If $B \to 0$,

$$M \to G^2\left(\frac{t-2m^2}{2m^2}\right)^N \frac{1}{m^2-s}. \quad (15)$$

4. DISCUSSION

It follows from the previous example that in infinite-component theories with nondegenerate mass spectra one might find discontinuities in the amplitude or in some of its derivatives whenever the four-momentum configuration of the external lines allows the four-momentum of an internal line to change from spacelike to timelike.

Although our result depends on the model we have used, and although the Born approximation (which is real in our case) is not the whole scattering amplitude, we believe that it is worth while to look at processes where the kinematics is the same as in our example, to see if cusps near $u=0$ are present. In particular, in meson-nucleon elastic backward scattering the kinematics is similar. Although preliminary evidence seems to show peaks near $u=0$ in pion-proton[11] and kaon-proton[12] elastic backward scattering, better data are needed.

Finally, it is worth remarking that the shifting of the effective position of the pole in the u channel with respect to the position of the pole in the s channel [see Sec. 3, (iii)] might be considered also in the exchange of bosons—in particular, in the vector-meson-dominance model for the electromagnetic form factors and in the one-boson-exchange baryon-baryon potentials.

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Determination of the Nucleon-Nucleon Scattering Matrix. VII. (p,p) Analysis from 0 to 400 MeV*

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All of the available (p,p) scattering data from 1 to 400 MeV have been analyzed, and a self-consistent set of 839 data has been chosen. Using this data selection, we investigated a number of different forms for the phase-shift energy dependence. The correct number of free parameters to use with each form was studied. The most suitable form, form A, gave the least-squares values $\chi^2=810$ and $\chi^2=858$ for 30- and 23-parameter solutions, respectively. A subset of 588 data in six narrow energy bands was used to obtain single-energy solutions. It is shown that this subset contains most of the physical content of the full set of 839 data. The value $g^2=14.72\pm0.83$ was obtained for the pion-nucleon coupling constant.

I. INTRODUCTION

IN previous papers in this series,[1–6] we have discussed phase-shift analyses of (p,p) and (n,p) data from 25 to 350 MeV. Subsequent to the publishing of these papers, a considerable amount of new data has become available,[7,8] both in the energy range we had previously considered and also at the higher energies. Thus it seemed to us worthwhile to update the previous analyses and to extend them to higher energies.

The (p,p) data in the elastic energy range up to about 400 MeV are now reasonably complete and accurate. Thus the isotopic spin $I=1$ scattering matrix can be reliably determined in this energy range. The aim of the present paper (paper VII) is to give the best possible values for the $I=1$ phase shifts from 0 to 400

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[8] The existing (p,p) and (n,p) experimental data from 0 to 400 MeV are illustrated in graphical form in Figs. 1 and 2 of M. H. MacGregor, Rev. Mod. Phys. **39**, 556 (1967).

MeV that we can obtain from the existing data. At energies above 400 MeV inelastic effects become important, and an accurate (p,p) phase-shift analysis is not yet possible. The existing data sets are not complete. However, a qualitative determination of the elastic scattering phase shifts can be made over the energy range 400–750 MeV. These results are presented in paper VIII. Finally, in paper IX we shall present the results of the (n,p) phase-shift analyses from 0 to 750 MeV. Since the (n,p) data are still woefully incomplete, this last analysis can be carried out only by first using (p,p) data to essentially fix the $I=1$ scattering matrix, and then using the corresponding (n,p) data to determine the $I=0$ scattering matrix. Charge independence provides the justification for setting the $I=1$ phases the same for (p,p) and for (n,p) scattering. The $I=0$ elastic scattering matrix thus determined is only qualitatively correct in the energy range from 0 to 400 MeV. Above 400 MeV the (n,p) data are so scarce that even a qualitative determination of the phase shifts is difficult to achieve.

In Sec. II, we discuss the (p,p) data selection from 1 to 400 MeV. We have examined essentially all of the available (p,p) data points and have selected the ones that form a self-consistent data set. Section III gives a discussion of the energy parametrization that we employ for the phase shifts. Section IV lists the phase-shift results. Section V gives the value for the pion-nucleon coupling constant g^2 that we obtain from this analysis. Section VI contains our conclusions.

II. DATA SELECTION

A. Compatibility of the Data

We attempt to answer in a reasonable and semi-mathematical way the question as to whether or not a specific experimental result is to be considered as "compatible" with other experiments. There are, of course, certain obvious criteria which are considered in such a judgment. If, for instance, two identical experiments are compared, then compatibility consists of the "overlapping" of the error bars. It is the nature of existing proton-proton scattering data, however, that such direct comparisons can in general seldom be made —the data tend to differ in type, energy or angle. Therefore we adopt as a general criterion for compatibility the ability of various experiments to be described by the same phase-shift representation (or in terms of a representation which is closely related to the phase shifts).

In making the compatibility study, the parameters are first adjusted to give a minimum least-squares value (χ_D^2) to some relatively complete and self-consistent subset D of the data. We now add an experiment E, which was not included in D, and readjust the parameters to obtain a new minimum (χ_{D+E}^2). We define a figure of compatibility f_c as

$$f_c=(\chi_{D+E}^2-\chi_D^2)/N_E, \tag{1}$$

where N_E is the number of experimental points in E. Thus f_c includes not only the χ^2 increase due to the points E, but also the χ^2 increase due to any readjustments forced on the data set D. For a compatible data set E, f_c must be of order unity.

It should be recognized that f_c is a measure of compatibility between D and E *only within the constraints of our model.* The comparison can only be as reasonable as the model. The model we use in the present paper is described in detail in Sec. III. It is chosen to produce a "smooth" energy dependence of the phase shifts, and to satisfy very general physical constraints such as correct threshold behavior and proper singularity structure. The model is sufficiently flexible, we believe, that "form-limiting" effects are unimportant in fitting to the existing nucleon-nucleon data.

The testing of large numbers of data for compatibility is speeded up tremendously if, instead of using the data set D directly, we replace it by a "reduced" matrix representation in the space of the parameters of the model. This requires that the data set D be large enough and complete enough to accurately fix the parameters (p) of the model. (The reduced matrix representation refers to the intrinsic minimization of the functional χ^2 with respect to the experimental normalization parameters.[4,9])

The detailed procedure for obtaining f_c is as follows. If p_0 is the phase-shift parameter column vector (in a vector notation) that gives the minimum value χ_D^2 for the data set D, then in the neighborhood of this vector, $\chi^2(p)$ is approximately

$$\chi^2(p)\equiv\chi^2(p_0+\Delta p)\simeq\chi_D^2+(\Delta p)^T\alpha(\Delta p), \tag{2}$$

where $(\ )^T\equiv$transposed (row) vector, and we use the matrix

$$\alpha_{jk}=\frac{1}{2}\left(\frac{\partial^2\chi^2}{\partial p_j\partial p_k}\right)\Bigg|_{p_0(\text{reduced})} \tag{3}$$

We can now add the χ^2 sum coming from experiment E to obtain

$$\chi^2(p_0+\Delta p)\approx\chi_D^2+(\Delta p)^T\alpha(\Delta p)+\sum_{i=1}^{N}\left(\frac{X\theta_i(p_0+\Delta p)-\theta_i^{\text{expt}}}{E_i}\right)^2+\left(\frac{X-1}{\Delta X}\right)^2, \tag{4}$$

where θ_i^{expt} is the experimentally measured quantity, N is the number of data in E, X is the normalization parameter for E, E_i is the error on ith data point, ΔX is the normalization error, and $\theta_i(p)$ is the value of the observable predicted by p. Δp is now adjusted to minimize χ^2. This gives χ_{D+E}^2, from which f_c can be derived.

In Eq. (4), some people associate the normalization constant X with the quantities θ_i^{expt} and E_i, rather

[9] R. A. Arndt and M. H. MacGregor, in *Methods of Computational Physics* (Academic Press Inc., New York, 1966), Vol. 6, p. 253.

than with $\theta_i(p)$ as we have done. However, computationally the form we use (which is also used by the Yale group) is much simpler. For values of X near unity, the two methods are essentially identical.

B. Final Data Selection

In making a data selection, we started with the set of data from paper IV as a reference set, and with a Q-function expansion (see Sec. III) for the energy dependence. Using the data-compatibility procedure outlined above, we examined essentially all of the available (p,p) data from 1 to 400 MeV. The sets of data having an M value (χ^2 average per datum) of one or less were combined to form a new reference set, and all of the data were reexamined. Second and third data selections were made that included sets with $M<1.5$ and then $M<1.6$. Using the $M<1.6$ reference set, we made one final compatibility study of the data to obtain the final data selection. In this process any individual data points that deviated from the predicted values (based on our reference set) by three standard deviations or more were discarded. In the final data selection, sets of data having M values greater than 2 were discarded.

Table I summarizes the results of the data compatibility studies. All of the (p,p) data that we examined are described in Tables I and II in the accompanying references. The final data selection includes all *independent data points* that are within three standard deviations of the theoretical values plus all *data sets* (two or more points taken from a common experiment) that have M values less than 2. Some experiments below 5 MeV were arbitrarily discarded, even though they met our M-value test, because our vacuum-polarization corrections are only approximate. Out of a total of 1084 (p,p) data considered, 839 were selected for the final set of data.

Using the final data selection, we obtained 20-parameter, 23-parameter, and 30-parameter phase-shift solutions, as described in Sec. III. The M values shown in Table I are based on the 30-parameter solution. The M values for the excluded data were obtained by comparing these data to the 30-parameter solution, with the normalization constants of the excluded data allowed to vary, but with no searching on the phase-shift parameters. The least-squares value χ^2 for the 30-parameter solution was 810. Although normalization contributions were included in the χ^2 sum [see Eq. (4)], the normalization errors were not counted as data.

Out of the 839 data in the complete selection, we selected 588 points in six energy bands for "single-energy" analyses and for constructing phase-shift matrices to use as a representation of the data. As we shall show in Sec. IV C, these 588 data contain essentially the full physical content of the information embodied in the larger 839-datum set. The subset of data contained in the 588-datum set is indicated in Table I by asterisks in front of the energies. The data contained in the full 839-datum set are all the data sets listed in Table I that do not have parentheses around the M values.

After this work was completed, we received some new (p,p) polarization data from Slobodrian and coworkers.[10] After a number of phase-shift analyses of the data at 19.7 MeV, we concluded[11] that they are not consistent with nearby R and A values at 27.6 MeV (see Table I). They are also not consistent with the type-I solution of Stapp[12] and of MacGregor[13] that is required to fit our 839-datum selection. For this reason, we have not incorporated them into the present paper. It is obviously of considerable interest to have this experimental puzzle resolved. *Note added in proof.* P. Catillon, J. Sura, and A. Tarrats [Phys. Rev. Letters **20**, 602 (1968)] have measured the (p,p) polarization at 20 MeV and obtained results that agree with the phase-shift analyses and that disagree with the Slobodrian measurements.

III. ENERGY PARAMETRIZATION

In our previous energy-dependent analysis,[4] we chose the energy-dependent phase-shift forms to be the one-pion-exchange (OPE) phase shift plus a sum of terms representing two-pion, three-pion, and so on, exchanges. As a basis set of functions for the representation of the higher-order meson exchanges, we chose Legendre functions of the second kind, $Q_l(x)$. These functions, which arise naturally from the partial-wave projection of pole terms ("driving forces") in crossed channels, have proper threshold behavior and can be made to have a singularity structure corresponding to the location of the cuts for two-pion exchange, three-pion exchange, etc. Of course these functions do not have the proper discontinuities across the cuts, and thus they are, strictly speaking, chosen phenomenologically. However, the facts that the functions thus chosen give good fits to the data with a small number of terms and that they give a good extrapolation to higher energies indicate that they are a reasonable basis set.

The phase shifts are written in the general form

$$\delta_l^{(S,J)}(T)=\delta_{l0}^{(S,J)}(T)+\sum_{i=1}^{N}\alpha_i^{(S,J)}F_{li}(T). \qquad (5)$$

Here l is the orbital angular momentum, J and S are the total momentum and total spin, and T is the laboratory kinetic energy. For the $^1S_0$ phase, δ_{00} is taken from effective-range theory[14]:

$$C^2K\cot\delta_{00}+2K\eta h(\eta)=-(1/a)+\tfrac{1}{2}r_0K^2, \qquad (6)$$

[10] R. J. Slobodrian, J. S. C. McKee, H. Bischel, and W. F. Tivol, Phys. Rev. Letters **19**, 704 (1967).
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[13] M. H. MacGregor, Phys. Rev. **113**, 1559 (1959).
[14] H. P. Noyes, Phys. Rev. Letters **12**, 171 (1964); M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Ann. Rev. Nucl. Sci. **10**, 291 (1960); **10**, 325 (1960).

