha^*(\beta^*-r),\alpha\beta}{}^s(s,0)\,,\\ (\beta\uparrow)^r\bar{f}_{\alpha^*\beta^*,\alpha\beta}{}^s(s,0)&\equiv \bar{g}(-\beta)\bar{g}(-\beta-1)\cdots\\ &\quad\times \bar{g}(-\beta-r+1)\bar{f}_{\alpha^*\beta^*,\alpha(\beta+r)}{}^s(s,0)\,,\\ g(\alpha)&\equiv[(J-\alpha+1)(J+\alpha)]^{1/2},\\ \bar{g}(\beta)&\equiv[(J'-\beta+1)(J'+\beta)]^{1/2}, \end{aligned}$$

and the operators $(\alpha\uparrow)^r$, $(\alpha^*\downarrow)^p$, $(\beta\downarrow)^{r'}$, and $(\beta^*\uparrow)^{p'}$ are defined in a similar way.

Equations (7) and (8) are derived in Appendix A. Note that the validity of these equations does not depend on the assumption of time-reversal invariance.

Equations (7) and (8) are equivalent to the following equations:

$$\sum_{\substack{\alpha',\alpha'',\beta',\beta''\\ \lambda-\mu=\text{even}}} d(\alpha''\alpha)d(\alpha'\alpha^*)d(\beta''\beta)d(\beta'\beta^*)f_{\alpha'\alpha'',\beta'\beta''}{}^t(s,t)$$
$$\begin{aligned} &=0 &&\text{if}\quad \lambda^*-\mu^*=\text{odd}\,,\\ &=t^{|\lambda^*-\mu^*|/2}[F_{\alpha^*\beta^*,\alpha\beta}(s)+O(t)] &&\text{if}\quad \lambda^*-\mu^*=\text{even}\,, \end{aligned} \qquad (9)$$

[9] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).
[10] I. Muzinich, J. Math. Phys. 5, 1481 (1964).

[11] These identities follow directly from parity symmetry [see Eq. (2)].

$$\sum_{\substack{\alpha',\alpha'',\beta',\beta'' \\ \lambda-\mu=\text{odd}}} d(\alpha''\alpha)d(\alpha'\alpha^*)d(\beta''\beta)d(\beta'\beta^*) \times[t^{-1/2}(K+st)^{-1/2}f_{\alpha'\alpha'',\beta'\beta''}{}^t(s,t)]$$
$$=0 \quad \text{if } \lambda^*-\mu^*=\text{even},$$
$$=t^{(|\lambda^*-\mu^*|-1)/2}[K^{-1/2}F_{\alpha^*\beta^*,\alpha\beta}(s)+O(t)] \quad \text{if } \lambda^*-\mu^*=\text{odd}, \tag{10}$$

where $O(t)$ means "of the order t as $t\to 0$."

The forward amplitudes F are all kinematically independent. (The amplitudes which are related by parity symmetry or time-reversal invariance are considered to be the same amplitude.) The amplitudes $F(s)$ have the same high-energy behavior as the forward amplitudes $\bar{f}(s,0)$. Equations (7) and (8) imply that certain linear combinations of $F(s)$ have kinematic factors of the form $K^{n/2}$ ($\sim s^n$ at high energy), where n is a positive integer. In the cases of low spin, these combinations can be found directly. We shall develop a general method to find these combinations.

In Regge-pole theory, Eqs. (7) and (8) imply certain conspiracy conditions among different Regge trajectories.[12] The special case of $J=J'=\frac{1}{2}$ (nucleon-nucleon scattering) has been discussed by Volkov and Gribov.[13]

III. FORWARD ELASTIC SCATTERING OF A SPINLESS PARTICLE BY A SPINNING PARTICLE

In this section, we shall discuss the superconvergence conditions for forward amplitudes in the case of $J'=0$. We use the following simplified notations:

$$f_{\alpha^*\alpha}(t)\equiv\bar{f}_{\alpha^*0,\alpha0}{}^s(s,t),$$
$$T_{\alpha'\alpha''}(t)\equiv f_{\alpha'\alpha'',00}{}^t(s,t) \quad \text{if } \lambda=\text{even},$$
$$\equiv K^{1/2}t^{-1/2}(K+st)^{-1/2}f_{\alpha'\alpha'',00}{}^t(s,t) \quad \text{if } \lambda=\text{odd},$$
$$\bar{T}_{\alpha'\alpha''}(t)\equiv(K+st)^{-|\lambda|/2}f_{\alpha'\alpha'',00}{}^t(s,t) \quad \text{if } \lambda=\text{even},$$
$$\equiv t^{-1/2}(K+st)^{-|\lambda|/2}f_{\alpha'\alpha'',00}{}^t(s,t) \quad \text{if } \lambda=\text{odd},$$
$$F(\alpha^*\alpha)\equiv F_{\alpha^*0,\alpha0}(s),$$
$$T_{\alpha'\alpha''}{}^n(t)\equiv(\partial^n/\partial t^n)T_{\alpha'\alpha''}(t),$$
$$v\equiv 0 \quad \text{if } J=\text{integer},$$
$$\equiv\tfrac{1}{2} \quad \text{if } J=\text{half integer}.$$

Using these notations, Eqs. (7) and (8) can be expressed in the form

$$\sum_{\substack{\alpha',\alpha'' \\ \lambda=\text{even}}} d(\alpha''\alpha)d(\alpha'\alpha^*)T_{\alpha'\alpha''}{}^n(0)=0 \quad \text{if } |\lambda^*|>2n \text{ or } \lambda^*=\text{odd},$$
$$=n!F(\alpha^*\alpha) \quad \text{if } |\lambda^*|=2n, \tag{11}$$

$$\sum_{\substack{\alpha',\alpha'' \\ \lambda=\text{odd}}} d(\alpha''\alpha)d(\alpha'\alpha^*)T_{\alpha'\alpha''}{}^n(0)$$
$$=0 \quad \text{if } |\lambda^*|>2n+1 \text{ or } \lambda^*=\text{even},$$
$$=n!F(\alpha^*\alpha) \quad \text{if } |\lambda^*|=2n+1. \tag{12}$$

We define

$$n!G^n(\alpha^*\alpha)\equiv\sum_{\substack{\alpha',\alpha'' \\ \lambda=\text{even}}} d(\alpha''\alpha)d(\alpha'\alpha^*)T_{\alpha'\alpha''}{}^n(0) \quad \text{if } |\lambda^*|=\text{even}<2n,$$
$$\equiv\sum_{\substack{\alpha',\alpha'' \\ \lambda=\text{odd}}} d(\alpha''\alpha)d(\alpha'\alpha^*)T_{\alpha'\alpha''}{}^n(0) \quad \text{if } |\lambda^*|=\text{odd}<2n+1. \tag{13}$$

The functions G are all kinematically independent and can be expressed in terms of the amplitudes $f(0)$ and their t derivatives. The orthonormal property of the d function implies that $T^n(0)$ for $n>0$ are linearly related to both F and G. However, certain linear combinations of $T^n(0)$ can be expressed in terms of F alone.

Note that the high-energy behavior of the forward amplitudes $f(0)$ is quite different from that of their t derivatives. For example, in Regge-pole theory, we have

$$[f(0)]^{-1}(\partial^r/\partial t^r)f(t)|_{t=0}\to C[\ln(s)]^r \quad \text{as } s\to\infty.$$

On the other hand, we have the following restrictions imposed by unitarity and analyticity[14]:

$$|(\partial^r/\partial t^r)f(t)|_{t=0}\leqslant Cf(0)[\ln(s)]^{2r}.$$

Therefore those linear combinations of $T^n(0)$ which do not depend on G have much better high-energy behavior than the individual terms.

The algebraic structure of Eqs. (11) and (12) is discussed in Appendix B. The case of $n=0$ has been discussed in Ref. 3. It is shown in Ref. 3 that those forward amplitudes $T(0)$ having different s-kinematic factors are all kinematically independent. Let us choose a set of T in the following way:

$$T_i(t)\equiv T_{\alpha_i',\alpha_i''}(t), \quad |\lambda_i|=i=0,1,\cdots,2J.$$

Then any amplitude $T(0)$ other than $T_i(0)$ is a linear combination of T_i[3]:

$$T_{\alpha'\alpha''}(0)-\sum_{\substack{i\geqslant|\lambda| \\ i-\lambda=\text{even}}} C(\alpha',\alpha'',i)T_i(0)=0, \tag{14}$$

where the C are constants. We apply this result to Eqs. (11) and (12) for $n=1$. Since these equations are linear, we have

$$T_{\alpha'\alpha''}{}^1(0)-\sum_i C(\alpha',\alpha'',i)T_i{}^1(0)$$
$$=\sum_{\substack{\alpha,\alpha^* \\ |\lambda^*|=2 \text{ if } \lambda=\text{even}, \\ =3 \text{ if } \lambda=\text{odd}}} F(\alpha^*\alpha)[d(\alpha''\alpha)d(\alpha'\alpha^*) -\sum_i C(\alpha',\alpha'',i)d(\alpha_i''\alpha)d(\alpha_i'\alpha^*)]. \tag{15}$$

[12] A general discussion on this subject was given by M. L. Goldberger, Comments Nucl. Part. Phys. **1**, 63 (1967).

[13] D. V. Volkov and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **44**, 1068 (1963) [English transl.: Soviet Phys.—JETP **17**, 720 (1963)].

[14] T. Kinoshita, *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colo., 1965), Vol. 7B.

Note that these linear combinations of $T^1(0)$ do not depend on G. Equation (14) implies that we can write the left-hand side of Eq. (15) in the form

$$\{t^{-1}[T_{\alpha'\alpha''}(t)-\sum_i C(\alpha',\alpha'',i)T_i(t)]\}_{t=0}$$
$$=K^{|\lambda|/2}\{t^{-1}[\bar{T}_{\alpha'\alpha''}(t)-\sum_i C(\alpha',\alpha'',i)\times(K+st)^{(i-|\lambda|)/2}\bar{T}_i(t)]\}_{t=0}.$$

In other words, these linear combinations of F with $|\lambda^*|=2(3)$ have s-kinematic factors of the form K^n ($K^{n+1/2}$), where $n=0,1,\cdots,J-v-1$ ($J+v-2$). Note that there is no linear combination of F with $|\lambda^*|=2(3)$ which has an s-kinematic factor K^{J-v} ($K^{J+v-1/2}$). The reason is that there is only one amplitude among all T such that $|\lambda|=2J$ or $2J-1$. In general, we can construct a new set of forward amplitudes by taking proper linear combinations of F with $|\lambda^*|=2r$ ($2r+1$) such that they have s-kinematic factors of the form K^n ($K^{n+1/2}$), where $n=0,1,\cdots,J-v-r$ ($J+v-r-1$). However, not all of them are kinematically independent, although the amplitudes having different $|\lambda^*|$ are all kinematically independent. We shall prove in Appendix B that these new forward amplitudes have the following properties (for a fixed $|\lambda^*|$): (a) Those amplitudes which have different s-kinematic factors are kinematically independent; therefore the corresponding superconvergence conditions are independent; (b) in general, there are several amplitudes which have the same s-kinematic factor; in order to obtain all independent superconvergence conditions, only one of them needs to be investigated.

