Approximations for Calculating Radiative Corrections

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We discuss a new approximation for calculating the radiative correction due to the emission of real photons. This approximation is an improvement on the previously used approximations, namely, the soft-photon and peaking approximations, both of which are also discussed.

INTRODUCTION

If we were to calculate exactly the radiative correction to a scattering process due to the radiation of undetected photons we would be faced by a double difficulty. First we would have to calculate the differential cross section $d\sigma^{\rm rad}/dl_0d\Omega$ for the radiation of a photon in a given direction and with given energy l_0. This task is usually extremely involved and the result inelegantly complicated. Secondly, we would have to integrate this complicated result over the photon angles and energy subject to specific kinematic restrictions, depending on the particular experimental situation at issue. This integration would most certainly be impossible to perform except by numerical methods.

Faced with such difficulties, it is natural to look for simplifying approximations. In the past, two such approximations have been used, namely, the soft-photon approximation[1–3] and the peaking approximation.[4–7] Since in most experiments the maximum energy of the radiated photon is restricted to be much less than the energy of the radiating particles, it is useful to expand $d\sigma^{\rm rad}/dl_0d\Omega$ in l_0. In the soft-photon approximation, only the first term in this expansion is kept. It is of order l_0^{-1}. The integration over $dl_0d\Omega$ is then trivial, except for the infrared divergence at $l_0=0$ which must be treated appropriately. The peaking approximation is an improvement of the soft-photon approximation in that it treats the terms of order l_0^{-1} as above and adds, in addition, an estimate of the higher-order terms in l_0. This estimate relies on the fact that a highly relativistic particle radiates dominantly in the direction of its motion. By considering only the dominant part, the integration over $d\Omega$ can be performed simply, leaving only the final integration over dl_0.

In this paper we present a new approximation which is an improvement on the above two approximations. It treats $d\sigma^{\rm rad}/dl_0d\Omega$ exactly to order l_0^0 and estimates the higher-order terms by way of the peaking approximation. It is an essential part of this approximation, as it is of the others, that the relevant part of $d\sigma^{\rm rad}/dl_0d\Omega$ can be written down by inspection in a simple form. Thereby the first difficulty mentioned above of finding $d\sigma^{\rm rad}/dl_0d\Omega$ is eliminated and the second difficulty of integrating over the photon angles and energy is simplified.

In the discussion below, we treat first the radiative correction to scattering processes involving two charged spin-zero particles. Thereafter, in Sec. II, we discuss spin-$\frac{1}{2}$ particles, and finally in Sec. III we show how the results are easily generalized to scattering processes involving four charged particles.

We are using the conventions of Bjorken and Drell.[8] The metric is $g_{00}=1$, $g_{ii}=-1$.

I. SPIN-ZERO PARTICLES

Consider a scattering process and its first-order real-photon radiative correction shown in Fig. 1. The scattering process involves an incoming and an outgoing pion of momentum p_1 and p_2, respectively, and any number of photons of total momentum k. Some of these photons may be virtual, as long as they connect to near static potentials which do not radiate significantly. Thus we are considering, among many others, such processes as Compton scattering and, at low momentum transfers to the nucleus, nuclear scattering and bremsstrahlung. Pair production at low momentum transfers is also included by way of the simple transformation $p_1\rightarrow -p_1$.

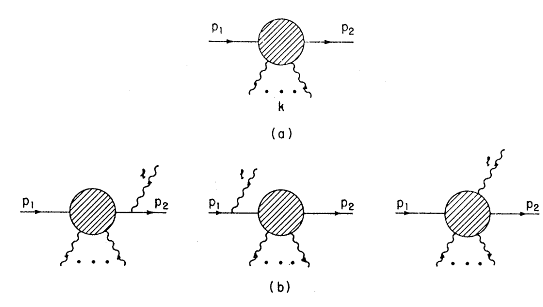

FIG. 1. (a) A general scattering process involving two charged particles and any number of incoming and outgoing, real, and virtual photons with total momentum k; and (b) its radiative corrections due to the radiation of a real photon.

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[1] Julian Schwinger, Phys. Rev. **76**, 790 (1949).
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[5] A. S. Krass, Phys. Rev. **125**, 2172 (1962).
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[7] R. Atkinson, III, thesis, Stanford University, 1965 (unpublished).
[8] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Co., New York, 1964).