TABLE I. (p,p) data from 1 to 400 MeV.

Energy (MeV)	No., type data[b]	Angular range (c.m.)	Data std. err.	Norm. std. err.	Deleted angles[b]	M value[c]	Predicted norm.[d]	Comment	Reference
1.397	11 σ	12°– 70°	~0.2%			(5.9)		e, f	Wisconsin (1966)
1.855	13 σ	12°– 90°	~0.2%			(3.1)		e, f	Wisconsin (1966)
2.425	14 σ	12°–100°	~0.2%			(0.7)		e, f	Wisconsin (1966)
3.037	13 σ	12°– 90°	~0.2%			(0.6)		e, f	Wisconsin (1966)
9.68	1 σ	90°	1.2%			1.2		g	Minnesota (1959A)
9.69	26 σ	10°– 90°	1%	0.73%		0.7	1.015	g	Minnesota (1959B)
9.73	9 σ	27°–112°	3%	Unknown		(···)		e, h	Berkeley (1954)
11.4	1 C_{NN}	90°	2%			0.6		i, j	Saclay (1966)
11.4	1 A_{xx}	90°	1%			0.0		i, j	Saclay (1966)
14.16	17 σ	18°–114°	~3%	10%		0.1	0.991		Tokyo (1960)
16.2	1 P	50°	~100%			0.7		g	Princeton (1959)
18.2	8 σ	30°– 90°	1%	1.5%		0.7	1.001	g	Princeton (1954)
19.2	1 C_{NN}	90°	2%			0.3		i, j	Saclay (1966)
19.2	1 A_{xx}	90°	2%			0.1		i, j	Saclay (1966)
19.8	15 σ	14°– 90°	3%	2.5%		(1.1)	0.999	e, h	Los Angeles (1955A)
19.8	7 σ	18°– 36°	3%	2.5%		(1.8)	1.005	e, h	Los Angeles (1955A)
20	1 C_{NN}	90°	6%			(2.2)		e, h	Saclay (1962)
21.95	1 σ	90°				(>10)		e, k	Rutherford (1964)
[a] 23.5	1 C_{NN}	90°	3%			0.0		i, j	Saclay (1966)
[a] 23.5	1 A_{xx}	90°	3%			0.5		i, j	Saclay (1966)
25.62	1 σ	90°	0.5%			(5.7)		e, k	Rutherford (1964)
[a] 25.63	23 σ	10°– 90°	0.8%	0.93%		0.6	1.009	g	Minnesota (1960A)
[a] 26.5	1 C_{NN}	90°	2%			0.1		i, j	Saclay (1966)
[a] 26.5	1 A_{xx}	90°	2%			2.9		i, j	Saclay (1966)
[a] 27	1 C_{NN}	90°	10%			0.1			Los Alamos (1967
27.4	1 P	45°	~150%			(···)		e, l	Harwell (1963A)
[a] 27.6	3 A	23°– 55°	>40%	3%		1.4	0.997		Rutherford (1965B)
[a] 27.6	2 R	23°, 55°	15%	3%	39°	0.5	0.991		Rutherford (1965B)
[a] 28.16	1 σ	90°	2%			1.4		g	Minnesota (1959A)
30	1 P	45°	Large			5.2			Rutherford (1963A)
30.33	1 σ	90°	0.6%			(7.0)		e, k	Rutherford (1964)
31.15	1 σ	90°	<2%			0.2		g	Minnesota (1959A)
34.2	1 σ	90°	<2%			0.3		g	Minnesota (1959A)
34.27	1 σ	90°	0.6%			(26.1		e, k	Rutherford (1964)
36.8	1 P	60°	30%			(···)		e, l	Harwell (1963A)
36.9	1 σ	90°	<2%			0.0		g	Minnesota (1959A)
38.3	1 P	70°	50%			(···)		e, l	Harwell (1963A)
39.4	27 σ	8°– 90°	<1%	0.93%		0.9	0.988	g	Minnesota (1958)
39.6	1 σ	90°	<2%			0.1		g	Minnesota (1959A)
40.75	1 σ	90°	0.6%			(14.3)		e, k	Rutherford (1964)
41	1 σ	90°	7%			0.8			Harvard (1956)
44.66	1 σ	90°	2%			1.3		g	Minnesota (1959A)
45.04	1 σ	90°	0.6%			(36.0)		e, k	Rutherford (1964)
46	1 σ	45°	5%			(3.1)		e, m	Harvard (1958)
[a] 46	1 P	45°	~100%			1.6		n	Harvard (1958)
[a] 47.5	5 A	23°– 87°	~50%	5%		1.6	0.982		Rutherford (1965A)
[a] 47.8	5 A	23°– 87°	~50%	5%		0.5	1.019		Rutherford (1965B)
[a] 47.8	5 R	23°– 87°	~10%	5%		1.1	1.004		Rutherford (1965B)
[a] 49.4	28 σ	13°– 90°	0.5%	0.4%	14°, 17°	1.1	0.999		Rutherford (1967)
49.7	1 P	45°	16%			(10.7)		e, b, n	Harwell (1963A)
[a] 49.9	1 P	45°	6%			0.0			Rutherford (1963A)
[a] 50	1 D	70°	30%			1.3		o	Rutherford (1963B)
50.02	1 σ	90°	0.6%			(11.2)		e, k	Rutherford (1964)
[a] 50.17	1 σ	90°	2%			0.3		g	Minnesota (1959A)
[a] 51.5	1 σ	90°	7%			1.0			Harvard (1956)
[a] 51.5	9 σ	16°– 35°	4%	4.5%	12°	0.9	0.941		Tokyo (1961)
[a] 51.7	1 P	60°	25%			0.6		n	Harwell (1963A)

TABLE I. (*Continued*).

Energy (MeV)	No., type data[b]	Angular range (c.m.)	Data std. err.	Norm. std. err.	Deleted angles[b]	M value[c]	Predicted norm.[d]	Comment	Reference	
[a] 51.8	9 σ	35°– 90°	2%	2.5%		1.7	0.950		Tokyo	(1961)
[a] 52	1 C_{NN}	90°	Large			2.6			Tokyo	(1963)
[a] 52	1 C_{KP}	90°	~100%			2.9			Tokyo	(1963)
[a] 52.3	25 σ	14°– 90°	0.5%	0.5%	16°, 18°, 20°, 26°	1.1	1.003	p	Tokyo	(1967)
[a] 53.2	1 P	75°	100%			1.8		n	Harwell	(1963A)
56	2 σ	45°	5%			(1.1)		e, m	Harvard	(1958)
[a] 56	1 P	45°	14%			0.1		n	Harvard	(1958)
[a] 56.15	1 σ	90°	<2%			0.0		g	Minnesota	(1959A)
[a] 58.5	1 P	45°	25%			0.7		n	Harwell	(1963A)
61.92	1 σ	90°	<2%			0.1		g	Minnesota	(1959A)
66	10 σ	25°– 71°	<2%	Float	20°	0.1	1.043	m	Harvard	(1958)
66	11 P	20°– 71°	15%	2.8%		1.0	1.031	n	Harvard	(1958)
68.3	26 σ	10°– 90°	<1%	1.1%		1.0	1.015	g	Minnesota	(1960B)
68.42	1 σ	90°	<2%			0.4		g	Minnesota	(1959A)
69.5	1 σ	90°	6%			0.0			Harvard	(1956)
70	6 σ	25°– 90°	Unknown	Float		(···)		e, q	Harvard	(1956)
70	1 P	45°	10%			2.4		n	Harwell	(1963A)
71	1 σ	45°	6%			3.0		e, m	Harvard	(1958)
71	1 P	45°	11%			0.5		n	Harvard	(1958)
73.5	1 C_{NN}	90°	25%			0.1		j	Harwell	(1965B)
78	1 σ	45°	6%			(0.8)		e, m	Harvard	(1958)
78	1 P	45°	8%			0.7		n	Harvard	(1958)
78.5	1 σ	90°	7%			0.0			Harvard	(1956)
86	1 σ	45°	5%			(0.4)		e, m	Harvard	(1958)
86	1 P	45°	7%			0.0		n	Harvard	(1958)
[a] 93.2	9 P	20°– 80°	5%	0.85%	90°	1.3	1.001	r	Harwell	(1967)
[a] 95	6 σ	40°– 90°	4%	Float		0.1	0.997		Harvard	(1956)
[a] 95	6 σ	25°– 90°	3%	Float		0.5	1.018	s	Harvard	(1956)
[a] 95	1 σ	90°	6%			0.0			Harvard	(1956)
95	4 σ	40°– 90°	4%	Float		(···)		e, t	Harvard	(1956)
[a] 95	13 σ	25°– 86°	<2%	Float	20°	0.2	0.979	m	Harvard	(1958)
[a] 95	14 P	20°– 86°	7%	2.8%		1.2	0.994	n	Harvard	(1958)
[a] 97	1 P	45°	5%			3.2		n	Harwell	(1963A)
98	10 σ	10°– 41°	3%	Float		(65.4)	0.927	e, c	Harwell	(1960A)
98	9 σ	20°– 81°	2.5%	Float		(4.5)	0.918	e, c	Harwell	(1960A)
[a] 98	14 P	10°– 81°	20%	2%		1.0	0.979	n, u	Harwell	(1960A)
98	1 C_{NN}	90°	8%			0.3		j	Harwell	(1965B)
[a] 98	5 R	31°– 72°	30%			1.3			Harwell	(1965A)
[a] 98	4 R'	31°– 62°	40%			0.2			Harwell	(1965A)
[a] 98	5 D	20°– 61°	100%			1.2			Harvard	(1960B)
[a]102	3 σ	30°– 66°	2%	Float		2.1	0.991	m	Harvard	(1958)
[a]102	3 P	30°– 66°	5%	2.8%		0.9	1.021	n	Harvard	(1958)
107	3 σ	31°– 67°	2%	Float		0.1	0.934	m	Harvard	(1958)
107	3 P	31°– 67°	5%	2.8%		(3.3)	0.980	e, c, n	Harvard	(1958)
118	15 σ	20°– 88°	<2%	Float	86°	1.2	0.953	m	Harvard	(1958)
118	15 P	20°– 87°	5%	2.8%		(2.6)	0.949	e, c, n	Harvard	(1958)
127	3 σ	31°– 67°	2%	Float		0.0	0.955	m	Harvard	(1958)
127	3 P	31°– 67°	4%	2.8%		(3.0)	0.976	e, c, n	Harvard	(1958)
130	4 P	20°– 82°	15%	3.3%		0.7	0.981	v	Rochester	(1957)
[a]137	3 σ	31°– 67°	2%	Float		0.1	0.978	m	Harvard	(1958)
[a]137	3 P	31°– 67°	3%	2.8%		1.8	0.967	n	Harvard	(1958)
[a]137.5	5 R'	43°– 82°	20%			0.1			Harvard	(1963B)
[a]138	4 D	31°– 82°	50%			1.2			Orsay	(1963)
138	21 P	20°– 88°	3%	2.3%		(···)		c, w	Orsay	(1963)
[a]139	6 A	31°– 82°	10%	4%		0.5	0.973		Harvard	(1963A)
[a]140	20 P	16°–107°	<2%	0.85%		1.0	1.005		Harwell	(1966)

TABLE I. (*Continued*).