The following forward amplitudes are all kinematically independent and free from s-kinematic singularities:

$$\bar{T}_{J,J-2n}(0)=K^{-n}[\sum_\alpha d(J\alpha)d(J-2n,\alpha)F(\alpha\alpha)],\qquad n=0,1,\cdots,J-v$$

$$\bar{T}_{J,J-2n-1}(0)=K^{-n-1/2}[\sum_{\substack{\alpha,\alpha^*\\|\lambda^*|=1}} d(J\alpha^*)d(J-2n-1,\alpha)F(\alpha^*\alpha)],\qquad n=0,1,\cdots,J+v-1$$
$$\cdots,$$

$$\{t^{-r}[\bar{T}_{J-r,J-r-2n}(t)-\sum_{\substack{i=1,2,\cdots,r\\j=0,1,\cdots,i}}\bar{C}(i,j,n,r)(K+st)^i\bar{T}_{J-r+i,J-r+i-2n-2j}(t)]\}_{t=0}$$
$$=K^{-n}\sum_{\substack{\alpha,\alpha^*\\|\lambda^*|=2r}}F(\alpha^*\alpha)[d(J-r,\alpha^*)d(J-r-2n,\alpha)-\sum_{i,j}\bar{C}(i,j,n,r)d(J-r+i,\alpha^*)d(J-r+i-2n-2j,\alpha)],$$
$$n=0,1,\cdots,J-v-r$$

$$\{t^{-r}[\bar{T}_{J-r,J-r-2n-1}(t)-\sum_{\substack{i=1,\cdots,r\\j=0,\cdots,i}}\bar{C}^*(i,j,n,r)(K+st)^i\bar{T}_{J-r+i,J-r+i-2n-1-2j}(t)]\}_{t=0}$$
$$=K^{-n-1/2}\sum_{\substack{\alpha,\alpha^*\\|\lambda^*|=2r+1}}F(\alpha^*\alpha)[d(J-r,\alpha^*)d(J-r-2n-1,\alpha)-\sum_{i,j}\bar{C}^*(i,j,n,r)d(J-r+i,\alpha^*)d(J-r+i-2n-1-2j,\alpha)],$$
$$\vdots\qquad n=0,1,\cdots,J+v-r-1\qquad(16)$$

where

$$\bar{C}(i,j,n,r)=C(i,j,n,r)[(2J-r)!(2J-r-2n)!r!(2n+r)!]^{1/2}\times[(2J-r+i)!(2J-r-2n+i-2j)!(r-i)!(2n+r-i+2j)!]^{-1/2},$$

$$\bar{C}^*(i,j,n,r)=C^*(i,j,n,r)[(2J-r)!(2J-r-2n-1)!r!(2n+1+r)!]^{1/2}\times[(2J-r+i)!(2J-r-2n-1-2j+i)!(r-i)!(r+2n+1-i+2j)!]^{-1/2},$$

$$C(m,m,n,r)=(-1)^{m+1}m^{-1}(2n+2m)\binom{2n+m-1}{m-1},\qquad m=1,\cdots,r$$

$$C(m,0,n,r)=(-1)^{m+1}\binom{A+m-1}{A-1}+(-1)^m\binom{A+m-3}{A-1},\qquad m=1,\cdots,r$$

$$A\equiv C(1,0,n,r)=2J-2n-2r+2,$$

$$C(m,m-m',n,r)=-C(m',0,n,r)C(m-m',m-m',n,r),\qquad m=m'+1,\cdots,r$$

$$C^*(m,m,n,r)=(-1)^{m+1}\binom{2n+m}{m},$$

$$C^*(m,0,n,r)=(-1)^{m+1}\binom{A+m-2}{m},$$

$$C^*(m,m-m',n,r)=-C^*(m',0,n,r)C^*(m-m',m-m',n,r).$$

From these amplitudes all independent superconvergence conditions for forward amplitudes can be obtained. The proof and a simple way to calculate the coefficients of F in the above expressions are given in Appendix B.

IV. FORWARD ELASTIC SCATTERING OF TWO PARTICLES WITH SPIN

We use the following simplified notations:

$$v(v')\equiv 0 \quad \text{if} \quad J(J')=\text{integer},$$
$$\equiv \tfrac{1}{2} \quad \text{if} \quad J(J')=\text{half integer},$$
$$T_{\alpha'\alpha'',\beta'\beta''}(t)\equiv f_{\alpha'\alpha'',\beta'\beta''}{}^{t}(s,t) \quad \text{if} \quad \lambda-\mu=\text{even},$$
$$\equiv K^{1/2}t^{-1/2}(K+st)^{-1/2}f_{\alpha'\alpha'',\beta'\beta''}{}^{t}(s,t) \quad \text{if} \quad \lambda-\mu=\text{odd},$$
$$T_{\alpha'\alpha'',\beta'\beta''}{}^{n}(t)\equiv(\partial^n/\partial t^n)T_{\alpha'\alpha'',\beta'\beta''}(t),$$
$$\bar{T}_{\alpha'\alpha'',\beta'\beta''}(t)\equiv K_{+}{}^{-|\lambda+\mu|/2}K_{-}{}^{-|\lambda-\mu|/2}t^{-w}f_{\alpha'\alpha'',\beta'\beta''}{}^{t}(s,t),$$

where

$$w\equiv 0\ (1) \quad \text{if} \quad \lambda-\mu=\text{even} \quad (\text{odd}),$$
$$K_{\pm}\equiv[K+st+(t-4M_a{}^2)(t-4M_b{}^2)4^{-1}]^{1/2} \pm[(t-4M_a{}^2)(t-4M_b{}^2)4^{-1}]^{1/2}.$$

The amplitudes $\bar{T}$ are free from s-kinematic singularities.[6,8]

Using these notations, Eqs. (7) and (8) can be expressed in the form

$$\sum_{\substack{\alpha',\alpha'',\beta',\beta''\\ \lambda-\mu=\text{even}}} d(\alpha''\alpha)d(\alpha'\alpha^*)d(\beta''\beta)d(\beta'\beta^*)T_{\alpha'\alpha'',\beta'\beta''}{}^{n}(0)$$
$$=0 \quad \text{if} \quad |\lambda^*-\mu^*|>2n \quad \text{or} \quad =\text{odd},$$
$$=n!F_{\alpha^*\beta^*,\alpha\beta}(s) \quad \text{if} \quad |\lambda^*-\mu^*|=2n, \tag{17}$$

$$\sum_{\substack{\alpha',\alpha'',\beta',\beta''\\ \lambda-\mu=\text{odd}}} d(\alpha''\alpha)d(\alpha'\alpha^*)d(\beta''\beta)d(\beta'\beta^*)T_{\alpha'\alpha'',\beta'\beta''}{}^{n}(0)$$
$$=0 \quad \text{if} \quad |\lambda^*-\mu^*|>2n+1 \quad \text{or} \quad =\text{even},$$
$$=n!F_{\alpha^*\beta^*,\alpha\beta}(s) \quad \text{if} \quad |\lambda^*-\mu^*|=2n+1. \tag{18}$$

The algebraic structure of Eqs. (17) and (18) is discussed in Appendix C. The case of $n=0$ has been discussed in Ref. 4. We shall describe a general method to construct a set of forward amplitudes which are all kinematically independent and free from s-kinematic singularities; from these amplitudes all independent superconvergence conditions for forward amplitudes can be obtained. Since the essential arguments are the same as those used in Sec. III, we shall give our result without proof.

We shall review briefly the result of Ref. 4. The t-channel forward amplitudes can be divided into two groups, according to whether $\lambda-\mu=\text{even}$ (group A) or odd (group B). The amplitudes of group A (B) are linearly related to each other. Among all amplitudes of group A (B), there exists a subgroup of amplitudes, say T_i, which satisfies the following properties: (a) They are kinematically independent; (b) any other amplitudes of group A (B) can be written in the form $T_{\alpha'\alpha'',\beta'\beta''}(0)=\sum_i C_iT_i$, where $C_i=0$ if $|\lambda_i+\mu_i|<|\lambda+\mu|$ or $|\lambda_i-\mu_i|<|\lambda-\mu|$, and not all of the C_i with $|\lambda_i\pm\mu_i|=|\lambda\pm\mu|$ are zero. The T_i of group A can be chosen in the following way:

$$(\lambda_i,\mu_i)^N=(2m+p,\pm p)^{m+1},\quad (p,\pm 2m\pm p)^m,\quad m=0,1,\cdots,J'-v',\quad p=0,1,\cdots,2J'-2m;$$
$$=(2J'+2p,\pm 2J'\mp 2m)^{m+1},\quad m=0,1,\cdots,J'-v',\quad p=1,\cdots,J-J'-|v-v'|;$$
$$=(2J'+2p-1,\pm 2J'\mp 2m\mp 1)^{m+1},\quad m=0,1,\cdots,J'+v'-1,\quad p=1,\cdots,J-J'+|v-v'|; \tag{19}$$

where N is the number of T_i having the same λ_i and μ_i, and any two amplitudes (say, T_i and T_j) having the same λ_i and μ_i satisfy the restriction (this result is more general than that of Ref. 4; see Lemma 9 of Appendix C)

$$(\beta_i'-\beta_j')(\beta_i'+\beta_j'-\mu_i)\neq 0 \quad \text{if} \quad |\lambda_i|>|\mu_i|,$$
$$(\alpha_i'-\alpha_j')(\alpha_i'+\alpha_j'-\lambda_i)\neq 0 \quad \text{if} \quad |\lambda_i|<|\mu_i|.$$