The matrix elements corresponding, respectively, to the three diagrams of Fig. 1(b) are

$$M^{\mathrm{rad}}=\frac{2p_2\cdot e}{2p_2\cdot l}M(p_1,\,p_2+l)-\frac{2p_1\cdot e}{2p_1\cdot l}M(p_1-l,\,p_2)+e^\nu A_\nu. \qquad (1)$$

Here $M(p_1,p_2)$ is the matrix element corresponding to scattering without radiation [Fig. 1(a)], $l=(l_0,\mathbf{l})$ is the four-momentum of the radiated photon, and e its polarization vector. A_ν is of order l_0^0 or higher and, by gauge invariance, satisfies the following equality:

$$l^\nu A_\nu=M(p_1-l,\,p_2)-M(p_1,\,p_2+l). \qquad (2)$$

Squaring M^{rad} and summing over the polarizations of the radiated photon we get

$$\sum_{\mathrm{spins}}|M^{\mathrm{rad}}|^2=-\frac{m^2}{(p_1\cdot l)^2}|M(p_1-l,\,p_2)|^2-\frac{m^2}{(p_2\cdot l)^2}|M(p_1,\,p_2+l)|^2+\frac{2p_1\cdot p_2}{(p_1\cdot l)(p_2\cdot l)}\mathrm{Re}[M(p_1-l,\,p_2)M^*(p_1,\,p_2+l)]$$
$$-\frac{2p_2^\nu}{p_2\cdot l}\mathrm{Re}[M(p_1,\,p_2+l)A_\nu^*]+\frac{2p_1^\nu}{p_1\cdot l}\mathrm{Re}[M(p_1-l,\,p_2)A_\nu^*]-g^{\mu\nu}A_\mu A_\nu^*\,. \qquad (3)$$

The first three terms are of order l_0^{-2} or higher, the next two terms of order l_0^{-1} or higher, and the final term of order l_0^0 or higher.

Before considering Eq. (3) in detail, we want to discuss briefly the infrared divergence. The cross sections are connected to the matrix elements as follows:

$$d\sigma_0(p_1,p_2)=(\mathrm{kin})\,|M(p_1,p_2)|^2\delta^4(p_1-p_2-k)\,,$$
$$d\bar{\sigma}^{\mathrm{rad}}=\frac{\alpha}{\pi}\int_\delta dl_0\int\frac{l_0d\Omega}{4\pi}$$
$$\times[(\mathrm{kin})\textstyle\sum_{\mathrm{spins}}|M^{\mathrm{rad}}|^2\delta^4(p_1-p_2-k-l)]. \qquad (4)$$

$d\sigma_0$ is the cross section for the basic scattering process Fig. 1(a). We have expressed as (kin) the kinematic factors which are the same for $d\bar{\sigma}^{\mathrm{rad}}$ and $d\sigma_0$. These factors, of course, depend on the particular scattering process considered. The integral in Eq. (4) over dl_0 is cut off at $l_0=\delta$, where δ is arbitrarily small. For the remaining integration over dl_0 we follow Yennie, Frautschi, and Suura.[9] We give the photon a small mass λ and cancel the terms divergent as $\lambda\to 0$ against similar terms in the radiative correction due to the exchange of virtual photons. From such a procedure we find a number of terms, some of which are appropriately included with the virtual-photon radiative corrections. Finally, we may write the finite result

$$d\sigma^{\mathrm{rad}}=\lim_{\delta\to 0}\left[\frac{\alpha}{\pi}F(p_1,p_2)\ln\frac{\delta^2}{E_1E_2}d\sigma_0+d\bar{\sigma}^{\mathrm{rad}}\right], \qquad (5)$$

where

$$F(p_1,p_2)=\frac{2p_1\cdot p_2}{(p_1+p_2)^2}C\ln\left|\frac{C+1}{C-1}\right|-1\,,$$
$$C=[(p_1+p_2)^2/-(p_1-p_2)^2]^{1/2}. \qquad (6)$$

[9] D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N. Y.) **13**, 379 (1961).