Energy (MeV)	No., type data[b]	Angular range (c.m.)	Data std. err.	Norm. std. err.	Deleted angles[b]	M value[c]	Predicted norm.[d]	Comment	Reference	
[a]140	6 R	31°– 82°	20%			1.1			Harwell	(1960C)
[a]140.4	6 R'	31°– 82°	15%			1.0			Harwell	(1964)
142	7 σ	5°– 10°	5%	Float		(6.8)	1.070	e, c	Harwell	(1960A)
142	7 σ	10°– 31°	5%	Float		(3.7)	1.020	e, c	Harwell	(1960A)
142	9 σ	10°– 41°	2%	Float		(9.3)	0.953	e, c	Harwell	(1960A)
142	9 σ	20°– 90°	2%	Float		(11.1)	1.026	e, c	Harwell	(1960A)
[a]142	27 P	5°– 82°	5%	2%	6°, 78°	1.4	1.044	n, u	Harwell	(1960A)
[a]142	8 R	24°– 90°	20%			1.4			Harwell	(1960B)
[a]142	8 D	12°– 82°	20%			1.5			Harvard	(1960A)
[a]143	7 D	31°– 92°	50%			0.4			Harwell	(1961)
[a]143	6 A	32°– 85°	15%			1.0			Harwell	(1963B)
[a]143	2 C_{NN}	60°, 90°	8%			0.1		j	Harwell	(1965B)
[a]144.1	6 σ	16°– 36°	0.7%	0.9%		0.7	1.000		Harwell	(1966)
[a]144.1	15 σ	41°–112°	0.5%	0.6%		1.1	0.998		Harwell	(1966)
147	8 σ	12°– 31°	3%	Float	4°, 6°, 8°, 10°	(0.6)	1.094	e, h, m, x	Harvard	(1958)
147	19 σ	20°– 88°	2%	Float		(1.2)	1.064	e, h, m, u, x	Harvard	(1958)
[a]147	28 P	6°– 88°	4%	2.8%		1.0	1.005	n, u	Harvard	(1958)
[a]155	23 σ	10°– 90°	2%	4%	8.4°	1.3	0.971		Orsay	(1961)
170	7 P	31°– 82°	5%	3.3%		0.5	0.973	u	Rochester	(1957)
170	6 σ	10°– 62°	3%	Float		(6.2)	0.933	e, t	Berkeley	(1956A)
174	7 σ	9°– 62°	3%	Float		(5.8)	1.043	e, t	Berkeley	(1956A)
174	5 P	20°– 72°	15%	6.6%		0.9	0.930		Berkeley	(1955)
210	9 P	13°– 83°	4%	3.3%		(1.8)	0.998	e, h, v	Rochester	(1957)
210	9 P	21°–113°	10%	3.5%		(···)		e, h, t	Rochester	(1957)
210	4 P	31°– 71°	15%	3.3%		(···)		e, h, t	Rochester	(1957)
[a]210	6 P	30°– 80°	3%	3.6%		0.4	0.976	y	Rochester	(1967)
[a]210	7 σ	30°– 90°	1.5%	Float		0.7	0.979		Rochester	(1967)
[a]213	13 σ	9°– 39°	2%	1.3%		1.3	1.002		Rochester	(1967)
[a]213	13 P	9°– 39°	5%	3.1%		0.8	0.999	y	Rochester	(1967)
[a]213	7 R	30°– 90°	20%			0.6		z, aa	Rochester	(1967)
[a]213	5 R'	30°– 90°	20%		60°, 70°	1.2		bb	Rochester	(1967)
[a]213	7 D	30°– 90°	10%			0.6			Rochester	(1967)
[a]213	5 E	30°– 70°	10%		80°, 90°	0.7		z	Rochester	(1967)
[a]213	2 A	80°, 90°	100%			0.9		aa, cc	Rochester	(1967)
217	6 P	60°– 80°	6%	2.2%		(3.3)	0.946	e, c, h	Rochester	(1967)
259	6 σ	10°– 64°	3%	8%		(···)		e, t	Berkeley	(1956A)
260	6 σ	9°– 63°	3%	8%		(···)		e, t	Berkeley	(1956A)
276	6 P	19°– 77°	10%	7.5%		1.5	0.882	dd	Berkeley	(1957)
305	14 C_{NN}	59°–104°	15%	9.6%		1.6	0.764		Chicago	(1967)
[a]310	7 σ	6°– 22°	7%	Float		1.8	1.192		Berkeley	(1958)
[a]310	7 P	6°– 22°	20%	4%		0.6	0.989	dd, ee	Berkeley	(1958)
[a]310	6 P	33°– 79°	3%	Float	83.7°	1.3	0.964		Berkeley	(1967)
[a]310	6 R	22°– 80°	30%			1.8			Berkeley	(1957)
[a]310	6 D	23°– 80°	15%			0.7			Berkeley	(1957)
[a]315	7 σ	21°– 90°	2%	Float		1.1	1.060		Berkeley	(1957)
[a]315	6 P	21°– 76°	7%	4%		1.1	0.930	dd	Berkeley	(1957)
[a]315	1 C_{NN}	90°	20%			0.5			Dubna	(1964)
[a]315	1 C_{NN}	45°	60%			0.0			Dubna	(1965)
[a]315	1 C_{KP}	45°	70%			0.0			Dubna	(1965)
[a]316	3 A	25°– 76°	20%			0.2			Berkeley	(1956B)
[a]320	1 C_{NN}	90°	15%			1.3			Liverpool	(1961)
[a]328	13 P	49°– 89°	10%	6.2%	85°	0.6	0.954		Berkeley	(1966)
[a]330	17 σ	5°– 30°	10%	Float	4°, 5.8°, 11.1°	1.0	1.066		Berkeley	(1958)
[a]330	13 C_{NN}	59°–100°	20%	10.9%	57°	0.6	0.833		Chicago	(1967)
[a]345	10 σ	15°– 53°	5%	Float	11°, 11°	1.2	1.008		Berkeley	(1951)

TABLE I. (*Continued*).

Energy MeV	No., type data[b]	Angular range (c.m.)	Data std. err.	Norm. std. err.	Deleted angles[b]	M value[c]	Predicted norm.[d]	Comment	Reference	
[a]345	17 σ	35°– 89°	3%	5%	44°, 88.6°	1.7	1.117		Berkeley	(1951)
358	14 C_{NN}	58°–102°	20%	9%		1.1	0.896		Chicago	(1967)
380	26 σ	4°– 90°	2%	1.6%		(41.6)	1.185	e, ff	Liverpool	(1958)
380	1 C_{NN}	45°	50%			(31.8)		e, b	Liverpool	(1966)
380	1 C_{KP}	45°	80%			(29.8)		e, b	Liverpool	(1966)
382	1 C_{NN}	90°	20%			0.6			Liverpool	(1961)
382	1 C_{KP}	90°	20%			3.2			Liverpool	(1961)
386	14 C_{NN}	58°–101°	15%	8.6%		1.6	0.889		Chicago	(1967)
400	2 C_{NN}	60°, 90°	60%, 15%			1.7			Princeton	(1963)
400	2 C_{KP}	60°, 90°	60%, 30%			0.6			Princeton	(1963)
400	7 P	33°– 83°	3%	3%		0.8	1.045		Berkeley	(1967)
400	7 P	33°– 83°	6%	3%		1.6	1.010		Berkeley	(1967)

[a] These energies show data used for single-energy calculations and for the matrix representations of the data.
[b] Data points more than three standard deviations from the theoretical values were deleted. The number of data points shown does not include deleted points nor any experimentally determined normalization constants.
[c] The M value is χ^2/No. of data. Data sets with $M>2$ were deleted. The quoted M value is taken from the 30-parameter solution. Deleted data sets have parentheses around the M values.
[d] This is the over-all *theoretical* normalization value arrived at in the final search problem. The *reciprocal* of this number gives the amount by which the experimental data should be changed to be consistent with the phase-shift solution.
[e] This data set was deleted.
[f] Since our vacuum polarization corrections were only approximate, these data were not included.
[g] Probable errors were changed to standard deviations.
[h] Older and possibly redundant data, usually with large statistical errors.
[i] Absolute normalization as given by H. P. Noyes and H. M. Lipinski, Phys. Rev. **162**, 884 (1967).
[j] The Saclay A_{xx} and A_{yy} data at 11.4, 19.2, 23.5, and 26.5 MeV should all refer to a common normalization factor. This was not possible with the existing code. Treating each energy separately gives essentially the same result. The same remark holds for the Harwell C_{NN} data at 73.5, 98, and 143 MeV.
[k] C. Batty (private communication) suggested considering these data as being one set. Since $M>2$ for the set, we have deleted all of the points.
[l] Excessive energy spread.
[m] Early Harvard cross-section normalizations were withdrawn.
[n] Renormalized as suggested by O. N. Jarvis and B. Rose [Harwell (1965C)].
[o] The correct value is $D=-0.241$.
[p] J. Sanada (private communication) assigned preliminary absolute errors of 0.7% to these data. We have arbitrarily set the normalization error as 0.5% and the statistical error as 0.5% (0.6% for the smallest angle).
[q] Inadequate error information.
[r] Final data by M. R. Wigan and P. Martin (private communication) have been taken at 97.7 MeV (13 polarization points) that replace the data described here. Our compatibility code gave a value $M=0.85$ for the new data set, and a predicted normalization of 0.996. The changes produced in the phases at 97.7 MeV by the addition of these data are less than 2% of the experimental errors in all cases, and hence are completely negligible. Thus the use of the preliminary data at 93.2 MeV, as we have done in the present paper, is essentially equivalent to using the final data at 97.7 MeV.
[s] Errors as recommended by R. Wilson (private communication).
[t] Inconsistent data.
[u] Independent measurements of data at the same angles have been averaged.
[v] These data should probably be renormalized, but are included here as originally published. The effects of renormalization are slight for imprecise data.
[w] Data not symmetric about 90°.
[x] Large data renormalization.
[y] Renormalized as recommended by Thorndike [Rochester (1967)].
[z] Errors changed as recommended by Thorndike [Rochester (1967)]. E data are A, R combination, corresponding to F data (R', R combination).
[aa] New data points, as given by Thorndike [Rochester (1967)].
[bb] In principle, F (Thorndike's notation) rather than R' should be analyzed. However, the inaccuracies in the data do not warrant this refinement in the analysis.
[cc] These points replace the E data at 80° and 90°.
[dd] Beam polarization error removed from individual errors.
[ee] Normalization uncertainty from O. Chamberlain *et al.* [Berkeley (1957)].
[ff] A phase-shift study showed that these data are inconsistent in shape with nearby cross sections at higher and lower energies. (But see paper VIII.)

where $a=-7.815$ F, $r_0=2.795$ F, and K is the c.m. momentum of a nucleon. For $l\neq 0$, we set[15]

$$\delta_{l0}{}^{(S,J)}=\text{OPE}. \tag{7}$$

The Q-function forms are written as

$$F_{li}(T)=\tfrac{1}{2}[(\beta_i\mu)^2/M](T^2+2MT)^{-1/2} \times Q_l[1+(\beta_i\mu)^2/MT], \tag{8}$$

where $\beta_i=2, 3, 4, \cdots$ for 2π, 3π, 4π, $\cdots$ exchange, and $(\mu,M)=$ (pion, nucleon) mass. For the analysis used in paper IV, we chose $\beta_1=2$, $\beta_2=3$, $\beta_3=5$, and $\beta_4=9$.

In the Q-function expansion just described, the free parameters are the coefficients α_i in Eq. (5). However, the β_i are in a sense hidden parameters. It is possible to write a form that avoids the β parametrization. If we start with a generalized Yukawa function

$$f(t)=\int_{4\mu^2}^{\infty}\frac{dt'}{(t'-t)}\rho(t'), \tag{9}$$

where t is the s-channel momentum transfer variable [$t=-MT(1-\cos\theta_s)$, with θ_s the s-channel c.m. scattering angle], then a partial-wave projection gives

$$f_l(t)=\frac{1}{MT}\int_{4\mu^2}^{\infty}Q_l\left(1+\frac{t'}{MT}\right)\rho(t')t'dt'. \tag{10}$$

Putting $x=1-4\mu^2/t$, $x_0=1+4\mu^2/MT$, we have

$$f_l(x_0)=(x_0-1)\int_0^1 Q_l\left(\frac{x_0-x}{1-x}\right)\frac{\rho(x)}{(1-x)^3}dx. \tag{11}$$

To obtain s-wave ($l=0$) convergence, we require

$$\rho(x)\underset{x\to 1}{\longrightarrow}(1-x)^2. \tag{12}$$

Thus we can take

$$\rho'(x)=\rho(x)/(1-x^2). \tag{13}$$

Then Eq. (11) becomes

$$f_l(x_0)=(x_0-1)\int_0^1 Q_l\left(\frac{x_0-x}{1-x}\right)\frac{\rho'(x)}{(1-x)}dx. \tag{14}$$

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TABLE II. Data references for Table I.

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This gives

$$\mathrm{Im}f_l(x_0)=\int_0^{\frac{1}{2}(x_0+1)} P_l\left(\frac{x_0-x}{1-x}\right)\frac{\rho'(x)}{(1-x)}dx,$$

$$-1<x_0<1. \quad (15)$$

As the left-hand discontinuity threshold ($x_0=-1$) is approached, we require[16]

$$\mathrm{Im}f_l(x_0)\to(x_0+1)^{3/2}, \quad (16)$$

TABLE III. χ^2 values for several energy-dependent forms. A matrix representation of 515 data at six energies was used in the fitting. The forms are explained in the text. Results obtained with redundant parameters (upper part of table) or with excessive form limiting (lower part) are set in italics.