The T_i of group B can be chosen in the following way [we define $r\equiv J-\alpha'$, $s\equiv J'-\beta'$, $s'\equiv J'-\beta''$, and use the convention $\lambda\geqslant 0$; we choose all possible values of r and s if $\mu\geqslant 0$ (r and s' if $\mu<0$)]:

$$(\lambda_i,\mu_i)^N=(2m+1+p,\pm p)^{2m+2},\quad m=0,\cdots,J'-1+v',\quad p=0,\cdots,2J'-2m-1;$$
$$=(p,\pm 2m\pm 1\pm p)^{2m+2},\quad m=0,\cdots,J'+v'-1,\quad p=0,\cdots,2J'-2m-1,\quad 2m+1+p\neq 2J';$$
$$=(2J'-2m-1,\pm 2J')^{m+1},\quad m=0,\cdots,J'-1+v';$$
$$=(2J,\pm 2J'\mp 1\pm 2|v-v'|\mp 2m)^{m+1},\quad m=0,\cdots,J'-v'-1+2v;$$
$$=(p,\pm 2J'\mp m)^{m+1},\quad m=0,\cdots,2J',\quad 2J'<p<2J,\quad 2J'-m+p=\text{odd}; \tag{20}$$

we choose all amplitudes with $\lambda=2J$;

$$r=0,1,\quad s(s')=0,\cdots,\tfrac{1}{2}N-1 \quad \text{if} \quad \lambda_i\neq 2J,\quad \lambda_i>|\mu_i|,\quad \text{and} \quad N=\text{even},$$
$$r=0,1,\quad s(s')=0,\cdots,\tfrac{1}{2}(N-1),\quad r+s(s')<\tfrac{1}{2}(N+1) \quad \text{if} \quad \lambda_i\neq 2J,\quad \lambda_i>|\mu_i|,\quad \text{and} \quad N=\text{odd},$$
$$r=0,1,\cdots,N-1 \quad \text{if} \quad |\mu_i|=2J'>\lambda_i,$$
$$s(s')=0,1,\quad r=0,\cdots,\tfrac{1}{2}N-1 \quad \text{if} \quad |\mu_i|\neq 2J' \quad \text{and} \quad |\mu_i|>\lambda_i.$$

Let us now consider the t-channel amplitudes $T(t)$, with $\lambda-\mu=$ even. We shall classify them in the following way. The amplitudes are divided into several subgroups. The amplitudes in each subgroup satisfy one of the following conditions (for a definite number n within the range $0\leqslant n\leqslant J+J'-|v-v'|$):

for $\lambda\geqslant|\mu|$:

$$r+s(s')=n+N-1,\quad r=n,\quad 2J-\lambda\geqslant 2r,\quad 2J'-|\mu|\geqslant 2s(s');$$
$$2J-\lambda-r+s(s')=n+N-2,\quad 2J-\lambda-r=n-1,\quad 2J-1-\lambda\geqslant 2r,\quad 2J'-1-|\mu|\geqslant 2s(s');$$

for $|\mu|>\lambda$:

$$r+s(s')=n+N-1,\quad s(s')=n,\quad 2J-\lambda\geqslant 2r,\quad 2J'-|\mu|\geqslant 2s(s');$$
$$2J-\lambda-r+s(s')=n+N-2,\quad s(s')=n-1,\quad 2J-1-\lambda\geqslant 2r,\quad 2J'-1-|\mu|\geqslant 2s(s'),$$

where the number N is a function of λ and μ given by Eq. (19). We shall denote the amplitudes corresponding to n by $I_{n,i}(t)$.

Similarly, we shall classify the t-channel amplitudes with $\lambda-\mu=$ odd by the following conditions ($0\leqslant n\leqslant J+J'+|v-v'|-1$): The amplitudes with $\lambda=2J$ belong to the subgroup corresponding to $n=0$;

for $\lambda\neq 2J$, $\lambda>|\mu|$, and $N=$ even:

$$r+s(s')=n+\tfrac{1}{2}N-1,\quad r=n,\quad 2J-\lambda\geqslant 2r,\quad 2J'-|\mu|\geqslant 2s(s');$$
$$2J-\lambda-r+s(s')=n+\tfrac{1}{2}N-1,\quad 2J-\lambda-r=n,\quad 2J-1-\lambda\geqslant 2r,\quad 2J'-1-|\mu|\geqslant 2s(s');$$

for $\lambda\neq 2J$, $\lambda>|\mu|$, and $N=$ odd:

$$r+s(s')=n+\tfrac{1}{2}N-\tfrac{1}{2},\quad r=n,\quad 2J-\lambda\geqslant 2r,\quad 2J'-|\mu|\geqslant 2s(s');$$
$$2J-\lambda-r+s(s')=n+\tfrac{1}{2}(N+1)-2,\quad 2J-\lambda-r=n,\quad 2J-1-\lambda\geqslant 2r,\quad 2J'-1-|\mu|\geqslant 2s(s');$$

for $|\mu|=2J'>\lambda$:

$$r=n+N-1,\quad 2J-\lambda\geqslant 2r;$$

for $|\mu|\neq 2J'$; and $|\mu|>\lambda$:

$$r+s(s')=n+\tfrac{1}{2}N-1,\quad s(s')=n,\quad 2J-\lambda\geqslant 2r,\quad 2J'-|\mu|\geqslant 2s(s');$$
$$2J-\lambda-r+s(s')=n+\tfrac{1}{2}N-1,\quad s(s')=n,\quad 2J-1-\lambda\geqslant 2r,\quad 2J'-1-|\mu|\geqslant 2s(s'),$$

where N is a function of λ and μ given by Eq. (20). We shall denote the amplitudes corresponding to n by $I_{n,p}^*(t)$.

The following forward amplitudes are kinematically independent and free from s-kinematic singularities:

$$K(\lambda,\mu)\equiv K_+^{-|\lambda+\mu|/2}K_-^{-|\lambda-\mu|/2}\Big|_{t=0};$$

$$K(\lambda_{n,i},\mu_{n,i})\{t^{-n}[I_{n,i}(t)-\sum_{\substack{m,j\\ m<n}}C(n,i,m,j)I_{m,j}(t)]\}_{t=0}$$
$$=K(\lambda_{n,i},\mu_{n,i})\sum_{\substack{\alpha,\alpha^*,\beta,\beta^*\\ |\lambda^*-\mu^*|=2n}}F_{\alpha^*\beta^*,\alpha\beta}(s)D_{n,i}(\alpha^*\beta^*,\alpha\beta),\quad n=0,1,\cdots,J+J'-|v-v'|;\quad(21)$$

$$D_{n,i}(\alpha^*\beta^*,\alpha\beta)\equiv d(\alpha_{n,i}'',\alpha)d(\alpha_{n,i}',\alpha^*)d(\beta_{n,i}'',\beta)d(\beta_{n,i},\beta^*)$$
$$-\sum_{m,j}C(n,i,m,j)d(\alpha_{m,j}'',\alpha)d(\alpha_{m,j}',\alpha^*)d(\beta_{m,j}'',\beta)d(\beta_{m,j'},\beta^*);$$

$$K(\lambda_{n,p},\mu_{n,p})\{t^{-n}[I_{n,p}^*(t)-\sum_{\substack{m,q\\ m<n}}C^*(n,p,m,q)I_{m,q}^*(t)]\}_{t=0}$$
$$=K(\lambda_{n,p},\mu_{n,p})\sum_{\substack{\alpha,\alpha^*,\beta,\beta^*\\ |\lambda^*-\mu^*|=2n+1}}F_{\alpha^*\beta^*,\alpha\beta}(s)D_{n,p}^*(\alpha^*\beta^*,\alpha\beta),\quad n=0,1,\cdots,J+J'-1+|v-v'|;$$

$$D_{n,p}^*(\alpha^*\beta^*,\alpha\beta)\equiv d(\alpha_{n,p}'',\alpha)d(\alpha_{n,p}',\alpha^*)d(\beta_{n,p}'',\beta)d(\beta_{n,p}',\beta^*)$$
$$-\sum_{m,q}C^*(n,p,m,q)d(\alpha_{m,q}'',\alpha)d(\alpha_{m,q}',\alpha^*)d(\beta_{m,q}'',\beta)d(\beta_{m,q}',\beta^*),$$

where

$$C(n,i,m,j)=0\quad\text{if}\quad|\lambda_{n,i}+\mu_{n,i}|>|\lambda_{m,j}+\mu_{m,j}|\quad\text{or}\quad|\lambda_{n,i}-\mu_{n,i}|>|\lambda_{m,j}-\mu_{m,j}|,$$
$$C^*(n,p,m,q)=0\quad\text{if}\quad|\lambda_{n,p}+\mu_{n,p}|>|\lambda_{m,q}+\mu_{m,q}|\quad\text{or}\quad|\lambda_{n,p}-\mu_{n,p}|>|\lambda_{m,q}-\mu_{m,q}|.$$

A general way to determine the numbers C and C^* is described in Appendix C.

