(C) Finally, we would like to make a short comment on the soft-pion-emission approach. The extrapolation $q_\mu \to 0$ is certainly more hazardous than the one[13] we are using, $-q^2=m_\pi^2 \to 0$. By using the soft-pion technique, Callan and Treiman obtained,[11] in our notation,

$$F_+(m_K^2)+F_-(m_K^2)=(f_K/f_\pi)(\sin\theta_A)/\sin\theta_V. \quad (15)$$

The values of F's at $s=m_K^2$ are not physical for the K_{l3} decay. The K_{e3} decay energy spectrum has a maximum around $q^2=0$. Therefore, physically important form factors are $F_+(0)$ and $F_-(0)$. Since there is no convincing way of making an extrapolation in the $q_\mu \to 0$ approach, let us assume, as in Ref. 1, $F_+(m_K^2) \simeq F_+(0)$ and $F_-(m_K^2) \simeq F_-(0)$. If we take, for example, $F_+(0) \simeq 1.05$ and assume $\theta_A=\theta_V$, we obtain, from (15), $F_-(0)=0.23$ and therefore $\xi=+0.22$. This must be compared with our prediction $-0.28<\xi<-0.026$. Of course, the value of ξ, $+0.22$, is not very trustworthy, since there is no guarantee that the extrapolation $F_+(m_K^2) \simeq F_+(0)$ and $F_-(m_K^2) \simeq F_-(0)$ is very good. However, since the s dependence of the F_+ and F_- form factors does not seem to be very large, both experimentally and theoretically the above extrapolation should not be very bad. Therefore, we think that the Callan-Treiman relation already indicates that the value of ξ is small, i.e., $|\xi|<1$. We now wish to show that our prediction of a small negative value of ξ is consistent with the Callen-Trieman relation obtained at $s=m_K^2$. Let us compute $F_+(m_K^2)+F_-(m_K^2)$ by using Eqs. (11) and (12′). Neglecting the $O(\epsilon^2)$ term, we obtain

$$F_+(m_K^2)+F_-(m_K^2)=\frac{f_K}{f_\pi}+\frac{m_\pi^2}{m_{K^*}{}^2-m_K^2}\simeq 1.30, \quad (16)$$

which is indeed very close to the Callan-Trieman relation (15). We particularly note that Eq. (16) is independent of the parameter m_κ. We therefore claim that our results include the soft-pion result given at $s=m_K^2$. It will be interesting to check our prediction (especially the sign of the parameter ξ) by more precise experiments. It is certainly encouraging to observe that all the polarization experiments of $K_{\mu3}$ decay (which seem cleaner than other types of K_{l3} decay experiments) give negative values for the parameter ξ.[14]

We wish to thank Professor J. Sucher for his stimulating discussions in the early stage of this work. We also thank Professor G. Snow for reading the manuscript.

[13] This approach has also been attempted recently by Okubo *et al.* from a different standpoint. S. Okubo, in *Proceedings of the 1967 International Conference on Particles and Fields* (Interscience Publishers Inc., New York, 1967), p. 469.

[14] At present, experiments do not seem to exclude the possibility $\xi<0$. J. W. Cronin, in *Proceedings of the 1967 International Conference on Particles and Fields* (Interscience Publishers Inc., New York, 1967), p. 1.

PHYSICAL REVIEW VOLUME 169, NUMBER 5 25 MAY 1968

Quark Models, Current Algebras, and Radiative Corrections to Beta Decay*

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(Received 10 January 1968)

We study radiative corrections to semileptonic decays. Special commutation relations between the weak and electromagnetic currents reduce the question of finiteness of $i \to f+e^-+\bar{\nu}$ to that of the finiteness of $\langle f|J_\mu{}^W|i\rangle/\sqrt{Z_2{}^e}$. In a quark model with interactions properly chosen, standard renormalization theory is then used to prove that these quantities, hence the order-α radiative corrections to any β decay, are finite. We conclude that commutators are not enough and that some small amount of dynamics is needed to remove divergent radiative corrections.