Energy form No. of free parameters	*A*	*Q*1	*Q*2	*Q*3	*Q*4
		Pion-mass values			
		2,3,5,9	1,2,3,5	2,4,7,13	5,10,15,20
38	*510*	*510*	*510*	*518*	*546*
37	*511*	*511*	*510*	*518*	*546*
36	*512*	*511*	*510*	*519*	*546*
35	*512*	*512*	*511*	*519*	547
34	*512*	*512*	*511*	*520*	550
33	*513*	*512*	512	520	550
32	*514*	*512*	515	522	551
31	*516*	*513*	518	523	552
30	*517*	513	524	527	554
29	517	516	535	530	563
28	519	518	552	532	565
27	523	521	562	550	568
26	525	523	592	551	579
25	530	534	592	571	598
24	538	550	*620*	590	*644*
23	544	566	*638*	*636*	*684*
22	549	569	*653*	*642*	*692*
21	559	589	*698*	*656*	*720*
20	565	*697*	*775*	*763*	*722*
19	*637*	*745*	*854*	*768*	*783*
18	*673*	*843*	*976*	*851*	*842*
17	*722*	*894*	*1205*	*891*	*861*
16	*838*	*1016*	*1320*	*941*	*939*
15	*945*	*1016*	*1636*	*996*	*1161*
14		*1256*	*2778*	*1172*	*1573*
13		*1417*			*2167*

[16] R. A. Arndt, R. A. Bryan, and M. H. MacGregor, Phys. Rev. **152**, 1490 (1966), Appendix I.

so that we must have $\rho'(x)\propto x^{1/2}$. Thus we have finally

$$F_{li}(x_0)=(x_0-1)\int_0^1 Q_l\left(\frac{x_0-x}{1-x}\right)\frac{x^{i-1/2}}{(1-x)}dx. \quad (17)$$

As i increases, the factor $x^{i-1/2}$ weights the integral towards the values $x\sim1$, which corresponds to high-energy t in the crossed channel. Thus increasing i in Eq. (17) has the same general effect as increasing the values β_i in Eq. (8).

To study the effect of different energy parametrizations, we chose a number of different forms, as summarized in Table III. Form *A* is the one using Eq. (17). Forms $Q1\cdots Q4$ use Eq. (8) and have different choices for the set $\beta_1\cdots\beta_4$, as noted in Table III. Since all of these forms have roughly the same theoretical framework, it is not surprising that they give rather similar results. As the number of free parameters is increased to the point of redundancy, forms *A*, *Q*1, and *Q*2 all give essentially the same fit to the data. As the number of parameters is decreased, *A* emerges as the best parametrization. Since we have taken account of OPE

TABLE IV. Breakdown of free parameters for form-*A* solutions. The χ^2 values shown are for a fit to the full set of 839 (p,p) data.

χ^2	810	858	874
No. of parameters	30	23	20
$^1S_0$	4	4	4
$^1D_2$	3	2	2
$^1G_4$	2	1	1
$^3P_0$	2	2	2
$^3P_1$	4	2	2
$^3P_2$	3	3	3
ϵ_2	3	2	2
$^3F_2$	2	1	1
$^3F_3$	2	1	1
$^3F_4$	1	1	1
ϵ_4	1	1	1
$^3H_4$	1	1	⋯
$^3H_5$	1	1	⋯
$^3H_6$	1	1	⋯

TABLE V. Phase-shift values from the 23-parameter form-A solution. The errors are calculated from the parameter error matrix. All phase shifts in this paper are Stapp nuclear-bar phase shifts in degrees. The value $g^2=15$ is used for all solutions. These results are illustrated in Fig. 2.

Lab energy (MeV)	$^1S_0$	$^1D_2$	$^1G_4$	$^3P_0$	$^3P_1$	$^3P_2$	ϵ_2	$^3F_2$	$^3F_3$	$^3F_4$	ϵ_4	$^3H_4$	$^3H_5$	$^3H_6$
5	54.65±0.03	0.06±0.00	0.00±0.00	1.77±0.02	−1.09±0.01	0.29±0.01	−0.06±0.00	0.00±0.00	−0.01±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00
10	54.97±0.07	0.20±0.00	0.00±0.00	3.83±0.04	−2.32±0.01	0.80±0.02	−0.23±0.00	0.01±0.00	−0.04±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00
15	53.01±0.09	0.38±0.00	0.01±0.00	5.61±0.07	−3.41±0.02	1.41±0.03	−0.44±0.00	0.04±0.00	−0.10±0.00	0.01±0.00	−0.02±0.00	0.00±0.00	0.00±0.00	0.00±0.00
20	50.75±0.11	0.57±0.01	0.03±0.00	7.09±0.10	−4.36±0.03	2.07±0.04	−0.66±0.01	0.07±0.00	−0.17±0.00	0.01±0.00	−0.03±0.00	0.00±0.00	−0.01±0.00	0.00±0.00
25	48.51±0.11	0.77±0.01	0.05±0.00	8.28±0.12	−5.20±0.04	2.75±0.04	−0.87±0.01	0.10±0.00	−0.26±0.00	0.02±0.00	−0.06±0.00	0.00±0.00	−0.02±0.00	0.00±0.00
30	46.36±0.11	0.98±0.01	0.07±0.00	9.23±0.14	−5.95±0.05	3.43±0.05	−1.08±0.01	0.14±0.00	−0.35±0.00	0.04±0.00	−0.08±0.00	0.01±0.00	−0.03±0.00	0.00±0.00
40	42.37±0.12	1.38±0.02	0.12±0.00	10.54±0.18	−7.28±0.05	4.76±0.05	−1.45±0.02	0.22±0.00	−0.55±0.00	0.08±0.00	−0.14±0.00	0.02±0.00	−0.06±0.00	0.00±0.00
50	38.78±0.13	1.77±0.02	0.17±0.00	11.25±0.20	−8.45±0.06	6.02±0.05	−1.76±0.02	0.30±0.01	−0.75±0.01	0.13±0.00	−0.21±0.00	0.03±0.00	−0.10±0.00	0.01±0.00
60	35.55±0.14	2.15±0.03	0.23±0.00	11.51±0.22	−9.51±0.06	7.19±0.05	−2.03±0.03	0.37±0.01	−0.94±0.01	0.18±0.01	−0.28±0.00	0.04±0.00	−0.14±0.00	0.01±0.00
70	32.62±0.16	2.53±0.04	0.29±0.00	11.45±0.23	−10.51±0.07	8.27±0.05	−2.25±0.03	0.43±0.02	−1.12±0.02	0.25±0.01	−0.35±0.00	0.06±0.00	−0.19±0.00	0.02±0.00
80	29.94±0.18	2.89±0.04	0.35±0.00	11.13±0.23	−11.47±0.07	9.26±0.05	−2.43±0.04	0.48±0.02	−1.28±0.02	0.32±0.01	−0.42±0.01	0.08±0.00	−0.24±0.01	0.03±0.00
90	27.48±0.20	3.25±0.05	0.41±0.01	10.62±0.23	−12.39±0.07	10.17±0.05	−2.57±0.04	0.53±0.03	−1.44±0.03	0.40±0.01	−0.49±0.01	0.10±0.01	−0.29±0.01	0.04±0.00
100	25.21±0.21	3.60±0.05	0.47±0.01	9.97±0.23	−13.29±0.07	10.99±0.05	−2.68±0.04	0.56±0.04	−1.58±0.04	0.48±0.02	−0.55±0.01	0.12±0.01	−0.35±0.01	0.05±0.00
120	21.08±0.23	4.27±0.06	0.59±0.01	8.36±0.23	−15.02±0.07	12.41±0.06	−2.83±0.04	0.61±0.06	−1.83±0.06	0.65±0.03	−0.66±0.01	0.16±0.01	−0.46±0.02	0.08±0.01
140	17.38±0.25	4.91±0.07	0.71±0.02	6.49±0.24	−16.70±0.09	13.58±0.06	−2.91±0.04	0.64±0.08	−2.04±0.08	0.83±0.03	−0.77±0.02	0.20±0.02	−0.57±0.04	0.11±0.01
160	13.96±0.27	5.52±0.08	0.82±0.03	4.50±0.26	−18.33±0.11	14.53±0.07	−2.91±0.05	0.64±0.10	−2.22±0.11	1.01±0.04	−0.86±0.03	0.25±0.03	−0.68±0.05	0.15±0.01
180	10.71±0.29	6.10±0.10	0.93±0.03	2.44±0.30	−19.91±0.13	15.30±0.08	−2.87±0.06	0.62±0.13	−2.37±0.13	1.20±0.05	−0.93±0.04	0.29±0.04	−0.78±0.07	0.19±0.02
200	7.58±0.31	6.66±0.11	1.04±0.04	0.38±0.34	−21.46±0.16	15.91±0.09	−2.79±0.08	0.59±0.15	−2.50±0.15	1.39±0.06	−1.00±0.05	0.34±0.05	−0.88±0.09	0.23±0.03
220	4.51±0.34	7.19±0.12	1.15±0.05	−1.65±0.40	−22.96±0.20	16.39±0.10	−2.68±0.09	0.55±0.17	−2.61±0.18	1.58±0.07	−1.06±0.06	0.38±0.07	−0.98±0.11	0.27±0.03
240	1.46±0.38	7.69±0.14	1.25±0.06	−3.64±0.46	−24.43±0.24	16.77±0.12	−2.55±0.12	0.49±0.20	−2.71±0.20	1.77±0.08	−1.11±0.07	0.42±0.08	−1.08±0.14	0.32±0.04
260	−1.57±0.43	8.17±0.16	1.35±0.07	−5.57±0.53	−25.86±0.27	17.04±0.15	−2.39±0.14	0.43±0.22	−2.79±0.23	1.96±0.09	−1.15±0.08	0.46±0.10	−1.17±0.16	0.36±0.05
280	−4.62±0.50	8.63±0.17	1.45±0.08	−7.43±0.60	−27.26±0.30	17.24±0.18	−2.23±0.17	0.37±0.24	−2.85±0.25	2.14±0.10	−1.19±0.09	0.50±0.12	−1.26±0.19	0.41±0.06
300	−7.67±0.59	9.07±0.19	1.55±0.09	−9.22±0.69	−28.62±0.34	17.37±0.21	−2.05±0.20	0.30±0.27	−2.91±0.27	2.32±0.11	−1.22±0.10	0.54±0.14	−1.34±0.22	0.46±0.06
320	−10.75±0.71	9.49±0.21	1.65±0.10	−10.93±0.76	−29.94±0.37	17.44±0.24	−1.86±0.23	0.23±0.29	−2.96±0.30	2.50±0.12	−1.24±0.11	0.58±0.16	−1.43±0.25	0.51±0.07
340	−13.85±0.84	9.89±0.23	1.74±0.11	−12.57±0.83	−31.23±0.40	17.46±0.27	−1.67±0.26	0.15±0.31	−3.01±0.32	2.67±0.13	−1.26±0.12	0.62±0.18	−1.51±0.28	0.57±0.08
360	−16.97±0.99	10.28±0.25	1.83±0.12	−14.12±0.90	−32.49±0.43	17.44±0.31	−1.47±0.29	0.08±0.34	−3.04±0.34	2.85±0.14	−1.28±0.14	0.65±0.20	−1.58±0.31	0.62±0.09
380	−20.11±1.15	10.64±0.27	1.92±0.13	−15.61±0.97	−33.72±0.46	17.38±0.35	−1.27±0.32	0.00±0.36	−3.07±0.37	3.01±0.15	−1.29±0.15	0.69±0.22	−1.66±0.34	0.67±0.10
400	−23.27±1.33	11.00±0.29	2.01±0.14	−17.02±1.04	−34.91±0.49	17.28±0.39	−1.07±0.35	−0.08±0.38	−3.10±0.39	3.18±0.16	−1.30±0.16	0.72±0.24	−1.73±0.38	0.72±0.11