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APPENDIX A: DERIVATION OF LINEAR RELATIONS AMONG DERIVATIVES OF t-CHANNEL AMPLITUDES

We shall derive Eqs. (7) and (8) in this Appendix. Several useful properties of the d function are summarized here, namely,[9]

$$d_{\lambda\mu}{}^{J}(\pi-\theta)=(-1)^{J-\mu}d_{-\lambda,\mu}{}^{J}(\theta)=(-1)^{J+\lambda}d_{\lambda,-\mu}{}^{J}(\theta)\,, \tag{A1}$$

$$2(\partial/\partial\theta)d_{\lambda\mu}{}^{J}(\theta)=g(\mu)d_{\lambda,\mu-1}{}^{J}(\theta)-g(-\mu)d_{\lambda,\mu+1}{}^{J}(\theta)\,, \tag{A2}$$

$$2^{-J}[(J+\mu)!(J-\mu)!]^{-1/2}(x+y)^{J+\mu}(x-y)^{J-\mu}=\sum_{\lambda} d_{\lambda\mu}{}^{J}(\tfrac{1}{2}\pi)x^{J-\lambda}y^{J+\lambda}[(J+\lambda)!(J-\lambda)!]^{-1/2}\,, \tag{A3}$$

$$d_{\lambda\mu}{}^{J}(\theta)d_{\lambda\mu'}{}^{J}(\theta)=\delta_{\mu\mu'}\,. \tag{A4}$$

Using these properties, we can rewrite Eq. (5) in the form

$$f_{\alpha^*\beta^*,\alpha\beta^s}(s,t)=\overset{(+)}{\sum} M^{\pm}f_{\alpha'\alpha'',\beta'\beta''}{}^{t}(s,t)+[t(K+st)]^{1/2}\overset{(-)}{\sum} M^{\mp}[t^{-1/2}(K+st)^{-1/2}f_{\alpha'\alpha'',\beta'\beta''}{}^{t}(s,t)] \quad \text{if} \quad \lambda^*-\mu^*=\pm\,, \tag{A5}$$

where

$$M^{\pm}\equiv 2^{-1}[d_{\alpha''\alpha}{}^{J}(\tfrac{1}{2}\pi-\phi)d_{\alpha'\alpha^*}{}^{J}(\tfrac{1}{2}\pi+\phi)d_{\beta''\beta}{}^{J'}(\tfrac{1}{2}\pi+\phi')d_{\beta'\beta^*}{}^{J'}(\tfrac{1}{2}\pi-\phi')$$
$$\pm d_{\alpha''\alpha}{}^{J}(\tfrac{1}{2}\pi+\phi)d_{\alpha'\alpha^*}{}^{J}(\tfrac{1}{2}\pi-\phi)d_{\beta''\beta}{}^{J'}(\tfrac{1}{2}\pi-\phi')d_{\beta'\beta^*}{}^{J'}(\tfrac{1}{2}\pi+\phi')]\,,$$

$$\overset{(+)}{\sum}\equiv \sum_{\substack{\alpha',\alpha'',\beta',\beta''\\ \lambda-\mu=\text{even}}}\,,$$

$$\overset{(-)}{\sum}\equiv \sum_{\substack{\alpha',\alpha'',\beta',\beta''\\ \lambda-\mu=\text{odd}}}\,.$$

The following functions are analytic functions of t at $t=0$ [6,8]:

$$f_{\alpha'\alpha'',\beta'\beta''}{}^{t}(s,t)\,,\quad \text{where}\quad \lambda-\mu=\text{even}\,,$$
$$t^{-1/2}f_{\alpha'\alpha'',\beta'\beta''}{}^{t}(s,t)\,,\quad \text{where}\quad \lambda-\mu=\text{odd}\,,$$
$$M^{+},$$

and

$$t^{-1/2}M^{-}\,.$$

Equations (7) and (8), which correspond to $m=0$, have been discussed in Ref. 3. Let us assume that they are true for all $m\leqslant n$, where n is a positive integer. At $t=0$, we have [see Eq. (6)]

$$(\partial^{n+1}/\partial t^{n+1})[\overset{(+)}{\sum} M^{+}f_{\alpha'\alpha'',\beta'\beta''}{}^{t}(s,t)+\overset{(-)}{\sum} M^{-}f_{\alpha'\alpha'',\beta'\beta''}{}^{t}(s,t)]=\sum_{r=0}^{n+1}\binom{n+1}{r}A_r^{+}+\sum_{r=1}^{n+1}\binom{n+1}{r}A_r^{-}$$
$$=0 \qquad \text{if}\quad |\lambda^*-\mu^*|=\text{an even integer}>2n+2\,,$$
$$=(n+1)!\bar{f}_{\alpha^*\beta^*,\alpha\beta^s}(s,0) \quad \text{if}\quad |\lambda^*-\mu^*|=2n+2\,, \tag{A6}$$

where

$$\binom{n}{r}\equiv n![(n-r)!r!]^{-1}\,,$$

$$A_r^{+}\equiv\overset{(+)}{\sum}(\partial^{r}/\partial t^{r})M^{+}(\partial^{n+1-r}/\partial t^{n+1-r})f_{\alpha'\alpha'',\beta'\beta''}{}^{t}(s,t)\,,$$

$$A_r^{-}\equiv\overset{(-)}{\sum}(\partial^{r}/\partial t^{r})[t^{1/2}(K+st)^{1/2}M^{-}](\partial^{n+1-r}/\partial t^{n+1-r})[t^{-1/2}(K+st)^{-1/2}f_{\alpha'\alpha'',\beta'\beta''}{}^{t}(s,t)]\,.$$

Using Eq. (A2), we can differentiate M^+ and $[t^{1/2}(K+st)^{1/2}M^-]$ in a straightforward manner and prove that (at $t=0$)

$$A_r^+=0 \quad\text{if}\quad |\lambda^*-\mu^*|=\text{an even integer}>2n+2 \text{ and } r\neq 0,$$

$$=[r!/(2r)!]H^{2r}(n+1-r)!\sum_{r'=0}^{2(n+1-r)}(r'!)^{-1}H^{r'}\bar{f}_{\alpha^*\beta^*,\alpha\beta^s}(s,0) \quad\text{if}\quad |\lambda^*-\mu^*|=2n+2 \text{ and } r\neq 0, \tag{A7}$$

$$A_r^-=0 \quad\text{if}\quad |\lambda^*-\mu^*|=\text{even}>2n+2 \quad\text{or}\quad r=0,$$

$$=(-1)[r!/(2r-1)!]H^{2r-1}(n+1-r)!\sum_{r'=0}^{2(n+1-r)+1}(r'!)^{-1}H^{r'}\bar{f}_{\alpha^*\beta^*,\alpha\beta^s}(s,0) \quad\text{if}\quad |\lambda^*-\mu^*|=2n+2 \text{ and } r\neq 0. \tag{A8}$$

It follows that $A_0^+=0$ at $t=0$ if $|\lambda^*-\mu^*|>2n+2$. If $|\lambda^*-\mu^*|=2n+2$, we have ($t=0$)

$$A_0^+\equiv \sum^{(+)} d(\alpha''\alpha)d(\alpha'\alpha^*)d(\beta''\beta)d(\beta'\beta^*)(\partial^{n+1}/\partial t^{n+1})f_{\alpha'\alpha'',\beta'\beta''}{}^t(s,t)=H^*\bar{f}_{\alpha^*\beta^*,\alpha\beta^s}(s,0), \tag{A9}$$

where

$$H^*\equiv(n+1)!+\sum_{r=1}^{n+1}\binom{n+1}{r}(r!)[(2r-1)!]^{-1}(n+1-r)!\sum_{r'=0}^{2(n+1-r)+1}(r'!)^{-1}H^{2r+r'-1}-\sum_{r=1}^{n+1}\binom{n+1}{r}$$

$$\times r![(2r)!]^{-1}(n+1-r)!\sum_{r'=0}^{2(n+1-r)}(r'!)^{-1}H^{2r+r'}$$

$$=(n+1)!\left\{1+\sum_{r=1}^{n+1}\sum_{r'=0}^{2n-2r+3}\binom{2r+r'-1}{2r-1}[(2r+r'-1)!]^{-1}H^{2r+r'-1}-\sum_{r=1}^{n+1}\sum_{r'=0}^{2n-2r+2}\binom{2r+r'}{2r}[(2r+r')!]^{-1}H^{2r+r'}\right\}$$

$$=(n+1)!\left[1+\sum_{p=1}^{2n+2}(p!)^{-1}H^p\sum_{0<2r-1\leqslant p}\binom{p}{2r-1}-\sum_{p=2}^{2n+2}(p!)^{-1}H^p\sum_{0<2r\leqslant p}\binom{p}{2r}\right]$$

$$=(n+1)!\sum_{p=0}^{2n+2}(p!)^{-1}H^p.$$

In the last step, we have used the identity

$$0=(1-1)^p=1+\sum_{r=1}^{p}\binom{p}{r}(-1)^r.$$

Equation (8) can be proved in a similar way.

APPENDIX B: ALGEBRAIC STRUCTURES OF EQS. (11) AND (12)

We shall generalize the method of Ref. 4. Let us consider the following linear equations:

$$\sum_{\alpha,\alpha^*} d(\alpha''\alpha)d(\alpha'\alpha^*)f(\alpha^*\alpha)=t(\alpha'\alpha''), \tag{B1}$$

under the conditions

$$f(\alpha\alpha^*)=f(\alpha^*\alpha)=f(-\alpha-\alpha^*),$$
$$f(\alpha\alpha^*)=0 \quad\text{if}\quad |\lambda^*|>2n \quad\text{or}\quad \lambda^*=\text{odd}. \tag{B2}$$

The conditions (B2) imply that $t(\alpha'\alpha'')=t(\alpha''\alpha')=t(-\alpha'-\alpha'')$ and $t(\alpha'\alpha'')=0$ if $\lambda=\text{odd}$. We need several linearly independent solutions of Eq. (B1) for our later discussion. Let us first prove the following lemmas.

Lemma 1. The numerical values of f and t which are defined by the generating function $g(a,b)$, with $a\leqslant n$, satisfy Eq. (B1) and conditions (B2), where

$$g(a,b)\equiv(x+y)^{2a}(1-xy)^{2b}(1+xy)^{2J-2a-2b}, \tag{B3}$$

$$g(a,b)\equiv\sum_{\alpha,\alpha^*} f(\alpha^*\alpha)x^{J-\alpha}y^{J-\alpha^*}[(J-\alpha)!(J+\alpha)!$$
$$\times(J-\alpha^*)!(J+\alpha^*)!]^{-1/2}, \tag{B4}$$

$$g(b,a)\equiv\sum_{\alpha',\alpha''} t(\alpha'\alpha'')x^{J-\alpha'}y^{J-\alpha''}[(J-\alpha')!(J+\alpha')!$$
$$\times(J-\alpha'')!(J+\alpha'')!]^{-1/2}. \tag{B5}$$

Proof. It is easy to check that the f defined by (B4) have the following properties:

$$f(\alpha^*\alpha)=f(\alpha\alpha^*)=f(-\alpha-\alpha^*),$$
$$f(\alpha^*\alpha)=0 \quad\text{if}\quad \lambda^*=\text{odd} \quad\text{or}\quad |\lambda^*|>2a,$$
$$f(\alpha^*\alpha)\neq 0 \quad\text{if}\quad |\lambda^*|=2a.$$