IT has recently been shown that, with the assumption of certain equal-time commutation relations between the electromagnetic and weak hadronic currents, the order-α radiative corrections to pion β decay are finite.[1,2] The algebra required differs from the standard current algebra only in the space-space part, where one needs

$$\delta(x_0)[J_i{}^{\text{e.m.}}(x),J_j(0)]=\delta(x)[\delta_{ij}J_0{}^W(0)-i\epsilon_{ijk}J_k{}^W(0)].$$

Such relations can actually be realized in several field-theory models involving fundamental integrally charged triplets rather than quarks, but which seem capable of reproducing all the known features of strong interactions. What remains to be shown is that this same scheme is sufficient to treat the order-α corrections to *any* β decay—for example, that the corrections to g_A are finite. In this article we address the question and

* Research supported in part by the U. S. Office of Naval Research, Contract No. Nonr-1866(55).
† Alfred P. Sloan Foundation Fellow.

[1] N. Cabibbo, L. Maiani, and G. Preparata, Phys. Letters **25B**, 132 (1967).

[2] K. Johnson, F. Low, and H. Suura, Phys. Rev. Letters **18**, 1224 (1967).

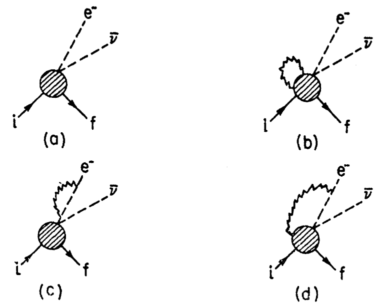

FIG. 1. The process $i \to f+e^- + \bar{\nu}$ up to order α.

show that within the framework of renormalized perturbation theory, with a proper choice of the fundamental triplet model, the general radiative correction is in fact finite.

The problem divides into two disjoint pieces. First, we use the commutation relations of $J^{\text{e.m.}}$ and J^W to show that the general radiative correction to $i \to f + e^- + \bar{\nu}$ is finite if $\langle f | J_\mu{}^W | i\rangle / \sqrt{Z_2{}^e}$ is finite, where $Z_2{}^e$ is the electronic wave-function renormalization. Then we use arguments which are independent of the algebra and depend only on the underlying field theory of the strong interactions to show that $\langle f | J_\mu{}^W | i\rangle / \sqrt{Z_2{}^e}$ is in fact finite (always to order α only).

In Fig. 1 we have indicated the four independent contributions to the process $i \to f + e^- + \bar{\nu}$, neglecting diagrams of order α^2 and higher. The sum of diagrams 1(a)–1(c) is just

$$(\sqrt{Z_2{}^e})\langle f | J_\mu{}^W | i\rangle \bar{u}_e \gamma^\mu (1-\gamma_5) v_\nu , \tag{1}$$

where $\langle f | J_\mu{}^W | i\rangle$ includes order-α radiative corrections and hence is cutoff-dependent. Diagram 1(d) may be expressed analytically as

$$M_d = \frac{e^2}{(2\pi)^4} \int \frac{dk}{k^2} \bar{u}_e \gamma^\mu \frac{1}{l - k - m_e} \gamma^\nu (1-\gamma_5) v_\nu T_{\mu\nu}(k) ,$$
$$T_{\mu\nu}(k) = \int dx\, e^{ik\cdot x} \langle f | T(J_\mu{}^{\text{e.m.}}(x) J_\nu{}^W(0)) | i\rangle , \tag{2}$$

where l is the electron momentum. Since $T_{\mu\nu} \to k^{-1}$ for large k, this integral is logarithmically divergent, but the divergent part is determined by the leading asymptotic behavior of $T_{\mu\nu}$.

Bjorken[3] has pointed out that the asymptotic behavior of a function like $T_{\mu\nu}$ is partially determined by the equal-time commutation relations of the currents involved. From the Low equation for $T_{\mu\nu}$ one can easily see that

$$T_{\mu\nu}(k) \xrightarrow[|k_0|\to\infty]{} \frac{i}{k_0} \int d\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \times \langle f | [J_\mu{}^{\text{e.m.}}(0,\mathbf{x}), J_\nu{}^W(0)] | i\rangle . \tag{3}$$