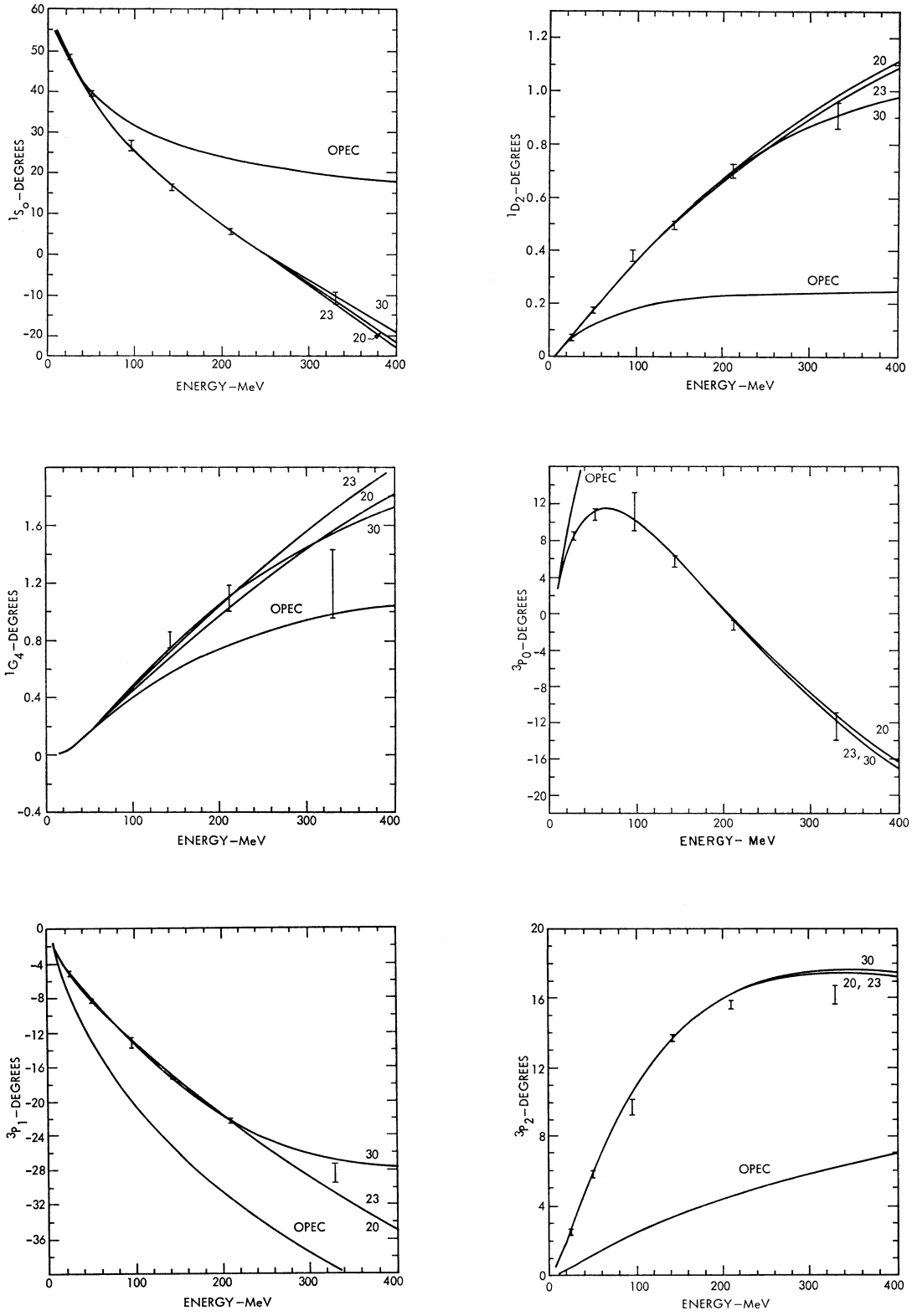

FIG. 1. Phase shifts from the single-energy solutions (Table VI), from the 20-, 23-, and 30-parameter form-*A* solutions, and from the OPE calculation with $g^2=15$. These curves are taken directly from computer cathode-ray-tube plots.

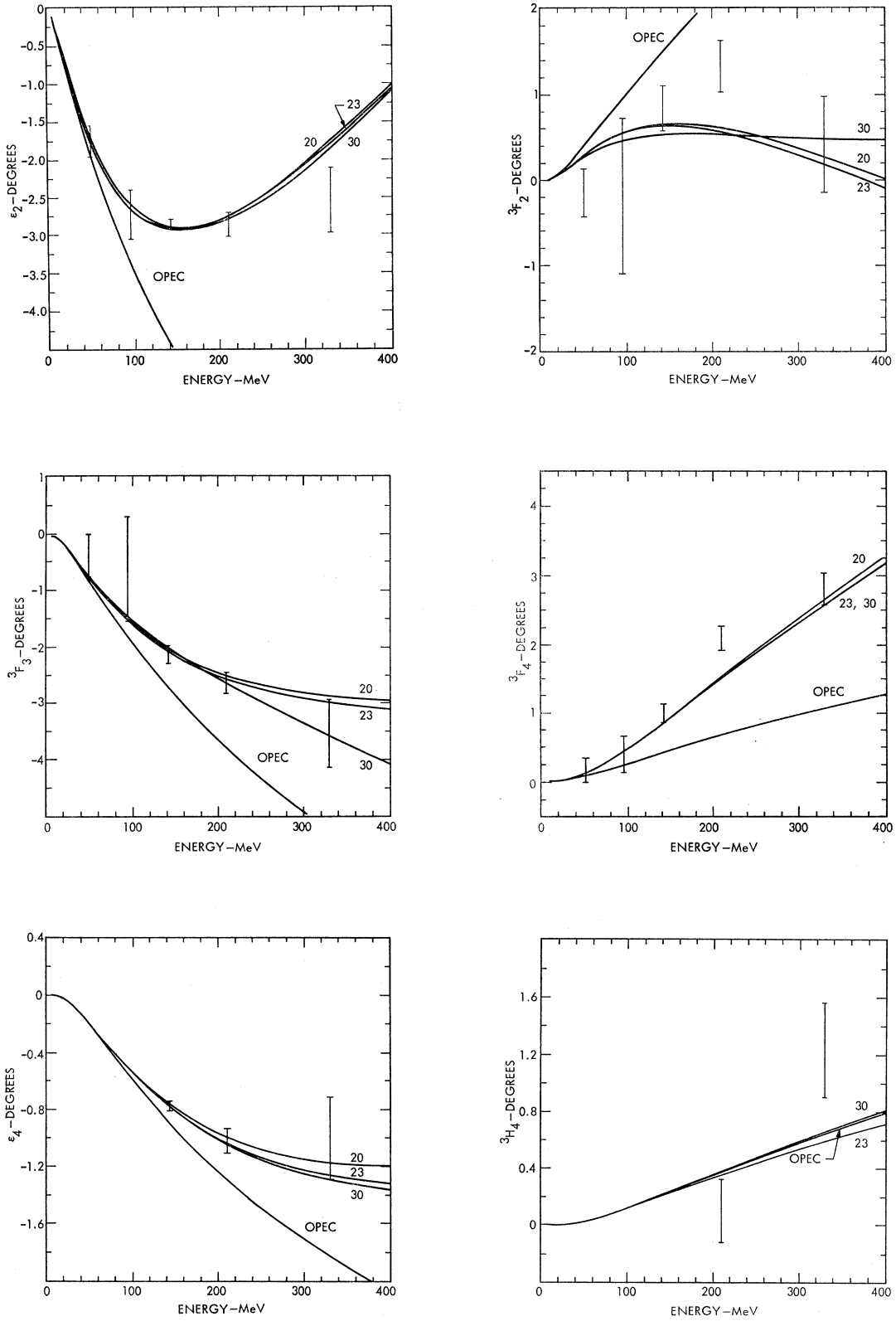

FIG. 1. (*Continued*).

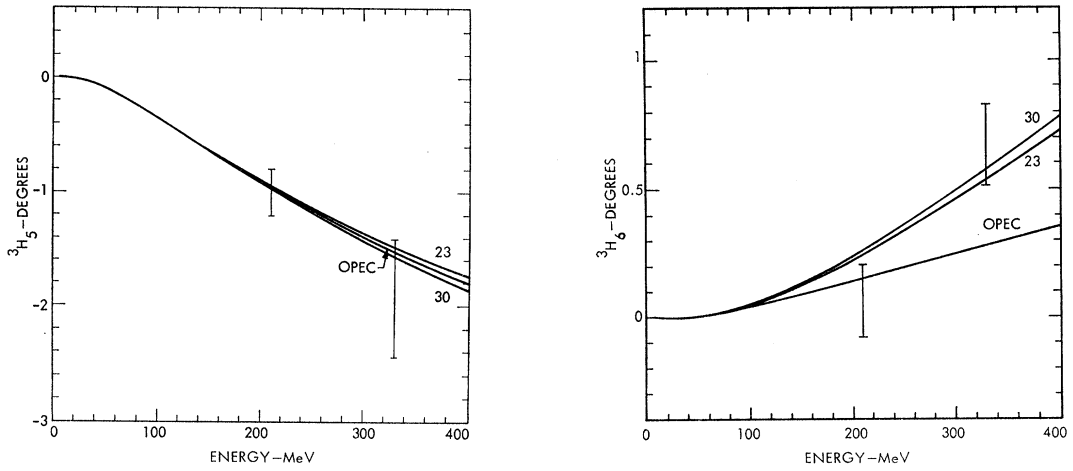

FIG. 1. (*Continued*).

($\beta_i=1$) explicitly in the δ_l [Eq. (7)], we would expect $Q1$ to be superior to $Q2$. We would also expect to see $Q3\sim Q1$ and to see $Q4$ as the poorest form. These expectations are more or less borne out in the results shown in Table III. This gives some credence to the argument, which has been made elsewhere,[17] for using a form that has the correct singularity structure. Form A has singularities starting from the 2π discontinuity ($\beta_i=2$), as can be noted in Eq. (9).

The results shown in Table III were based on a set of 515 (p,p) data, and were obtained by using a preliminary six-energy matrix representation of these data. With each form, the number of parameters was initially set at 38, and after each χ^2 minimization the least-sensitive parameter, as determined by the computer, was deleted. As the result of these calculations, we selected A as the most suitable form for the phase-shift energy dependence.

After completion of Table III, the data set included in the matrix representations was expanded to include a total of 588 (p,p) data. When the study shown in Table III was repeated for the form-A energy dependence, the 30-parameter and 20-parameter solutions appeared as the limiting parameter choices. Thirty free parameters give a solution that has enough flexibility to permit a precision fit to the data. Twenty free parameters give a solution in which form-limiting is still small, and in which an excellent fit to the data is still maintained (χ^2 per data point$=1.04$ for the entire set of 839 data). Thus for those who fit potential models to phase shifts, the 20-parameter solution is by all standards a completely adequate representation. The 30-parameter solution gives $\chi^2/[$(No. of data)$-$(No. of parameters)$]=1.00$ for the 839 data. From a statistical point of view, this is as good a value as we should hope to obtain if the experimenters have been accurate in their statements of experimental errors. Also, as Table III shows, increasing the number of parameters beyond 30 does not decrease χ^2 appreciably, and the redundant parameters increase the error limits, particularly for the high-l phase shifts.

Table IV shows how the free parameters are distributed for 20-, 23-, and 30-parameter form-A solutions. The values in Table IV can be taken as a rough measure of the "non-OPE contribution" to thee nergy dependence of the phase shifts. In particular, $^3P_2$, which receives a large contribution from the $I=0$, $J=0$ 2π channel, exhibits large deviations from the OPE phase shift. It is interesting to note that the 30-parameter solution includes all phases through H waves in the free parameters, and the 20-parameter solution has all H waves set equal to the OPE value. The 23-parameter solution differs from the 20-parameter solution in that it includes one free parameter for each of the H waves.

When the energy-dependent form $Q1$ from Table III (which is the form we used in our previous work[4]), was fitted to the full set of 839 data, the χ^2 value for a 30-parameter fit was 882, as compared to the value 810 for the 30-parameter form A fit. Thus form A is clearly a better parametrization than $Q1$, and it is used in all of our final energy-dependent analyses.

The results shown in Table III were run in a few minutes on the computer, since we used a matrix representation for the data. Direct use of the data would have involved a prohibitive amount of computer time. We show in Sec. IV that the subset of 588 data used for the matrix representation includes most of the physical content contained in the entire data set, and that the matrix representation of the data is an accurate description of the data.

IV. PHASE-SHIFT RESULTS

A. Energy-Dependent Analyses

Our final data selection was described in Sec. II and Tables I and II. Of some 1084 (p,p) data considered,

[17] R. A. Arndt and M. J. Moravcsik, Nuovo Cimento **51A**, 108 (1967).

TABLE VI. Single-energy analyses. The phases at each energy were given a fixed energy derivative as determined on the computer from the energy-dependent solutions.