Using the generating function of $d(\alpha''\alpha)$ given by

(A3), we can rewrite Eq. (B1) in the form

$$\sum_{\alpha',\alpha''} t(\alpha'\alpha'')x^{J-\alpha''}y^{J+\alpha''}\bar{x}^{J-\alpha'}\bar{y}^{J+\alpha'}[(J+\alpha')!(J-\alpha')!(J+\alpha'')!(J-\alpha'')!]^{-1/2}$$

$$=\sum_{\alpha,\alpha^*} f(\alpha^*\alpha)\{\sum_{\alpha''} d(\alpha''\alpha)x^{J-\alpha''}y^{J+\alpha''}[(J+\alpha'')!(J-\alpha'')!]^{-1/2}\}\{\sum_{\alpha'} d(\alpha'\alpha^*)\bar{x}^{J-\alpha'}\bar{y}^{J+\alpha'}[(J+\alpha')!(J-\alpha')!]^{-1/2}\}$$

$$=2^{-2J}\sum_{\alpha,\alpha^*} f(\alpha^*\alpha)(x+y)^{J+\alpha}(x-y)^{J-\alpha}(\bar{x}+\bar{y})^{J+\alpha^*}(\bar{x}-\bar{y})^{J-\alpha^*}[(J+\alpha)!(J-\alpha)!(J+\alpha^*)!(J-\alpha^*)!]^{-1/2}$$

$$=2^{-2J}[(x+y)(\bar{x}+\bar{y})]^{2J}\sum_{\alpha,\alpha^*} f(\alpha^*\alpha)[(x-y)(x+y)^{-1}]^{J-\alpha}[(\bar{x}-\bar{y})(\bar{x}+\bar{y})^{-1}]^{J-\alpha^*}\times[(J+\alpha)!(J-\alpha)!(J+\alpha^*)!(J-\alpha^*)!]^{-1/2}$$

$$=(x\bar{x}-y\bar{y})^{2a}(x\bar{y}+\bar{x}y)^{2b}(x\bar{x}+y\bar{y})^{2J-2a-2b}.$$

We have used Eq. (B4) in the last step. It is obvious now that the $t(\alpha'\alpha'')$ satisfy Eq. (B5).

Lemma 2. We choose a set of $t(\alpha'\alpha'')$ in the following way:

$$T_i \equiv t(\alpha_i',\alpha_i''),$$

where

$$\begin{aligned}\tfrac{1}{2}|\lambda_i| = {} & 0, 1, 2, \cdots, J-v-n, \cdots, J-v-1, J-v,\\ & 0, 1, 2, \cdots, J-v-n, \cdots, J-v-1,\\ & \cdots,\\ & 0, 1, 2, \cdots, J-v-n.\end{aligned}$$

Then any other nonvanishing $t(\alpha'\alpha'')$ can be written as a linear combination of the T_i such that

$$t(\alpha'\alpha'')=\sum_{|\lambda_i|\geqslant|\lambda|} C^n(\alpha',\alpha'',i)T_i, \tag{B6}$$

where the C are constants. For those C having $|\lambda_i|=|\lambda|$, at least one of them differs from zero.

Proof. Equation (14) is a special case of Eq. (B6). First, let us prove that if all nonvanishing f are linearly independent, then the T_i are linearly independent. To see this, let us assume that the T are linearly related. Since there exists a solution of Eq. (B1) such that all f and t are zero, it is clear that the linear relation among the T must be homogeneous. We can write $\sum_i d_iT_i=0$, where the coefficients d_i are constants to be determined.

According to Lemma 1, each of the generating functions $g(a,b)$ generates a special solution of Eq. (B1). We define

$$\begin{aligned}G(b,a)&\equiv\sum_{p=0}^{a} g(b,p)(-1)^p\binom{a}{p}\\ &=(4xy)^a(x+y)^{2b}(1+xy)^{2J-2a-2b}\\ &=\sum_{\alpha',\alpha''} t^{a,b}(\alpha'\alpha'')x^{J-\alpha'}y^{J-\alpha''}[(J-\alpha')!(J+\alpha')!\\ &\qquad\times(J-\alpha'')!(J+\alpha'')!]^{-1/2}.\end{aligned} \tag{B7}$$

Each special solution gives a linear relation among the constants d_i. We want to show that all of the d_i are zero. It is easy to check that $t^{a,0}(\alpha'\alpha'')=0$ if $\alpha'\neq\alpha''$. Therefore, we have

$$\sum_{i,\lambda_i=0} d_it^{a,0}(\alpha_i',\alpha_i'')=0,\quad a=0,1,\cdots,n. \tag{B8}$$

We have $n+1$ linear homogeneous equations and $n+1$ unknown d_i. These d_i must be zero if the determinant of Eq. (B8) does not vanish. Up to a nonvanishing proportional constant, we can write the determinant of Eq. (B8) in the form

$$\begin{vmatrix}\binom{2J}{m_0} & \binom{2J}{m_1} & \cdots & \binom{2J}{m_n}\\ \binom{2J-2}{m_0-1} & \binom{2J-2}{m_1-1} & \cdots & \binom{2J-2}{m_n-1}\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ \binom{2J-2n}{m_0-n} & \binom{2J-2n}{m_1-n} & \cdots & \binom{2J-2n}{m_n-n}\end{vmatrix}$$

$$=(2J)!(2J-2)!\cdots(2J-2n)![m_0!m_1!\cdots m_n!\times(2J-m_0)!(2J-m_1)!\cdots(2J-m_n)!]^{-1}$$

$$\times\begin{vmatrix}1 & \cdots & 1\\ m_0(2J-m_0) & \cdots & m_n(2J-m_n)\\ m_0^2(2J-m_0)^2 & \cdots & m_n^2(2J-m_n)^2\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ m_0^n(2J-m_0)^n & \cdots & m_n^n(2J-m_n)^n\end{vmatrix},$$

where we denote $J-\alpha_i'$ by $m_0, m_1, \cdots, m_n$. It is clear now that the determinant vanishes if and only if there exists a pair of i and j such that $(J-\alpha_i')(J+\alpha_i')=(J-\alpha_j')(J+\alpha_j')$, namely, $\alpha_i'=\pm\alpha_j'$. Therefore we conclude that all d_i with $\lambda_i=0$ must be zero. Using the generating functions $G(1,a)$, one can prove in a similar way that all d_i with $\frac{1}{2}|\lambda_i|=1$ must be zero, etc.

Since the number of f is the same as the number of T_i, any $t(\alpha'\alpha'')$ other than T_i can be written in the form (B6) without the restriction $|\lambda_i|\geqslant|\lambda|$. We want to prove that $C^n(\alpha',\alpha'',i)=0$ if $|\lambda_i|<|\lambda|$, and at least one of $C^n\neq0$, where $|\lambda|=|\lambda_i|$. Each special solution of Eq. (B1) gives a set of linear relations among the C. Let us consider first the case $\alpha'=\alpha''$. We have [see Eq. (B7)]

$$\begin{aligned}t^{0,0}(\alpha'\alpha'')&=0 \quad\text{if}\quad \alpha'\neq\alpha'',\\ &\neq0 \quad\text{if}\quad \alpha'=\alpha''.\end{aligned}$$

Therefore at least one of the $C^n(\alpha',\alpha'',i)$ with $\lambda_i=0$

does not vanish. Next, we consider the case $|\lambda|>0$. The special solutions generated by $G(0,a)$ imply

$$\sum_{i,\lambda_i=0} C^n(\alpha',\alpha'',i)t^{a,0}(\alpha_i',\alpha_i'')=0\,,\quad a=0,1,\cdots,n.$$

In other words, we have $C=0$ if $|\lambda|>|\lambda_i|=0$ [see Eq. (B8)]. Now we turn to the case $\frac{1}{2}|\lambda|=1$. Since we have

$$\begin{aligned} t^{0,1}(\alpha'\alpha'')&=0 \quad \text{if} \quad \tfrac{1}{2}|\lambda|>1\,,\\ &\neq 0 \quad \text{if} \quad \tfrac{1}{2}|\lambda|=1\,,\end{aligned}$$

it is clear that at least one of the C with $\frac{1}{2}|\lambda_i|=1$ differs from zero. The more general cases can be treated in the same way.

We apply these lemmas to Eq. (11). According to Lemma 2, there are several different ways to choose T_i; we use the symbol p to distinguish them. We have

$$\begin{aligned} T_{\alpha'\alpha''}{}^m(0)-\sum_{i,|\lambda_i|\geqslant|\lambda|} C^{n-1,p}(\alpha',\alpha'',i)T_{\alpha_i',\alpha_i''}{}^m(0)\\ =0 \qquad\qquad \text{if} \quad m=0,1,\cdots,n-1\,,\\ =n!F^*(\alpha',\alpha'',n,p) \quad \text{if} \quad m=n\,, \end{aligned}\tag{B9}$$

where

$$F^*(\alpha',\alpha'',n,p)\equiv\sum_{\substack{\alpha,\alpha^*\\ |\lambda^*|=2n}} F(\alpha^*\alpha)[d(\alpha''\alpha)d(\alpha'\alpha^*) -\sum_i C^{n-1,p}(\alpha',\alpha'',i)d(\alpha_i''\alpha)d(\alpha_i'\alpha^*)].$$

The new forward amplitude F^* has an s-kinematic factor $K^{|\lambda|/2}$. To see this, let us rewrite Eq. (B9) in the form

$$\begin{aligned} &\{t^{-n}[T_{\alpha'\alpha''}(t)-\sum_i C^{n-1,p}(\alpha',\alpha'',i)T_{\alpha_i',\alpha_i''}(t)]\}_{t=0}\\ &=K^{|\lambda|/2}\{t^{-n}[\bar{T}_{\alpha'\alpha''}(t)-\sum_i C^{n-1,p}(\alpha',\alpha'',i)\\ &\qquad\times(K+st)^{(|\lambda_i|-|\lambda|)/2}\bar{T}_{\alpha_i',\alpha_i''}(t)]\}_{t=0}\\ &=F^*(\alpha',\alpha'',n,p)\,.\end{aligned}$$

The forward amplitudes $F^*(\alpha',\alpha'',n,p)$ are linear combinations of the forward amplitudes $F(\alpha^*\alpha)$ with $|\lambda^*|=2n$. For a fixed $|\lambda^*|=2n$, the number of amplitudes F^* is bigger than the number of independent forward amplitudes F. In other words, the amplitudes F^* are not all kinematically independent. We can select a set of forward amplitudes $F^*(\alpha_i',\alpha_i'',n,p_i)$ such that $\frac{1}{2}|\alpha_i'-\alpha_i''|=0,\,1,\,\cdots,\,J-v-n$. Note that they have different s-kinematic factors. We shall prove that this set of forward amplitudes has the following properties: (a) They are kinematically independent; therefore the corresponding superconvergence conditions are independent; (b) any other forward amplitude $F^*(\alpha',\alpha'',n,p)$ can be expressed in terms of them, such that

$$F^*(\alpha',\alpha'',n,p)=\sum D_i(\alpha',\alpha'',n,p)F^*(\alpha_i',\alpha_i'',n,p_i)\,,\tag{B10}$$

where

$$\begin{aligned} D_i&=0 \quad \text{if} \quad |\lambda_i|<|\lambda|\,,\\ &\neq 0 \quad \text{if} \quad |\lambda_i|=|\lambda|\,.\end{aligned}$$

The second property implies that the superconvergence conditions for $F^*(\alpha',\alpha'',n,p)$ are linear combinations of the superconvergence conditions for this particular set of amplitudes.[3] The proof follows directly from the following lemma.