[3] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

The higher terms in this expansion may or may not be determined in terms of other more complicated equal-time commutators. The commutation relations we have adopted then guarantee that if we define

$$t_{\mu\nu} = (i/k^2)(-g_{\mu\nu} j\cdot k + j_\mu k_\nu + j_\nu k_\mu - i\epsilon_{\mu\nu\lambda\sigma} j^\lambda k^\sigma) , \tag{4}$$
$$j_\mu = \langle f | J_\mu{}^W | i\rangle ,$$

then both $T_{\mu\nu}$ and $t_{\mu\nu}$ have the same leading asymptotic behavior. Therefore, if $T_{\mu\nu} = t_{\mu\nu} + \hat{T}_{\mu\nu}$, the remainder term has the asymptotic behavior

$$\hat{T}_{\mu\nu} \xrightarrow[|k_0|\to\infty]{} O\left(\frac{1}{|k_0|^{1+\epsilon}}\right).$$

It is then possible to use arguments from dispersion theory to guarantee that $\hat{T}_{\mu\nu}$ decreases faster than k^{-1} in all directions in k space,[4] in which case its contribution to M_d is finite. The divergent part of M_d is then determined by $t_{\mu\nu}$, and we easily find

$$M_d{}^{\text{divergent}} = \frac{-ie^2}{(2\pi)^4} \int \frac{dk}{k^4} \langle f | J_\mu{}^W | i\rangle \bar{u}_e \gamma^\mu (1-\gamma_5) v_\nu$$
$$= -(Z_2{}^e - 1)\langle f | J_\mu{}^W | i\rangle \bar{u}_e \gamma^\mu (1-\gamma_5) v_\nu .$$

Finally, when this is combined with Eq. (1), we find that to order α, the weak decay amplitude is

$$(Z_2{}^e)^{-1/2} \langle f | J_\mu{}^W | i\rangle \bar{u}_e \gamma^\mu (1-\gamma_5) v_\nu + (\text{finite parts}) . \tag{5}$$

Therefore, the condition for finiteness is that $\langle f | J_\mu{}^W | i\rangle / \sqrt{Z_2{}^e}$ be finite.

As we shall soon see, the finiteness of this last object has not as much to do with the commutation relations of $J_\mu{}^W$ with the electromagnetic current, as with properties of its divergence. Our method will be to prove theorems about matrix elements of $J_\mu{}^W$ in a standard quark model with the interactions chosen in a special way and then to generalize to a triplet model in which the desired $J_\mu{}^W$, $J_\mu{}^{\text{e.m.}}$ algebra can be made to hold. The starting point is the following Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\gamma\cdot\partial - m)\psi + \Delta\bar{\psi}\lambda^8\psi + gB^\mu\bar{\psi}\gamma_\mu\psi - \tfrac{1}{4}F_{\mu\nu}{}^B F_B{}^{\mu\nu} + \tfrac{1}{2}\mu^2 B_\mu B^\mu , \tag{6}$$

where $\psi = (\psi_{\mathcal{P}}, \psi_{\mathfrak{N}}, \psi_\lambda)$ is the quark field, B_μ is a massive neutral vector-meson field, and $F_{\mu\nu}{}^B$ is the corresponding field tensor. Notice that this vector field couples to a conserved vector current so that the theory is renormalizable, and that SU_3 symmetry breaking is introduced through the quark mass term only. In spite of its special features (whose utility will soon

[4] For a discussion of this point we refer the reader to Bjorken's Varenna lectures of 1967. The sort of arguments developed there justify the assertion in the text for reasonable, but unfortunately not absolutely necessary, assumptions about the high-energy behavior of the discontinuities of the amplitudes in question. In order to see what follows from current algebra in the most favorable case, we assume that Nature is kind and the arguments of this reference go through for the general β-decay process.

appear), this model seems broad enough to reproduce all the known features of strong interactions. Electromagnetism is introduced by minimal coupling, and both the weak and electromagnetic currents are constructed out of the $SU_3 \otimes SU_3$ generators,