Energy (MeV)	25	50	95	142	210	330
Energy spread	23.5–28.2	44.7–58.5	93.2–102	137–155	210–213	310–345
No. of data	34	99	85	183	65	122
Free phases	5	9	9	11	14	14
χ^2	17.6	97.4	69.3	188.4	49.1	115.8
$^1S_0$	48.61±0.26	39.55±0.46	26.87±1.44	16.50±0.58	5.59±0.53	−10.70±1.46
$^1D_2$	0.76±0.03	1.74±0.10	3.77±0.26	4.95±0.17	7.03±0.29	9.09±0.52
$^1G_4$	(0.04)[a]	(0.17)	(0.39)	0.81±0.07	1.10±0.10	1.20±0.25
$^3P_0$	8.53±0.45	10.78±0.69	11.17±2.15	5.74±0.54	−1.23±0.55	−12.40±1.57
$^3P_1$	−5.01±0.21	−8.16±0.31	−13.12±0.66	−17.07±0.17	−22.2 ±0.32	−28.36±1.20
$^3P_2$	2.43±0.16	5.70±0.15	9.70±0.50	13.73±0.11	15.59±0.23	16.18±0.56
ϵ_2	(−0.90)	−1.74±0.21	−2.72±0.32	−2.85±0.07	−2.86±0.16	−2.54±0.44
$^3F_2$	(0.12)	−0.15±0.29	−0.19±0.92	0.84±0.27	1.33±0.31	0.42±0.57
$^3F_3$	(−0.27)	−0.39±0.39	−0.62±0.92	−2.13±0.17	−2.63±0.20	−3.54±0.61
$^3F_4$	(0.02)	0.17±0.18	0.40±0.28	0.98±0.14	2.09±0.18	2.80±0.24
ϵ_4	(−0.05)	(−0.22)	(−0.56)	−0.77±0.03	−1.02±0.09	−0.99±0.29
$^3H_4$	(0.005)	(0.03)	(0.11)	(0.22)	0.10±0.23	1.23±0.34
$^3H_5$	(−0.02)	(−0.10)	(−0.32)	(−0.59)	−1.00±0.20	−1.92±0.52
$^3H_6$	(0.001)	(0.008)	(0.04)	(0.08)	0.07±0.14	0.67±0.16

[a] The phases in parentheses are OPE contribution phases with $g^2=15$ and $M_\pi=135.04$ MeV.

839 were included in the final set. The energy parametrization selected was form A, as described in Sec. III and Tables III and IV. A total of 30 free parameters gives a χ^2 value of 810. Since, with 839 data points and 30 adjustable parameters we expect a χ^2 of 809 for an ideal statistical fit, the 30-parameter solution is flexible enough that form-limiting does not occur, a conclusion that was also illustrated in Table III.

When the number of free parameters is reduced from 30 to 20, as selected by the computer, a solution is obtained in which some form-limiting occurs, but which is nevertheless a very precise fit to the entire data set. For this solution, the average M value for the entire data selection is 1.04. Since the solution contains only S-G waves, it is a convenient one to use. The addition of H waves gives the 23-parameter solution listed in Table IV.

The phase-shift values for the 20-, 23-, and 30-parameter solutions, and also the OPE phase-shifts for comparison, are illustrated in Fig. 1. As can be seen, the 20- and 23-parameter phases are very similar, and the agreement with the 30-parameter solution is good. Tabulated values for the 23-parameter solution, together with the phase-shift errors as deduced from the 23-parameter correlation matrix, are given in Table V. Figure 2 shows the phase shifts and error corridors, as plotted by the computer, for the 23-parameter solution.

B. Energy-Independent Analyses

As a further check on the subject of form-limiting, we carried out phase-shift analyses in six narrow energy bands centered around 25, 50, 95, 142, 210, and 330 MeV. The data used for these analyses are indicated in Table I. Energy-dependent phase-shift slopes were assigned in each energy band as determined by the computer from the energy-dependent analyses.

The single-energy phase shifts are listed in Table VI. They are also shown on the phase-shift plots in Fig. 1. The general agreement between the energy-dependent and energy-independent phases is good, which indicates again that form-limiting is not an important factor in the energy-dependent solutions. There is some scatter at the highest energy, 330 MeV. The second-derivative matrices for the six energy-independent solutions are given in Table VII, and the corresponding error matrices are given in Table VIII.

In the present "single-energy" analyses, a total of 588 data were used. This compares to the total of 365 data used in our previous single-energy analyses.[1–6] As discussed in Sec. IV C, these 588 data contain most of the physical content that is inherent in the full set of 839 data.

C. Validity of the Matrix Representation

As a test of the validity of using a matrix representation of a set of data, we carried out the calculations summarized in Table IX. Selecting first the 588 data used in our six single-energy analyses, we obtained a matrix representation[4,9] of these data, labeled matrix A in Table IX. These are the matrices given in Table VII. Then, using form A with 20 and with 30 parameters, we obtained solution A by fitting against matrix A. Solution A was first tested against the actual data, set A, and then allowed to be searched against set A. As can be seen, the decrease in χ^2 (641–631 for 20 parameters, and 609–571 for 30 parameters) was quite small, showing that matrix A is a "faithful representation" of set A. In general, χ^2 is a very sensitive function of the phase-shift parameters. Thus the change in the actual phase shifts as obtained from matrix A and then from set A is miniscule.

TABLE VII. Second-derivative matrices for the single-energy solutions of Table VI. The units for this table are deg^{-2}.

	$^1S_0$	$^3P_0$	$^3P_1$	$^1D_2$	$^3P_2$	ϵ_2	$^3F_2$	$^3F_3$	$^3F_4$	$^1G_4$	ϵ_4			
								25 MeV						
$^1S_0$	21.28													
$^3P_0$	6.075	19.65												
$^3P_1$	−3.880	27.11	103.4											
$^1D_2$	40.58	−48.92	−157.3	1341.4										
$^3P_2$	13.82	73.06	170.1	−345.4	391.6									
								50 MeV						
$^1S_0$	187.1													
$^3P_0$	43.11	70.68												
$^3P_1$	−185.1	53.24	491.7											
$^1D_2$	565.1	−368.9	420.9	16354										
$^3P_2$	14.51	329.5	522.8	−3459	2036									
ϵ_2	257.2	−270.5	−85.64	11373	−2350	8314								
$^3F_2$	−54.76	123.7	319.1	−1601	823.2	−1265	620.6							
$^3F_3$	−231.5	342.4	737.1	−6388	2426	−4531	1338	3793						
$^3F_4$	58.71	302.4	920.8	2145	1455	1807	591.2	960.5	3531					
								95 MeV						
$^1S_0$	6.244													
$^3P_0$	−1.069	1.261												
$^3P_1$	12.08	−3.057	29.62											
$^1D_2$	26.38	3.856	51.33	259.9										
$^3P_2$	−16.97	4.841	−34.54	−54.06	66.16									
ϵ_2	18.05	11.04	33.51	258.1	−19.23	337.6								
$^3F_2$	15.95	−6.693	27.85	20.09	−69.40	−40.19	107.1							
$^3F_3$	−8.731	−3.091	−20.11	−99.44	9.873	−130.2	25.67	60.68						
$^3F_4$	−1.351	7.236	10.21	109.5	49.29	167.9	−132.5	−93.20	286.9					
								142 MeV						
$^1S_0$	42.45													
$^3P_0$	25.09	32.78												
$^3P_1$	6.069	16.40	150.5											
$^1D_2$	149.6	87.06	17.39	742.9										
$^3P_2$	2.096	−3.648	−90.58	−83.03	258.3									
ϵ_2	223.2	214.6	192.7	918.7	−60.44	2345								
$^3F_2$	−96.43	−107.2	−129.9	−231.1	−89.44	−1148	1084							
$^3F_3$	−177.3	−162.7	−132.6	−574.2	−5.102	−1592	1103	1412						
$^3F_4$	259.6	275.1	327.1	736.7	162.9	2537	−1959	−2168	4113					
$^1G_4$	−24.74	−110.0	−216.1	353.2	−211.4	−771.1	921.8	797.8	−1877	2114				
ϵ_4	−357.8	−479.5	−564.3	−571.8	−215.2	−4108	3445	3833	−6685	4148	14074			
	$^1S_0$	$^3P_0$	$^3P_1$	$^1D_2$	$^3P_2$	ϵ_2	$^3F_2$	$^3F_3$	$^1G_4$	$^3F_4$	ϵ_4	$^3H_4$	$^3H_5$	$^3H_6$
									210 MeV					
$^1S_0$	17.52													
$^3P_0$	−0.4406	13.46												
$^3P_1$	0.1930	−5.076	29.10											
$^1D_2$	30.03	−1.015	−19.46	127.7										
$^3P_2$	−1.750	−0.1476	14.73	−23.61	72.43									
ϵ_2	14.13	19.09	−27.51	29.79	−2.743	144.6								
$^3F_2$	−10.48	6.419	−0.6659	−4.072	−3.245	−23.57	88.09							
$^3F_3$	−28.96	−9.068	20.08	−49.84	22.81	−70.05	58.68	136.0						
$^1G_4$	1.448	1.040	−60.26	130.8	−61.03	50.44	3.661	−53.30	395.7					
$^3F_4$	22.73	20.77	27.28	−22.71	100.9	100.1	−112.7	−53.74	−151.1	446.5				
ϵ_4	8.124	−43.02	−16.24	131.5	−31.41	−85.27	101.9	69.22	202.4	−335.6	639.5			
$^3H_4$	4.476	−1.994	8.150	−26.87	−5.197	37.78	−13.90	−8.956	−22.54	55.64	−118.1	213.5		
$^3H_5$	−20.72	8.359	6.143	−86.80	6.968	−9.178	28.00	66.20	−88.32	38.83	−122.9	104.1	181.0	
$^3H_6$	11.49	18.70	12.35	55.82	70.06	4.677	57.76	82.78	−94.20	135.7	96.07	−209.1	−14.43	582.4

TABLE VII. (*Continued*).

	$^1S_0$	$^3P_0$	$^3P_1$	$^1D_2$	$^3P_2$	ϵ_2	$^3F_2$	$^3F_3$	$^1G_4$	$^3F_4$	ϵ_4	$^3H_4$	$^3H_5$	$^3H_6$
						330 MeV								
$^1S_0$	2.641													
P_0	0.7781	1.400												
$^3P_1$	−1.478	−0.8034	2.333											
$^1D_2$	0.7867	1.175	−0.3821	20.75										
$^3P_2$	−0.4642	−1.003	0.1506	−1.938	6.874									
ϵ_2	5.913	2.257	−5.282	0.3686	−0.7852	24.77								
$^3F_2$	−5.357	−0.4220	3.959	13.53	0.1095	−14.20	33.98							
$^3F_3$	−8.281	−3.792	6.619	5.573	4.364	−21.80	29.24	41.07						
$^1G_4$	5.656	1.870	−6.217	−3.774	−2.425	17.79	−21.28	−24.66	49.28					
$^3F_4$	10.17	3.347	−7.300	−24.95	4.292	27.05	−58.85	−53.10	39.36	139.1				
ϵ_4	−4.635	−0.4463	4.215	40.92	−2.578	−14.40	50.88	39.91	−28.15	−102.8	134.6			
$^3H_4$	1.567	−0.9477	−3.021	−32.20	3.430	9.919	−28.83	−18.34	33.16	57.36	−90.95	106.4		
$^3H_5$	−4.826	−1.979	2.821	−17.68	6.372	−13.52	−0.6349	13.48	1.224	11.62	−28.77	41.65	39.94	
$^3H_6$	5.708	4.694	−2.343	41.08	3.300	11.14	24.20	3.623	−38.14	−28.81	73.90	−110.8	−58.11	225.9

Table IX illustrates another important fact, which is shown by the continuation of the above process with data sets B and C, namely, that the subset of 588 "single-energy" data contains virtually the entire physical information content that is included in the complete set of 839 data. This is shown by the excellent fit of solution A against sets B and C. Set B contains 207 data that were not contained in set A, and yet the χ^2 change for the unsearched solution A with 20 parameters on set B as against the searched solution on set B is only from 840 to 823. A solution that gives a good fit to set A will also give a good fit to set B.

The difference between sets B and C is that the latter contains 44 data above 360 MeV. Since this lies beyond the energy range in which solution A was obtained, the solution-A fit to these data is not as precise as at the lower energies. In particular, the 30-parameter solution does not extrapolate as well as does the 20-parameter solution. This confirms our expectation that the smaller-dimensional representation should extrapolate more smoothly. The extra flexibility in the 30-parameter solution allows it to develop wiggles that are often only of local significance. However, the extrapolations to 400 MeV are reasonable qualitatively, even for the 30-parameter solution. This was not true of early forms used for energy-dependent phase-shift representations.[18]

D. Recommended Phase-Shift Solutions

From our energy-dependent analyses, we have selected the 23-parameter solution as being the most useful representation of the (p,p) scattering data from 9 to 400 MeV. The M value for this solution is 1.02 for the entire set of 839 data. Although the lowest energy included in the data set is at 9.68 MeV, the solution gives precision fits ($M<1$) to the Wisconsin differential cross-section data at 2.425 and 3.037 MeV (see Table I). Thus the solution can be used with confidence at energies well below 10 MeV. This is, of course, a consequence of the fact that we have chosen a form for the S wave that has the correct effective-range expansion [see Eq. (6)]. The single-energy solutions of Table VI were selected from studies using different choices for the free phases. The parametrizations given in Table VI appear to us to be the most useful ones.