Lemma 3. Let us consider Eq. (B1), with the following conditions:

(i) $f(\alpha\alpha^*)=f(\alpha^*\alpha)=f(-\alpha-\alpha^*)$;

(ii) $f(\alpha^*\alpha)=0$ if $|\lambda^*|>2n+2$ or $\lambda^*=\text{odd}$;

(iii) the nonvanishing f are all linearly independent.

According to Lemma 2, we have

$$\begin{aligned} &t(\alpha'\alpha'')-\sum_i C^{n,p}(\alpha',\alpha'',i)t(\alpha_i',\alpha_i'')\\ &=f^*(\alpha',\alpha'',p)\\ &\equiv\sum_{\substack{\alpha,\alpha^*\\ |\lambda^*|=2n+2}} f(\alpha^*\alpha)[d(\alpha''\alpha)d(\alpha'\alpha^*)-\sum_i C^{n,p}(\alpha',\alpha'',i)\\ &\qquad\qquad\times d(\alpha_i''\alpha)d(\alpha_i'\alpha^*)].\end{aligned}$$

We choose a set of $f^*(\alpha_i',\alpha_i'',p_i)\equiv f_i^*$ such that $\frac{1}{2}|\lambda_i|=0,\,1,\,\cdots,\,J-v-n$. This set of f^* has the following properties: (i) They are linearly independent; (ii) any other f^* can be written in the form

$$f^*(\alpha',\alpha'',p)=\sum_i D_i(\alpha',\alpha'',p)f_i^*\,,\tag{B11}$$

where

$$\begin{aligned} D_i&=0 \quad \text{if} \quad |\lambda_i|<|\lambda|\,,\\ &\neq 0 \quad \text{if} \quad |\lambda_i|=|\lambda|\,.\end{aligned}$$

Proof. We shall use the arguments of Ref. 3. If the f_i^* are not linearly independent, we can write

$$\sum_i d_if_i^*=0.\tag{B12}$$

According to Lemma 2, we can choose a particular set of $t(\alpha_j'\alpha_j'')\equiv T_j$ such that they are linearly independent; besides, any other $t(\alpha'\alpha'')$ can be expressed in terms of them [see Eq. (B6)]. The left-hand side of Eq. (B12) can be written as a linear combination of T_j. The linear independence of T_j implies that all of their coefficients must be zero. Let us first prove $d_0=0$. Note that, if $i\neq 0$, f_i^* is a linear combination of those T where $\frac{1}{2}|\lambda_j|>0$. The choice of T is arbitrary. In fact, we can choose those $t(\alpha'\alpha'')$ appearing in f_0^* as part of T. In other words, we have $d_0=0$. Similar arguments can be used to prove that all of the d_i are zero. Since the number of f_i^* is the same as the number of independent $f(\alpha^*\alpha)$, any other f^* is a linear combination of f_i^*. The conditions which are satisfied by the coefficients D_i can be proved by using essentially the same arguments.

The algebraic structure of Eq. (12) is very similar to that of Eq. (11). Let us consider Eq. (B1) under the following conditions:

$$f(\alpha\alpha^*) = -f(\alpha^*\alpha) = f(-\alpha^*-\alpha)\,,$$
$$f(\alpha\alpha^*) = 0 \quad \text{if} \quad |\lambda^*| > 2n+1 \quad \text{or} \quad \lambda^* = \text{even}. \tag{B13}$$

The conditions (B13) imply that $t(\alpha'\alpha'') = -t(\alpha''\alpha') = t(-\alpha''-\alpha')$ and $t=0$ if $\lambda=$even. We shall omit the proofs of the following lemmas.

We define

$$t'(\alpha'\alpha'') \equiv t(\alpha'\alpha'')[(J-\alpha')!(J+\alpha')!(J-\alpha'')!(J+\alpha'')!]^{-1/2}.$$
$$f'(\alpha^*\alpha) \equiv f(\alpha^*\alpha)[(J-\alpha)!(J+\alpha)!(J-\alpha^*)!(J+\alpha^*)!]^{-1/2}.$$

Lemma 4. The numerical values of f and t which are defined by the generating function $g^*(a,b)$, with $a \leqslant n$, satisfy Eq. (B1) and conditions (B13), where

$$g^*(a,b) \equiv (x-y)(x+y)^{2a}(1-xy)^{2b} \times(1+xy)^{2J-1-2a-2b}, \tag{B14}$$

$$g^*(a,b) \equiv \sum_{\alpha,\alpha^*} f'(\alpha^*\alpha)x^{J-\alpha}y^{J-\alpha^*}, \tag{B15}$$

$$g^*(b,a) \equiv \sum_{\alpha',\alpha''} t'(\alpha'\alpha'')x^{J-\alpha'}y^{J-\alpha''}. \tag{B16}$$

Lemma 5. We choose a set of t in the following way:

$$T_i \equiv t(\alpha_i',\alpha_i'')\,,$$

where

$$\begin{aligned}\tfrac{1}{2}(|\lambda_i|-1)\\ &= 0, 1, \cdots, J+v-n-1, \cdots, J+v-2, J+v-1,\\ &\quad 0, 1, \cdots, J+v-n-1, \cdots, J+v-2,\\ &\quad \cdots \quad \cdots \quad \cdots \quad \cdots,\\ &\quad 0, 1, \cdots, J+v-n-1.\end{aligned}$$

Then any other t can be written as a linear combination of them such that

$$t(\alpha'\alpha'') = \sum_{|\lambda_i| \geqslant |\lambda|} \bar{C}^n(\alpha',\alpha'',i)T_i\,, \tag{B17}$$

where the $\bar{C}$ are constants. For those $\bar{C}$ having $|\lambda_i| = |\lambda|$, at least one of them differs from 0.

Lemma 6. Let us consider Eq. (B1) with the following conditions:

(i) $f(\alpha\alpha^*) = -f(\alpha^*\alpha) = f(-\alpha^*-\alpha)$;

(ii) $f=0$ if $|\lambda^*| > 2n+3$ or $\lambda^* =$ even;

(iii) the nonvanishing f are all linearly independent.

According to Lemma 5, we have

$$\begin{aligned}t(\alpha'\alpha'') - \sum_i \bar{C}^{n,p}(\alpha',\alpha'',i)t(\alpha_i',\alpha_i'')\\ &= \bar{f}^*(\alpha',\alpha'',p)\\ &= \sum_{\substack{\alpha,\alpha^*\\ |\lambda^*|=2n+3}} f(\alpha^*\alpha)[d(\alpha''\alpha)d(\alpha'\alpha^*) - \sum_i \bar{C}^{n,p}(\alpha',\alpha'',i)\\ &\qquad \times d(\alpha_i''\alpha)d(\alpha_i'\alpha^*)].\end{aligned}$$

We choose a set of $\bar{f}^*(\alpha_i',\alpha_i'',p_i)$ such that $\frac{1}{2}(|\lambda_i|-1) = 0, 1, \cdots, J+v-n-1$. This set of $\bar{f}^*$ has the following properties: (i) They are linearly independent; (ii) any other $\bar{f}^*$ can be written in the form

$$\bar{f}^*(\alpha',\alpha'',p) = \sum_i \bar{D}_i(\alpha',\alpha'',p)f_i^*\,, \tag{B18}$$

where $\bar{D}_i = 0$ if $|\lambda_i| < |\lambda|$, but $\neq 0$ if $|\lambda_i| = |\lambda|$.

In Lemma 6, we use p to distinguish different ways of choosing T_i.

We shall now prove that the forward amplitudes (16) are all kinematically independent and free from s-kinematic singularities. Let us go back to Eq. (B1) with the conditions (B2). For convenience, we modify (B2) such that $f=0$ if $|\lambda^*| > 2r-2$ (instead of $2n$). We choose the T_i of Lemma 2 such that $\alpha_i' = J, J-1, \cdots, J-r+1$ and $\alpha_i' - \alpha_i'' = 0, 2, 4, \cdots$. We have [see Eq. (B6)]

$$t'(J-r, J-r-2n) = \sum_{\substack{i=1,\cdots,r\\ j\geqslant 0}} C(i,j,n,r) \times t'(J-r+i, J-r+i-2n-2j). \tag{B19}$$

The special solutions generated by $G(b,a)$ can be used to determine the coefficients $C(i,j,n,r)$ in a simple way. It is easy to check that $C=0$ if $j>i$.

Since Eq. (B19) is homogeneous, we shall ignore the factor 4^a of $G(b,a)$. The linear relations among the C corresponding to (a,b) are

$$\begin{aligned}
&(r-1,\, n), && 2J-2n-2r+2 \equiv A = C(1,0,n,r)\,,\\
&(r-1,\, n+1), && 2n+2 = C(1,1,n,r)\,,\\
&(r-2,\, n), && \binom{A+2}{2} = \binom{A+2}{1}C(1,0,n,r) + C(2,0,n,r)\,,\\
&(r-2,\, n+1), && A = (2n+2)C(1,0,n,r) + AC(1,1,n,r) + C(2,1,n,r)\,,\\
&(r-2,\, n+2), && \binom{2n+4}{2} = \binom{2n+4}{1}C(1,1,n,r) + C(2,2,n,r)\,,\\
& && \cdots.
\end{aligned} \tag{B20}$$

The simplicity of these equations is obvious. One can calculate all of C in a straightforward manner. The result is given in Sec. III.