$$\bar{\psi}(\tfrac{1}{2}\lambda_\mu)(1\pm\gamma_5)\psi.$$

The weak current is a linear combination of $V_\mu^{\pi^+}$, $V_\mu^{K^+}$, $A_\mu^{\pi^+}$, and $A_\mu^{K^+}$, so that we want to show that the matrix elements of each of these currents independently are rendered finite by dividing out a factor of $\sqrt{Z_2^e}$. Let us first consider the single-quark matrix elements of the vector current in a theory in which $e\neq 0$, but $\Delta=0$, so that SU_3 symmetry is broken only by electromagnetism. Since the theory is renormalizable, we know that the general off-mass-shell quark-quark vertex function is rendered finite by dividing out one cutoff-dependent, momentum-independent renormalization constant (we include in the definition of the vertex function appropriate square roots of quark wave-function renormalizations so that it reduces on the mass shell to the matrix element of the current between physical quark states). This renormalization constant may then be determined on the mass-shell, i.e., by studying $\langle \mathcal{P} | V_\mu^{\pi^+} | \mathfrak{N} \rangle$ and $\langle \mathcal{P} | V_\mu^{K^+} | \lambda \rangle$ (as usual, the quarks are denoted by $\mathcal{P}$, $\mathfrak{N}$, and λ). Since SU_3 symmetry is broken only by electromagnetism, and the photon is minimally coupled, we know that the vector currents satisfy the divergence equations[5]

$$\partial^\mu V_\mu^{\pi^+} = ieA^\mu V_\mu^{\pi^+}, \quad \partial^\mu V_\mu^{K^+} = ieA^\mu V_\mu^{K^+}, \qquad (7)$$

where A_μ is the photon field. Actually, this is true only if the electromagnetic mass differences come out finite. If they are infinite, we need in the basic Lagrangian a mass-difference counterterm which breaks the SU_2 symmetry as well, leading to the addition of a term of the form $\delta m\bar{\psi}\lambda\psi$ to the divergence equations. Whether renormalization is necessary or not, we take the mass differences between $\mathcal{P}$, $\mathfrak{N}$, and λ to be all of order α. We can then use methods first invented by Fubini and Furlan, and extended by Bjorken[3] in his work on pion β decay, to calculate the renormalization constant in question. Define Z_π and Z_K by

$$\langle \mathcal{P}(0) | V_0^{\pi^+} | \mathfrak{N}(0) \rangle = Z_\pi/\sqrt{2},$$
$$\langle \mathcal{P}(0) | V_0^{K^+} | \mathfrak{N}(0) \rangle = Z_K/\sqrt{2},$$

so that $Z \to 1$ as $e \to 0$. Define also

$$T_{\mu\nu}(q) = \int dx\, e^{iq\cdot x} \langle \mathfrak{N}(0) | T(V_\mu^{\pi^-}(x) V_\nu^{\pi^+}(0)) | \mathfrak{N}(0) \rangle.$$

Then

$$q^\mu q^\nu T_{\mu\nu} = \int dx\, e^{iq\cdot x} \times \langle \mathfrak{N} | T(\partial^\mu V_\mu^{\pi^-}(x) \partial^\nu V_\nu^{\pi^+}(0)) | \mathfrak{N} \rangle + i\frac{q\cdot p}{2m} + C,$$

where C is unknown, but certainly independent of q. The left-hand side is $O(q^2)$ as $q \to 0$ because $T_{\mu\nu}$ has no poles precisely at $q=0$. If we set $\bar{q}=0$, differentiate the whole equation with respect to q_0, and pass to the limit $q_0=0$, we have

$$0 = \frac{i}{m} + \frac{\partial}{\partial q_0} \int dx\, e^{iq\cdot x} \times \langle \mathfrak{N} | T(\partial^\mu V_\mu^{\pi^-}(x) \partial^\nu V_\nu^{\pi^+}(0)) | \mathfrak{N} \rangle |_{q_0=0}.$$

We evaluate the integral by expanding it in a Low equation. The contribution of intermediate states containing no photons is easily seen to be $\frac{1}{2}iZ_\pi^2$. If we are interested in a calculation correct to order α, the only other intermediate states to consider are those containing one photon. But then the contribution of the mass renormalization term $(\delta m\bar{\psi}\lambda\psi)$ in $\partial^\mu V_\mu$ is automatically of order α^2 (δm is of order α to start with; since there is a photon in the intermediate state, another α appears from creating and then absorbing it), and we may set $\partial^\mu V_\mu^{\pi^+} = +ieA^\mu V_\mu^{\pi^+}$. Then the argument in no way differs from that of Ref. 3 and we have the expressions, correct to order α,

$$Z_\pi = 1 - \frac{2e^2}{(2\pi)^4} \int \frac{dk}{k^4} k_0 \int dx\, e^{ik\cdot x} \times \langle \mathfrak{N}(0) | T(V_\mu^{\pi^-}(x) V^{\mu\pi^+}(0)) | \mathfrak{N}(0) \rangle, \qquad (8a)$$

$$Z_K = 1 - \frac{2e^2}{(2\pi)^4} \int \frac{dk}{k^4} k_0 \int dx\, e^{ik\cdot x} \times \langle \lambda(0) | T(V_\mu^{K^-}(x) V^{\mu K^+}(0)) | \lambda(0) \rangle. \qquad (8b)$$