E. Use of the Matrix Representations

For a fit to potential models, it is often accurate enough to make a fit to phase shifts in "diagonal form." In this approximation, we have

$$\chi^2=\sum_{T,l}\left(\frac{\delta_m{}^{T,l}-\delta_0{}^{T,l}}{\Delta\delta_0{}^{T,l}}\right)^2, \tag{18}$$

where T is the energy, l represents the phase-shift index at this energy, δ_m is the phase shift calculated

[18] M. J. Moravcsik, *The Two-Nucleon Interaction* (Oxford University Press, New York, 1963), Figs. 34–47.

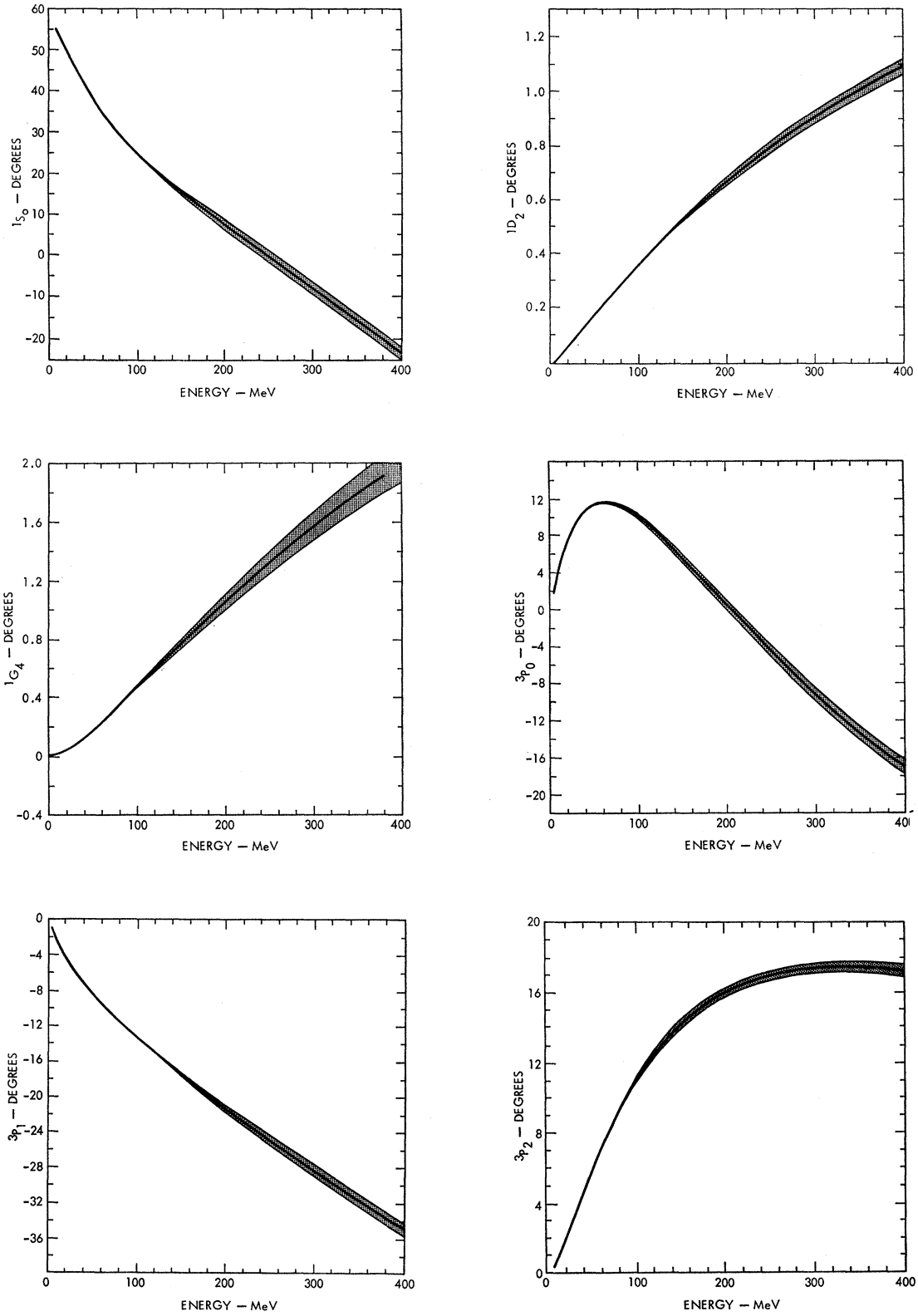

FIG. 2. Computer-calculated plots of the corridor of errors for the 23-parameter form-*A* solution of Table V. The phase-shift error limits were obtained from the parameter error matrix.

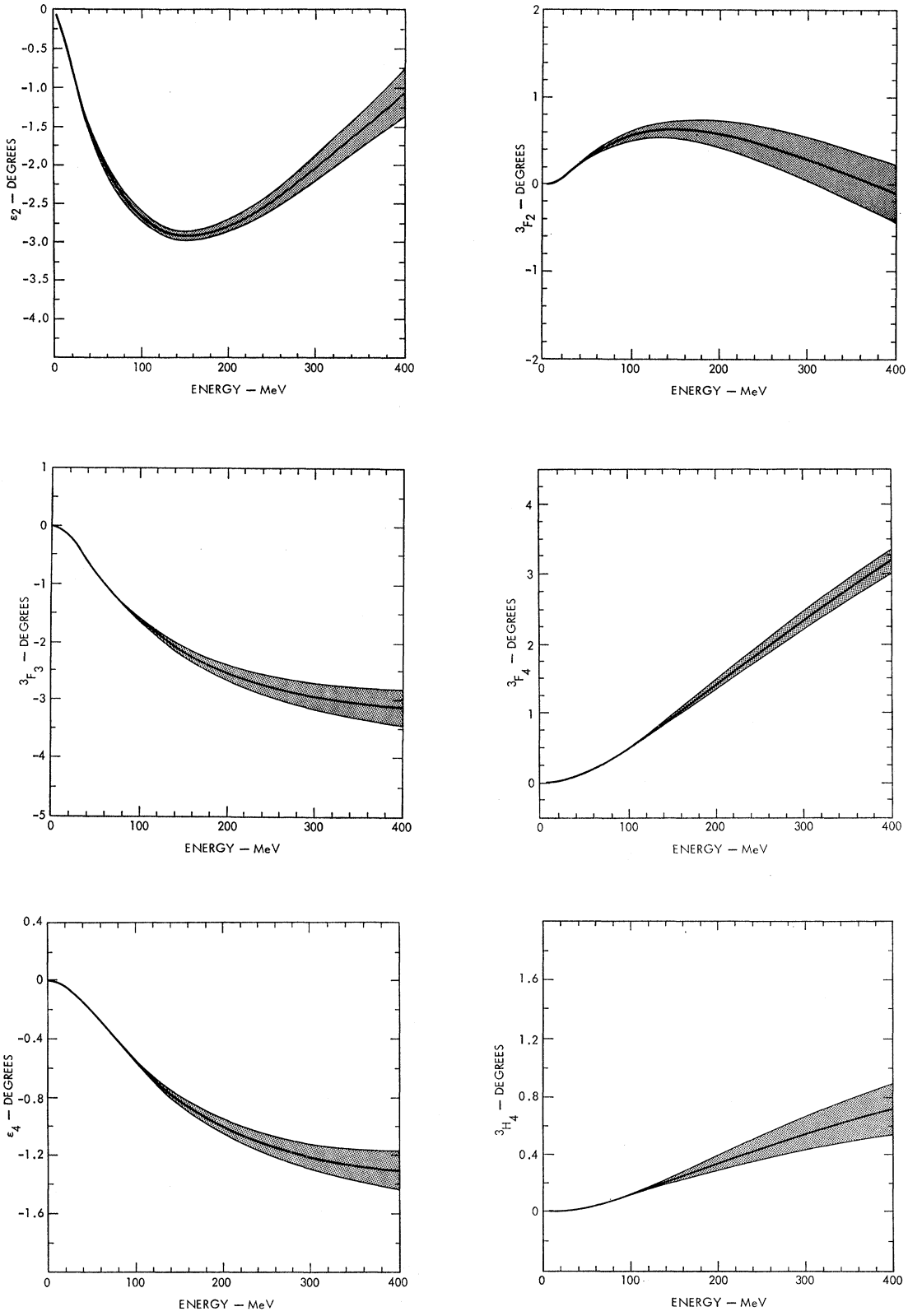

FIG. 2. (*Continued*).

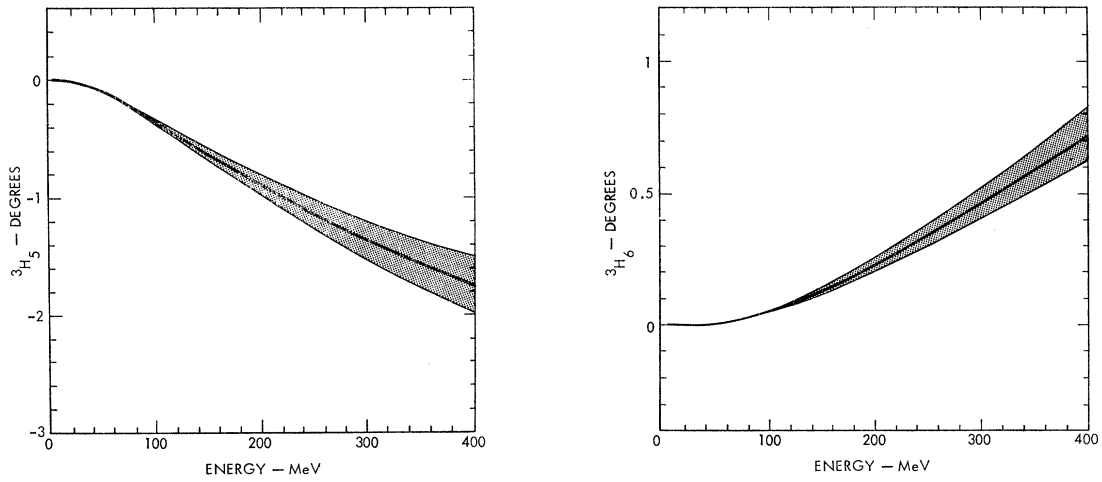

FIG. 2. (*Continued*).

from the model, and δ_0 and $\Delta\delta_0$ are the phase shift and its associated error as given in the present paper. The set $\delta_0{}^T$ and $\Delta\delta_0{}^T$ can be selected either from the energy-dependent values given in Table V, or else as the set of single-energy phase shifts and errors given in Table VI. Since the diagonal errors $\Delta\delta_0$ only partially reflect the correlations among the phase-shift uncertainties, χ^2 as calculated from Eq. (18) will differ by roughly a factor of 2 from the χ^2 obtained by fitting the set of phases $\delta_m{}^{T,l}$, obtained from Eq. (18), directly against the data. However, the model parameters will be quite accurately determined by the minimization of χ^2 as given in Eq. (18).

For a precision fit, it is necessary to use the full correlation matrices of Table VII. In Table VII we have given the values for the matrices $a_{l,l'}{}^T$, where

$$\alpha_{l,l'}=\tfrac{1}{2}\partial^2\chi^2/\partial\delta_l\partial\delta_{l'}. \tag{19}$$

The units for $\alpha_{l,l'}$ in Table VII are deg^{-2}. A model fit that minimizes χ^2 in the equation

$$\chi^2=\sum_{T,l,l'}\alpha_{l,l'}{}^T(\delta_m{}^{T,l}-\delta_0{}^{T,l})(\delta_m{}^{T,l'}-\delta_0{}^{T,l'}) \tag{20}$$

will give a precision fit to the actual data. This follows from the facts (a) that the matrix $\alpha_{l,l'}$ is a good representation of the data from which it was determined, and (b) that the subset of 588 data which are represented by the matrices of Table VII includes most of the physical content of the entire set of 839 data. These points were discussed in detail above.