Let us assume that $f(\alpha^*\alpha)=0$ if $|\lambda^*|>2r$ instead of $2r-2$. Then we have

$$t'(J-r,\,J-r-2n)-\sum_{i,j}C(i,j,n,r)t'(J-r+i,\,J-r+i-2n-2j)=\sum_{\substack{\alpha,\alpha^*\\|\lambda^*|=2r}}f'(\alpha^*\alpha)[d(J-r-2n,\,\alpha)d(J-r,\,\alpha^*)$$
$$-\sum_{i,j}C(i,j,n,r)d(J-r+i-2n-2j,\,\alpha)d(J-r+i,\,\alpha^*)]\equiv\sum_{\alpha,\alpha^*}f'(\alpha^*\alpha)D(\alpha^*\alpha)\,,\quad\text{(B21)}$$

where

$$D(\alpha^*\alpha)=D(\alpha\alpha^*)=D(-\alpha-\alpha^*)\,.$$

We shall describe a simple way to calculate the coefficients D. If the numerical values of t' are generated by $G(b,a)$, then the corresponding numerical values of f' are generated by [see Eq. (B7)]

$$\sum_{p=0}^{a}g(p,b)(-1)^p\binom{a}{p}=(1-xy)^{2b}(1+xy)^{2J-2a-2b}[(1-x^2)(1-y^2)]^a=\sum_{\alpha,\alpha^*}f'^{(b,a)}(\alpha^*\alpha)x^{J-\alpha}y^{J-\alpha^*}.$$

At $a=r$, we have

$$4^r\delta_{b,n}=\sum_{\substack{\alpha,\alpha^*\\|\lambda^*|=2r}}f'^{(b,r)}(\alpha^*\alpha)D(\alpha^*\alpha)\,.$$

It can be shown that the D satisfy the following equations:

$$D(r+v,\,-r+v)=(-1)^{n+r}4^{2r-J}\binom{J-r-v}{n}\,,$$
$$D(r+v+1,\,-r+v+1)+D(r+v,\,-r+v)=(-1)^{n+r}4^{2r-J+1/2}\binom{J-r-v-1}{n}\,,\quad\text{(B22)}$$
$$\cdots,$$
$$2D(r+v+m,\,-r+v+m)+2\sum_{i=1}^{m-1}\binom{2m}{i}D(r+v+m-i,\,-r+v+m-i)$$
$$+\binom{2m}{m}D(r+v,\,-r+v)=(-1)^{n+r}4^{2r-J+m}\binom{J-r-v-m}{n}\,,$$
$$\cdots,$$

where we define

$$\binom{a}{b}\equiv0\quad\text{if}\quad a<b.$$

From these equations one can calculate the D step by step.

Let us consider Eq. (B1) with the conditions (B13). For convenience, we change the notation such that $f=0$ if $|\lambda^*|>2r+1$ (instead of $2n+1$). We have

$$t'(J-r,\,J-r-2n-1)-\sum_{i,j}C^*(i,j,n,r)t'(J-r+i,\,J-r+i-2n-1-2j)=\sum_{\substack{\alpha,\alpha^*\\|\lambda^*|=2r+1}}f'(\alpha^*\alpha)D^*(\alpha^*\alpha)\,,\quad\text{(B23)}$$

where $D^*(\alpha^*\alpha)=-D^*(\alpha\alpha^*)=D^*(-\alpha-\alpha^*)$. The special solutions generated by $G^*(b,a)$ can be used to determine the D^* in a straightforward way, where G^* is given by Eq. (B7), with g replaced by g^*. The result is given in Sec.

III. Using the following equations, the D^* can be calculated step by step:

$$
\begin{aligned}
&D^*(r+1-v,\ -r-v)=(-1)^{n-r}4^{2r-J+1/2}\binom{J-r-1+v}{n},\\
&D^*(r+2-v,\ -r+1-v)+D^*(r+1-v,\ -r-v)=(-1)^{n+r}4^{2r-J+1}\binom{J-r-2+v}{n},\\
&\qquad\cdots,\\
&2D^*(r+1+m-v,\ -r+m-v)+2\sum_{i=1}^{m-1}\binom{2m}{i}D^*(r+1+m-v-i,\ -r+m-v-i)\\
&\qquad+\binom{2m}{m}D^*(r+1-v,\ -r-v)=(-1)^{n+r}4^{2r-J+m+1/2}\binom{J-r+v-m-1}{n},\\
&\qquad\cdots.
\end{aligned}
\tag{B24}
$$

APPENDIX C: ALGEBRAIC STRUCTURE OF EQS. (17) AND (18)

We shall generalize the results of Appendix B. Since the arguments used here are essentially the same, we shall give our results without proof, unless some new arguments are used.

Let us consider the following equations:

$$\sum_{(\alpha)} d(\alpha''\alpha)d(\alpha'\alpha^*)d(\beta''\beta)d(\beta'\beta^*)f(\alpha^*\beta^*,\alpha\beta)=t(\alpha'\alpha'',\beta'\beta''),\tag{C1}$$

where α means all possible values of α, α^*, β, and β^*. We define

$$f'(\alpha^*\beta^*,\alpha\beta)\equiv f(\alpha^*\beta^*,\alpha\beta)[(J-\alpha)!(J+\alpha)!(J-\alpha^*)!(J+\alpha^*)!(J'-\beta)!(J'+\beta)!(J'-\beta^*)!(J'+\beta^*)!]^{-1/2},\tag{C2}$$

$$t'(\alpha'\alpha'',\beta'\beta'')\equiv t(\alpha'\alpha'',\beta'\beta'')[(J-\alpha')!(J+\alpha')!(J-\alpha'')!(J+\alpha'')!(J'-\beta')!(J'+\beta')!(J'-\beta'')!(J'+\beta'')!]^{-1/2},\tag{C3}$$

$$g^f(x,\bar{x},y,\bar{y})\equiv\sum_{(\alpha)} f'(\alpha^*\beta^*,\alpha\beta)x^{J-\alpha}\bar{x}^{J-\alpha^*}y^{J'-\beta}\bar{y}^{J'-\beta^*},\tag{C4}$$

$$g^t(x,\bar{x},y,\bar{y})\equiv\sum_{(\alpha')} t'(\alpha'\alpha'',\beta'\beta'')x^{J-\alpha''}\bar{x}^{J-\alpha'}y^{J'-\beta''}\bar{y}^{J'-\beta'},\tag{C5}$$

where (α') means all possible values of α', α'', β', and β''.

Lemma 7. Equation (C1) implies

$$g^t(x,\bar{x},y,\bar{y})=[\tfrac{1}{2}(1+x)(1+\bar{x})]^{2J}[\tfrac{1}{2}(1+y)(1+\bar{y})]^{2J'}g^f(x^*,\bar{x}^*,y^*,\bar{y}^*),\tag{C6}$$

where $u^*\equiv(u-1)(u+1)^{-1}$, $u=x,\bar{x},y,\bar{y}$.

We are interested in Eq. (C1), with the following subsidiary conditions:

$$f(\alpha^*\beta^*,\alpha\beta)=f(\alpha\beta,\alpha^*\beta^*)=f(-\alpha-\beta,\ -\alpha^*\beta^*),\quad f(\alpha^*\beta^*,\alpha\beta)=0\quad\text{if}\quad|\lambda^*-\mu^*|>2n\quad\text{or}\quad=\text{odd}.\tag{C7}$$

The conditions (C7) imply that $t(\alpha'\alpha'',\beta'\beta'')=t(\alpha''\alpha',\beta''\beta')=t(-\alpha'-\alpha'',\ -\beta'-\beta'')$ and $t=0$ if $\lambda-\mu=$ odd. We define

$$
\begin{aligned}
g(a,b,c,a',b',c',a'',b'',c'')\equiv{}&(x+\bar{x})^a(1-x\bar{x})^b(1+x\bar{x})^c(y+\bar{y})^{a'}(1-y\bar{y})^{b'}(1+y\bar{y})^{c'}\\
&\times[(x+y)(\bar{x}+\bar{y})]^{a''}[(1-xy)(1-\bar{x}\bar{y})]^{b''}[(1+xy)(1+\bar{x}\bar{y})]^{c''},
\end{aligned}
\tag{C8}
$$

where $a+b+c+a''+b''+c''=2J$ and $a'+b'+c'+a''+b''+c''=2J'$.

Lemma 8. The f' given by $g^f=g(a,b,c,a',b',c',a'',b'',c'')$, where $a+a'=$ even, $b+b'=$ even, and $a+a'+2a''\leqslant 2n$, satisfy conditions (C7). It follows from Eq. (C6) that $g^t=g(b,a,c,b',a',c',b'',a'',c'')$.

The special case where $n=0$ has been discussed in Ref. 4. If $n>0$, it is more convenient to use the following generating functions, which are linear combinations of the above functions:

$$g^t=(x\bar{x})^r(y\bar{y})^s g(b,0,c,b',0,c',b'',0,c''),\tag{C9}$$

where $b+b'=$ even, $2r+b+c+b''+c''=2J$, $2s+b'+c'+b''+c''=2J'$, $r+s\leqslant n$;

$$g^t=(x\bar{x})^r(y\bar{y})^s(x\bar{x}+y\bar{y})g(b,0,c,b',0,c',b'',0,c''),\tag{C10}$$

where $b+b'=$ even, $2r+b+c+b''+c''=2J-1$, $2s+b'+c'+b''+c''=2J'-1$, $r+s\leqslant n-1$.