The divergent part of these Z's is determined by the leading asymptotic behavior of the two-current matrix elements, which is itself determined by equal-time current commutators. The commutators in question, $\delta(x_0)[V_\mu^{\pi^+}(x), V^{\mu\pi^-}(0)]$, are the same in the standard quark model and in the integrally charged triplet model and allow us to conclude that

$$Z_\pi = 1 + \frac{ie^2}{2(2\pi)^4} \int \frac{dk}{k^4} + e^2(\text{finite}),$$

$$Z_K = 1 + \frac{ie^2}{4(2\pi)} \int \frac{dk}{k^4} + e^2(\text{finite}).$$

One then verifies without trouble that both $Z_\pi/\sqrt{Z_2^e}$ and $Z_K/\sqrt{Z_2^e}$ are finite, to order α. Therefore the general off-mass-shell single-quark vector vertex functions are rendered finite by dividing out a factor $\sqrt{Z_2^e}$, when explicit SU_3 breaking is absent.

Now we must see if this remains true when $\Delta\neq 0$. Once again, it sufficies to study the on-mass-shell single-quark current matrix elements $\langle \mathcal{P} | V_\mu^{\pi^+} | \mathfrak{N} \rangle$ and $\langle \mathcal{P} | V_\mu^{K^+} | \mathfrak{N} \rangle$. Turning on Δ amounts to including the interaction with an external source via the interaction

[5] S. L. Adler, Phys. Rev. 139, B1638 (1965).

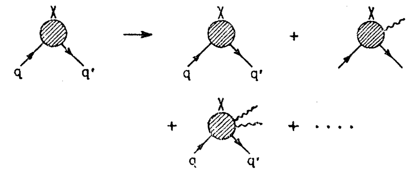

FIG. 2. Effect on the vertex of adding a mass term to the Lagrangian.

Lagrangian $\Delta\bar{\psi}\lambda^8\psi$. The new perturbation series for the current matrix elements is obtained from the old by inserting the source in all ways and to all orders. The new perturbation series is graphically represented in Fig. 2, where a wavy line stands for the source Δ. Clearly, only the diagram with no external source is primitively divergent. The possibly divergent insertions in any diagram correspond to mass and coupling-constant renormalizations, renormalization of Δ, and the vertex with no external sources itself. Imagine that the m, g, e, and Δ renormalizations have been performed. Then only the vertex remains. But if the whole series is multiplied by $(Z_2{}^e)^{-1/2}$, the vertex insertion in each diagram becomes finite. Then the whole series is finite term by term since the diagrams with external sources are primitively convergent. This disposes of the single-quark vector vertex functions in the theory with both $e\neq 0$ and explicit SU_3 symmetry breaking.

A rather similar trick allows us to draw the same conclusion for the axial current. Consider first $A_\mu{}^{\pi^+}=\bar{\psi}_{\mathcal{P}}\gamma_\mu\gamma_5\psi_{\mathfrak{N}}/\sqrt{2}$. Under the transformation $\psi_{\mathcal{P}}\rightarrow\psi_{\mathcal{P}}$, $\psi_{\mathfrak{N}}\rightarrow\gamma_5\psi_{\mathfrak{N}}$, $\psi_\lambda\rightarrow\psi_\lambda$ the current $V_\mu{}^{\pi^+}=\bar{\psi}_{\mathcal{P}}\gamma_\mu\psi_{\mathfrak{N}}/\sqrt{2}$ goes into $A_\mu{}^{\pi^+}$ and nothing happens to the Lagrangian except that the bare mass of the $\mathfrak{N}$ quark changes sign.[6] Therefore the renormalization constant of the axial current is the same as the renormalization constant of the vector current calculated from the Lagrangian of Eq. (6) modified by the addition of an extra mass term. The argument of the last paragraph, however, tells us that the addition of such an interaction changes renormalization constants only by a finite factor, so that dividing the single-quark axial vertex function by $(Z_2{}^e)^{-1/2}$ renders it finite as well. The same trick evidently works for $A_\mu{}^{K^+}$, and we finally conclude that any single-quark vertex function of the complete weak current, when divided by $\sqrt{Z_2{}^e}$, is finite.