In addition to the second-derivative matrices of Table VII, we have given the corresponding inverse matrices (the conventional error matrices) of Table VIII. These are the matrices $(\alpha^{-1})_{l,l'}$ in units of deg^2. These matrices are useful in calculating the theoretical errors for any observables. If we define

$$\beta_l=\partial O/\partial\delta_l \tag{21}$$

for an observable O, then the theoretical uncertainty in that observable, as given from the present analysis, is

$$\Delta O=(\beta^T\alpha^{-1}\beta)^{1/2}, \tag{22}$$

where β and β^T are the observable derivative vector and its transpose, respectively. If O represents one of the phase shifts δ_l, then (21) and (22) show immediately that

$$\Delta\delta_l=(\alpha_{ll}{}^{-1})^{1/2}, \tag{23}$$

which is a well-known result. More usefully, O can represent any one of the many different kinds of (p,p) observables.

V. VALUE FOR PION-NUCLEON COUPLING CONSTANT g^2

In previous papers (see Refs. 1–6 and 15), we have published a number of determinations of g^2, the pion-nucleon coupling constant. Since the present analysis is based on a fit to 839 selected (p,p) data, and since the energy-dependent parametrization includes the OPE contribution in a very plausible manner, we believe that the g^2 determination from this analysis is the most accurate one that we know how to obtain from the existing nucleon-nucleon data.

In Table X are listed the values for $\chi^2(g^2)$ obtained from the 20-, 23-, and 30-parameter solutions. It is a somewhat empirical fact that as the number of phase-shift parameters is reduced, or more precisely, as the OPE contribution is used at lower and lower l values, the minimum in the $\chi^2(g^2)$ curve tends to shift to lower values for g^2. The trend can be noticed in Table X, although the shift is rather erratic. On the other hand, as more and more freedom is given to the phases, the OPE contribution dependence decreases and the $\chi^2(g^2)$ parabola flattens out. The error in the g^2 determination increases. The 30-parameter solution shown in Table X probably represents a good compromise between the

TABLE VIII. Error matrices for the single-energy solutions of Table VI. The units for this table are deg².

	$^1S_0$	$^3P_0$	$^3P_1$	$^1D_2$	$^3P_2$	ϵ_2	$^3F_2$	$^3F_3$	$^3F_4$	$^1G_4$	ϵ_4
$^1S_0$	0.0679										
$^3P_0$	−0.0165	0.2006						25 MeV			
$^3P_1$	0.0194	0.0255	0.0442								
$^1D_2$	−0.0031	−0.0020	−0.0011	0.0011							
$^3P_2$	−0.0105	−0.0497	−0.0256	0.0019	0.0250						
$^1S_0$	0.2121										
$^3P_0$	−0.1833	0.4692						50 MeV			
$^3P_1$	0.0558	0.0820	0.0966								
$^1D_2$	−0.0108	0.0311	0.0156	0.0094							
$^3P_2$	−0.0427	0.0463	0.0019	0.0066	0.0228						
ϵ_2	0.0585	−0.0744	−0.0136	−0.0123	−0.0105	0.0448					
$^3F_2$	−0.0993	0.0814	−0.0322	−0.0069	0.0179	−0.0258	0.0847				
$^3F_3$	0.1401	−0.1569	0.0055	−0.0056	−0.0277	0.0656	−0.0974	0.1516			
$^3F_4$	−0.0296	−0.0294	−0.0325	−0.0060	−0.0073	−0.0158	0.0254	−0.0340	0.0315		
$^1S_0$	2.081										
$^3P_0$	−0.093	4.622						95 MeV			
$^3P_1$	−0.045	0.674	0.441								
$^1D_2$	0.030	0.154	0.050	0.066							
$^3P_2$	−0.222	−0.090	−0.069	−0.042	0.252						
ϵ_2	0.121	−0.369	0.007	−0.046	−0.039	0.105					
$^3F_2$	−0.769	−0.205	−0.306	−0.132	0.354	−0.051	0.846				
$^3F_3$	0.715	−0.179	0.253	0.027	−0.306	0.200	−0.628	0.841			
$^3F_4$	−0.153	−0.121	−0.103	−0.049	−0.063	0.014	0.218	−0.093	0.080		
$^1S_0$	0.3362										
$^3P_0$	0.1477	0.2956						142 MeV			
$^3P_1$	0.0356	0.0536	0.0287								
$^1D_2$	−0.0540	−0.0087	−0.0012	0.0291							
$^3P_2$	0.0014	0.0205	0.0132	0.0008	0.0117						
ϵ_2	0.0214	0.0060	0.0015	−0.0072	0.0005	0.0046					
$^3F_2$	−0.0943	−0.1101	−0.0237	−0.0053	−0.0058	−0.0042	0.0705				
$^3F_3$	0.0570	0.0429	0.0056	0.0033	−0.0013	0.0039	−0.0344	0.0291			
$^3F_4$	−0.0402	−0.0547	−0.0152	−0.0042	−0.0060	−0.0019	0.0326	−0.0128	0.0184		
$^1G_4$	0.0032	−0.0051	−0.0002	−0.0071	0.0007	0.0012	0.0065	−0.0018	0.0038	0.0043	
ϵ_4	0.0066	0.0085	0.0016	−0.0021	0.0003	0.0006	−0.0029	−0.0007	−0.0012	−0.0006	0.0012

	$^1S_0$	$^3P_0$	$^3P_1$	$^1D_2$	$^3P_2$	ϵ_2	$^3F_2$	$^3F_3$	$^1G_4$	$^3F_4$	ϵ_4	$^3H_4$	$^3H_5$	$^3H_6$
$^1S_0$	0.281													
$^3P_0$	0.053	0.298									210 MeV			
$^3P_1$	−0.024	0.028	0.102											
$^1D_2$	−0.072	−0.010	0.035	0.084										
$^3P_2$	0.002	0.034	0.023	0.025	0.052									
ϵ_2	0.001	0.001	0.028	0.020	0.014	0.026								
$^3F_2$	−0.018	−0.115	−0.025	−0.003	−0.036	−0.012	0.095							
$^3F_3$	0.062	0.038	−0.004	0.003	0.003	0.011	−0.019	0.042						
$^1G_4$	0.023	−0.007	0.000	−0.019	−0.004	−0.005	0.004	−0.001	0.010					
$^3F_4$	−0.008	−0.040	−0.020	−0.006	−0.030	−0.010	0.045	−0.004	0.001	0.034				
ϵ_4	−0.004	0.022	−0.001	−0.005	−0.005	−0.001	−0.004	−0.001	−0.002	0.005	0.008			
$^3H_4$	−0.011	0.021	−0.025	−0.038	−0.007	−0.019	−0.016	−0.012	0.007	−0.010	0.001	0.053		
$^3H_5$	−0.008	−0.002	0.032	0.040	0.018	0.017	−0.007	0.001	−0.006	−0.004	0.000	−0.036	0.041	
$^3H_6$	−0.005	0.004	−0.009	−0.024	−0.002	−0.009	−0.012	−0.010	0.006	−0.010	−0.001	0.029	−0.018	0.021
$^1S_0$	2.125													
$^3P_0$	0.286	2.451									330 MeV			
$^3P_1$	−0.511	0.097	1.436											
$^1D_2$	−0.033	−0.128	0.050	0.271										
$^3P_2$	−0.117	0.310	0.164	−0.009	0.317									
ϵ_2	−0.063	−0.139	0.124	0.086	−0.047	0.192								
$^3F_2$	0.010	−0.485	−0.033	−0.032	−0.019	0.003	0.319							
$^3F_3$	0.447	0.534	−0.287	−0.039	−0.036	−0.034	−0.139	0.368						
$^1G_4$	−0.093	−0.002	0.111	−0.034	0.024	−0.017	0.013	−0.020	0.061					
$^3F_4$	−0.032	−0.084	−0.055	−0.032	−0.024	−0.029	0.073	0.011	−0.005	0.059				
ϵ_4	−0.125	0.019	0.031	−0.083	−0.001	−0.046	−0.024	−0.028	0.002	0.025	0.085			
$^3H_4$	−0.159	0.079	0.052	−0.058	0.017	−0.077	−0.024	−0.003	0.004	0.024	0.070	0.112		
$^3H_5$	0.085	−0.301	0.032	0.125	−0.107	0.137	−0.009	−0.112	−0.017	−0.053	−0.047	−0.105	0.271	
$^3H_6$	−0.097	−0.044	0.052	−0.024	−0.017	−0.013	−0.011	−0.030	0.016	−0.002	0.021	0.029	0.003	0.025

Scylla of systematic error and the Charybdis of large experimental error. Thus the value $g^2=14.72\pm0.83$ is believed to be the most accurate value we can obtain for g^2 from the existing nucleon-nucleon data. This value for g^2 corresponds to the value $f^2=0.0762\pm0.0043$, which is (just barely) in agreement with the value for

TABLE IX. Check on the validity of the matrix representation. Solution A was obtained by fitting to matrix A, and then was tested against sets A, B, and C, first unsearched and then searched.

Data set used	Solution used	χ^2 values obtained 20 parameter	30 parameter
Matrix A[a]	Solution A[e]	624	562
Set A[b]	Solution A	641	609
Set A	Search	631	571
Set B[c]	Solution A	840	792
Set B	Search	823	751
Set C[d]	Solution A	943	931
Set C	Search	874	810

[a] Matrix A is the matrix representation of the 588 data comprising the six single-energy analyses.
[b] Set A is the 588 data described in footnote a.
[c] Set B is the 795 data extending from 10–360 MeV.
[d] Set C is the 839 data extending from 10–400 MeV.
[e] Solution A is the 20(30)-parameter solution obtained by fitting to matrix A, using form-A energy dependence.

f^2 obtained from pion-nucleon scattering,[19] $f^2=0.0822 \pm 0.0018$.

VI. CONCLUSIONS

The present paper represents the conclusion of a decade of effort at Livermore to determine the (p,p) elastic scattering matrix. Our belief is that the data are now complete enough to definitely point to a unique type-I solution and to define quite accurately the scattering amplitudes over the whole elastic energy range. It is unlikely that any future experiments will cause more than slight modifications in the phases we have presented here, although, from the discussion at the end of Sec. II, one is never sure of this fact.

We have been in direct contact with the principal experimental groups all over the world, and Tables I and II include all of the data that we could obtain as of July 1967. The recent Gainesville conference on nucleon-nucleon interactions[7] gave us an opportunity to meet personally with many of the workers in this field and to update our knowledge of each other's activities. Some of the pioneer experimental groups in this area, notably at Harvard and at Rochester, are discontinuing such work. Since we expect to do no further analyses of these data at Livermore, and since we have considerable confidence in the present results, we have presented the results in Sec. IV in more detail than would otherwise have been the case.

We have made no effort in this paper to compare our phase-shift results with those carried out recently at Yale, Harwell, Dubna, and Kyoto. Our belief here is that all groups are in essential agreement with regard to the (p,p) elastic scattering matrix. It should be emphasized that with the accuracies now obtainable it is important to pay careful attention to details such as data normalization constants and matrix search procedures.

Note added in proof. Final C_{NN} data from the Chicago Group give normalizations that agree with the phase shift predictions at 305 and 330 MeV, but a discrepancy still remains at 386 and 415 MeV (see Paper VIII).

TABLE X. Values for $\chi^2(g^2)$ and the values for g^2, the pion-nucleon coupling constant, obtained from the 20-, 23-, and 30-parameter form-A solutions. The value $g^2=14.72\pm0.83$ is in our opinion the best value that we can obtain from the 839-point (p,p) elastic data collection.

g^2	20 parameter 14.27 ± 0.47	23 parameter 13.64 ± 0.52	30 parameter 14.72 ± 0.83
13	878.4	853.0	814.1
15	873.6	858.4	809.9
17	904.8	893.5	817.3
19	970.3	959.5	837.1
19[a]	(975.0)	(969.7)	(836.4)

[a] χ^2 predicted by fitting a parabola through the points at $g^2=13$, 15, and 17.

[19] V. K. Samaranayake and W. S. Woolcott, Phys. Rev. Letters **15**, 936 (1965).

ACKNOWLEDGMENTS

The first energy-dependent phase-shift analyses at Livermore were carried out as the MIDPOP program by M. J. Moravcsik, H. P. Noyes, and H. P. Stapp. Although the results of these investigations were reported at the High-Energy Conferences of 1960 and 1962, they were never published. The present work (papers I–IX in this series) was conceived and executed independently of the MIDPOP program. However, we have benefitted considerably from the experience gained by the MIDPOP workers, particularly with respect to the problems encountered in carrying out many-dimensional search procedures. Since our present work has now superseded the results obtained in the MIDPOP program, we have been asked by the MIDPOP workers to announce here that the MIDPOP results will not be published.

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