Lemma 9. Let us consider Eq. (C1), with subsidiary conditions (C7), where $n=0$. If the nonvanishing f are all linearly independent, then there exists a set of t, say T_i, which has the following properties: (a) The T_i are linearly independent; (b) any other t can be written in the form $t(\alpha'\alpha'',\beta'\beta'')=\sum_i C_iT_i$, where $C_i=0$ if $|\lambda_i+\mu_i|<|\lambda+\mu|$ or $|\lambda_i-\mu_i|<|\lambda-\mu|$, and not all of the C_i with $|\lambda_i\pm\mu_i|=|\lambda\pm\mu|$ are zero. The T_i can be chosen in the following way:

$$\begin{aligned}(\lambda_i,\mu_i)^N &= (2m,0)^{m+1},\quad (0,2m)^m \quad\text{or}\quad (2m,0)^m,\quad (0,2m)^{m+1}\\ &= (2m+r,r)^{m+1},\quad (r,2m+r)^m \quad\text{or}\quad (2m+r,r)^m,\quad (r,2m+r)^{m+1}\\ &= (2m+r,-r)^{m+1},\quad (-r,2m+r)^m \quad\text{or}\quad (2m+r,-r)^m,\quad (-r,2m+r)^{m+1},\\ &\qquad m=0,1,\cdots,J'-v',\quad r=1,2,\cdots,2J'-2m\\ &= (2J'+2p,2J'-2m)^{m+1},\ (2J'+2p,-2J'+2m)^{m+1},\quad m=0,1,\cdots,J'-v',\quad p=1,2,\cdots,J-J'-|v-v'|\\ &= (2J'+2p-1,2J'-2m-1)^{m+1},\quad (2J'+2p-1,-2J'+2m+1)^{m+1},\\ &\qquad m=0,1,\cdots,J'+v'-1,\quad p=1,2,\cdots,J-J'+|v-v'|\end{aligned}$$

where we use N to represent the number of T_i having the same λ_i and μ_i, and any two t (say, T_i and T_j) having the same λ and μ satisfy the restriction

$$\begin{aligned}(\beta_i'-\beta_j')(\beta_i'+\beta_j'-\mu)&\neq 0 \quad\text{if}\quad |\lambda|>|\mu|,\\ (\alpha_i'-\alpha_j')(\alpha_i'+\alpha_j'-\lambda)&\neq 0 \quad\text{if}\quad |\lambda|<|\mu|.\end{aligned}\tag{C11}$$

Proof. The restriction (C11) is more general than that given in Ref. 4. To illustrate the new argument, let us consider those T_i corresponding to $(4,0)^3$ and prove that they are linearly independent of each other. If these T_i are not linearly independent, we have

$$\sum_{i=1}^{3} d_iT_i=0.$$

According to Ref. 4, we can use the following generating functions:

$$\begin{aligned}g^t=(1+x\bar{x})^{2J-4}(1+y\bar{y})^{2J'}(x+\bar{x})^4,\quad &(1+x\bar{x})^{2J-4}(1+y\bar{y})^{2J'-2}(x+\bar{x})^2(1+xy)(1+\bar{x}\bar{y})(x+y)(\bar{x}+\bar{y}),\\ &(1+x\bar{x})^{2J-4}(1+y\bar{y})^{2J'-4}[(1+xy)(1+\bar{x}\bar{y})(x+y)(\bar{x}+\bar{y})]^2.\end{aligned}$$

These functions imply three homogeneous linear relations among the d_i. The argument used in the proof of Lemma 2 [see Eq. (B8)] can be used here to prove that all of the d_i must be zero.

Lemma 10. Let us consider Eq. (C1), with subsidiary conditions (C7), where $J+J'-|v-v'|\geqslant n>0$. If the nonvanishing f are all linearly independent, then there exists a set of t, say T_i, which satisfies the two properties mentioned in Lemma 9. These T_i can be chosen in the following way [we define $r\equiv J-\alpha'$, $s\equiv J'-\beta'$, $s'\equiv J'-\beta''$, and use the convention $\lambda\geqslant 0$; we choose all possible values of r and s if $\mu\geqslant 0$ (r and s' if $\mu<0$) which satisfy the restrictions indicated below; we use the classification of (λ_i,μ_i) defined in Lemma 9, and N is the number of T_i having the same λ_i and μ_i for $n=0$]:

for $\lambda_i\geqslant|\mu_i|$:

$$\begin{aligned}&r+s(s')\leqslant n+N-1,\quad r\leqslant n,\quad 2J-\lambda_i\geqslant 2r,\quad 2J'-|\mu_i|\geqslant 2s(s');\\ &2J-\lambda_i-r+s(s')\leqslant n+N-2,\quad 2J-\lambda_i-r\leqslant n-1,\quad 2J-1-\lambda_i\geqslant 2r,\quad 2J'-1-|\mu_i|\geqslant 2s(s');\end{aligned}$$

for $|\mu_i|>\lambda_i$:

$$\begin{aligned}&r+s(s')\leqslant n+N-1,\quad s(s')\leqslant n,\quad 2J-\lambda_i\geqslant 2r,\quad 2J'-|\mu_i|\geqslant 2s(s');\\ &2J-\lambda_i-r+s(s')\leqslant n+N-2,\quad s(s')\leqslant n-1,\quad 2J-1-\lambda_i\geqslant 2r,\quad 2J'-1-|\mu_i|\geqslant 2s(s').\end{aligned}$$

The choice described by Lemma 10 is not the most general one; however, it is the simplest choice.

We now consider Eq. (C1), with the following subsidiary conditions:

$$f(\alpha^*\beta^*,\alpha\beta)=-f(\alpha\beta,\alpha^*\beta^*)=f(-\alpha-\beta,-\alpha^*-\beta^*),\qquad f(\alpha^*\beta^*,\alpha\beta)=0\quad\text{if}\quad|\lambda^*-\mu^*|=\text{even or}>2n+1.\tag{C12}$$

The conditions (C12) imply that $t(\alpha'\alpha'',\beta'\beta'')=-t(\alpha''\alpha',\beta''\beta')=t(-\alpha''-\alpha',-\beta''-\beta')$ and $t=0$ if $\lambda-\mu=$ even.

Lemma 11. The f given by $g^f=(x-\bar{x})g(a,b,c,a',b',c',a'',b'',c'')$ [or $(y-\bar{y})g(a,b,c,a',b',c',a'',b'',c'')$], where $a+a'=$ even, $b+b'=$ even, $a+b+c+a''+b''+c''=2J-1$ (or $2J$), $a'+b'+c'+a''+b''+c''=2J'$ (or $2J'-1$), and $a'+a+2a''\leqslant 2n$, satisfy conditions (C12); from Eq. (C6) we have $g^t=(x-\bar{x})g(b,a,c,b',a',c',b'',a'',c'')$ [or $(y-\bar{y})g(b,a,c,b',a',c',b'',a'',c'')$].

The special case where $n=0$ has been discussed in Ref. 4.

Lemma 12. Let us consider Eq. (C1), with subsidiary conditions (C12) and $n=0$. If the f are all linearly independent, then there exists a set of t, say T_i, which satisfies the two properties mentioned in Lemma 9. These T_i can be chosen in the following way (we use the convention of Lemma 10):

$$\begin{aligned}(\lambda_i,\mu_i)^N &= (2m+1+p,\ \pm p)^{2m+2}, \quad m=0,1,\cdots,J'+v'-1, \quad p=0,1,\cdots,2J'-2m-1\\ &= (p,\ \pm 2m\pm 1\pm p)^{2m+2}, \quad m=0,\cdots,J'+v'-1, \quad p=0,\cdots,2J'-2m-1, \quad 2m\neq 2J'-1-p\\ &= (2J'-2m-1,\ \pm 2J')^{m+1}, \quad m=0,\cdots,J'-1+v'\\ &= (2J,\ \pm 2J'\mp 1\pm 2|v-v'|\mp 2m)^{m+1}, \quad m=0,\cdots,J'-v'-1+2v\\ &= (p,\ \pm 2J'\mp m)^{m+1}, \quad m=0,\cdots,2J', \quad 2J'<p<2J, \quad 2J'-m+p=\text{odd};\end{aligned}$$

we choose all t with $\lambda=2J$;

$$\begin{aligned}&r=0,1, \quad s(s')=0,\cdots,\tfrac{1}{2}N-1 \quad \text{if} \quad \lambda\neq 2J, \quad \lambda>|\mu|, \quad \text{and} \quad N=\text{even},\\ &r=0,1, \quad s(s')=0,\cdots,\tfrac{1}{2}(N-1), \quad r+s(s')<\tfrac{1}{2}(N+1) \quad \text{if} \quad \lambda\neq 2J, \quad \lambda>|\mu|, \quad \text{and} \quad N=\text{odd},\\ &s(s')=0, \quad r=0,\cdots,N-1 \quad \text{if} \quad |\mu|=2J'>\lambda,\\ &s(s')=0,1, \quad r=0,\cdots,\tfrac{1}{2}N-1 \quad \text{if} \quad |\mu|>\lambda \quad \text{and} \quad |\mu|\neq 2J'.\end{aligned}$$

Lemma 13. Let us consider Eq. (C1), with subsidiary conditions (C12), where $J+J'-1+|v-v'|\geq n>0$. If the f are all linearly independent, then there exists a set of t, say T_i, which satisfies the two properties mentioned in Lemma 9. These T_i can be chosen in the following way [we use the classification of (λ_i,μ_i) defined in Lemma 12, and N is the number of T_i having the same λ_i and μ_i for $n=0$]:

for $\lambda_i=2J$: all t in this class; for $\lambda_i\neq 2J$, $\lambda_i>|\mu_i|$, and $N=\text{even}$:

$$\begin{aligned}&r+s(s')\leq n+\tfrac{1}{2}N-1, \quad r\leq n, \quad 2J-\lambda_i\geq 2r, \quad 2J'-|\mu_i|\geq 2s(s');\\ &2J-\lambda_i-r+s(s')\leq n+\tfrac{1}{2}N-1, \quad 2J-\lambda_i-r\leq n, \quad 2J-1-\lambda_i\geq 2r, \quad 2J'-1-|\mu_i|\geq 2s(s');\end{aligned}$$

for $\lambda_i\neq 2J$, $\lambda_i>|\mu_i|$, and $N=\text{odd}$:

$$\begin{aligned}&r+s(s')\leq n+\tfrac{1}{2}(N+1)-1, \quad r\leq n, \quad 2J-\lambda_i\geq 2r, \quad 2J'-|\mu_i|\geq 2s(s');\\ &2J-\lambda_i-r+s(s')\leq n+\tfrac{1}{2}(N+1)-2, \quad 2J-\lambda_i-r\leq n, \quad 2J-1-\lambda_i\geq 2r, \quad 2J'-1-|\mu_i|\geq 2s(s');\end{aligned}$$

for $|\mu_i|=2J'>\lambda_i$:

$$r\leq n+N-1, \quad 2J-\lambda_i\geq 2r;$$

for $|\mu_i|>\lambda_i$ and $|\mu_i|\neq 2J'$:

$$\begin{aligned}&r+s(s')\leq n+\tfrac{1}{2}N-1, \quad s(s')\leq n, \quad 2J-\lambda_i\geq 2r, \quad 2J'-|\mu_i|\geq 2s(s');\\ &2J-\lambda_i-r+s(s')\leq n+\tfrac{1}{2}N-1, \quad s(s')\leq n, \quad 2J-1-\lambda_i\geq 2r, \quad 2J'-1-|\mu_i|\geq 2s(s').\end{aligned}$$