To complete the proof, we must show that $\langle f|J_\mu{}^W|i\rangle/\sqrt{Z_2{}^e}$ is finite, where $|i\rangle$ and $|f\rangle$ are ordinary hadronic states. We exploit the fact that the mesons and baryons are bound states of multiple quark-antiquark systems so that the hadronic matrix elements of $J_\mu{}^W$ appear as residues of pole terms in off-mass-shell multiple quark-antiquark vertices of $J_\mu{}^W$. Then the general hadron vertex of $J_\mu{}^W$ is finite when divided by $\sqrt{Z_2{}^e}$ if the same is true of the general quark vertex.[7] But the general quark vertex is primitively convergent and contains as possible divergent insertions the standard mass and coupling-constant renormalizations and one insertion of the single quark vertex of $J_\mu{}^W$. This latter is rendered finite by dividing out $\sqrt{Z_2{}^e}$, and the others are made finite by performing the usual redefinitions of physical masses and coupling constants. Therefore $\langle f|J_\mu{}^W|i\rangle/\sqrt{Z_2{}^e}$ is finite for arbitrary hadron states $|i\rangle$ and $|f\rangle$, which was to be shown.

We must now see if these arguments go through in a model capable of realizing the commutators of $J_\mu{}^W$ and $J_\mu{}^{\text{e.m.}}$ used in the first part of the paper. Let us follow Ref. 1, in which the current is constructed out of three fundamental triplets of integrally charged particles, called S, U, and B. The $\mathfrak{N}$ and λ members of each triplet are one unit of charge lower than the $\mathcal{P}$ member, while the $\mathcal{P}$ and $\mathfrak{N}$ members are one unit of hypercharge higher than the λ member—i.e., the charge structure is the same as quarks, but shifted. Furthermore, the time-time and time-space commutators of the weak current are the same as in the quark model. Therefore, the arguments which led to Eq. (8) go through as before, provided that we are dealing with current vertices within the same triplet—*not* $\langle U|J_\mu{}^W|B\rangle$, for example. The matrix elements like $\langle U|J_\mu{}^W|B\rangle$ can be made to vanish by choosing the strong interaction to be a vector field coupled to a conserved current which causes no transitions between different triplets. The weak current conserves S, U, and B triplets separately, so that the whole theory must do likewise. Then the only fundamental current vertices which occur are between members of the same triplet and the arguments used earlier in the paper show that divergences are tamed by dividing out $\sqrt{Z_2{}^e}$. Finally, if strong interactions occur through intermediate vector particles and SU_3 breaking is introduced by mass differences only, the arguments of all of the paper after Eq. (8) go through without any trouble. Therefore, we have proven that with proper choice of the strong interaction, the model for the weak current which guarantees the finiteness to order α of pion β decay radiative corrections guarantees the same for arbitrary semileptonic processes.

I would like to thank Professor S. R. Coleman for a valuable conversation and suggestion.

[6] At this point we see the utility of a vector field. If we take a scalar field, coupling through $gB(\bar{\psi}_{\mathcal{P}}\psi_{\mathcal{P}}+\bar{\psi}_{\mathfrak{N}}\psi_{\mathfrak{N}}+\bar{\psi}_\lambda\psi_\lambda)$, the transformation $\psi_{\mathfrak{N}}\rightarrow\gamma_5\psi_{\mathfrak{N}}$ causes $\bar{\psi}_{\mathcal{P}}\psi_{\mathcal{P}}\rightarrow\bar{\psi}_{\mathcal{P}}\psi_{\mathcal{P}}$, $\bar{\psi}_{\mathfrak{N}}\psi_{\mathfrak{N}}\rightarrow -\bar{\psi}_{\mathfrak{N}}\psi_{\mathfrak{N}}$, $\bar{\psi}_\lambda\psi_\lambda\rightarrow\bar{\psi}_\lambda\psi_\lambda$, and $\delta\mathcal{L}=-2gB\bar{\psi}_{\mathfrak{N}}\psi_{\mathfrak{N}}$. This, however, is a far from trivial change in $\mathcal{L}$, and our theorem may not go through.

[7] Here we assume that no new infinities creep in when bound states appear. We of course cannot prove this.