In the limit that $(2p_1\cdot p_2)/m^2\gg 1$ (where m is the mass of the pion), we have

$$F(p_1,p_2)=\ln\left|\frac{2p_1\cdot p_2}{m^2}\right|-1\,. \qquad (7)$$

We are now in a position to consider exactly what is meant by the soft-photon approximation and the peaking approximation. In the soft-photon approximation we keep only terms of order l_0^{-2} in Eq. (3) and, in addition, ignore the l in the δ function of Eq. (4). The result is then simply

$$d\sigma^{\mathrm{rad}}(\mathrm{soft})=\frac{\alpha}{\pi}F(p_1,p_2)\ln[l_{0(\mathrm{max})}{}^2/E_1E_2]d\sigma_0\,, \qquad (8)$$

where $l_{0(\mathrm{max})}$ is the maximum energy of the radiated photon allowed kinematically. The essential assumption involved in the soft-photon approximation is that $d\sigma_0(p_1-l,\,p_2)$ and $d\sigma_0(p_1,\,p_2+l)$ are not significantly dependent on l for l in the kinematically allowed region. We will see this more clearly in what follows.

The peaking approximation has not been treated in a general manner in the literature, although variations of it have been used in particular situations.[4–7,10] We will therefore treat it in some detail here.

The approximation relies on the fact that the factors $1/(p_1\cdot l)$ and $1/(p_2\cdot l)$ in Eq. (3) peak strongly for $\mathbf{l}\|\mathbf{p}_1$ and $\mathbf{l}\|\mathbf{p}_2$, respectively, provided the pions are highly relativistic. Thus near $\mathbf{l}\|\mathbf{p}_1$ we have

$$2p_1\cdot l\cong E_1l_0[\theta_{p_1l}{}^2+m^2/E_1{}^2]\,, \qquad (9)$$

where $\cos\theta_{p_1l}=(\mathbf{p}_1\cdot\mathbf{l})/|\mathbf{p}_1|l_0$ and $m^2/E_1{}^2\ll 1$. To take advantage of this peaking, let us rewrite Eq. (3) as

[10] E. L. Lomon, Phys. Letters **21**, 555 (1966).

follows, using Eq. (2):

$$\sum_{\text{spins}} |M^{\text{rad}}|^2 = -\frac{m^2}{(p_1\cdot l)^2}\{|M(p_1-l,\,p_2)|^2\} - \frac{m^2}{(p_2\cdot l)^2}\{|M(p_1,\,p_2+l)|^2\} + \frac{2p_1\cdot p_2}{[(p_1+p_2)\cdot l](p_2\cdot l)}\,\text{Re}\{|M(p_1,\,p_2+l)|^2$$
$$-\left[p_2^{\nu}A_{\nu}\frac{(p_1+p_2)\cdot l}{p_1\cdot p_2} - l^{\nu}A_{\nu}\right]M^*(p_1,\,p_2+l)\} + \frac{2p_1\cdot p_2}{[(p_1+p_2)\cdot l](p_1\cdot l)}\,\text{Re}\{|M(p_1-l,\,p_2)|^2$$
$$+\left[p_1^{\nu}A_{\nu}\frac{(p_1+p_2)\cdot l}{p_1\cdot p_2} - l^{\nu}A_{\nu}\right]M^*(p_1-l,\,p_2)\} - g^{\mu\nu}A_{\mu}A_{\nu}^{*}. \quad (10)$$

The term $p_2^{\nu}A_{\nu}(p_1\cdot l)/(p_1\cdot p_2)-l^{\nu}A_{\nu}$ is of the order of $m^2/(p_1\cdot p_2)$ for $\mathbf{l}\|\mathbf{p}_2$, and $p_1^{\nu}A_{\nu}(p_2\cdot l)/(p_1\cdot p_2)-l^{\nu}A_{\nu}$ is of the order of $m^2/(p_1\cdot p_2)$ for $\mathbf{l}\|\mathbf{p}_1$. We now assume that because of the strong peaking of the factors in front of the curly brackets in Eq. (10), we may set $\mathbf{l}\|\mathbf{p}_1$ or $\mathbf{l}\|\mathbf{p}_2$ inside the brackets. In the same approximation we would ignore the $g^{\mu\nu}A_{\nu}A_{\nu}^{*}$ term, since it contains no peaking. Thus we get

$$\sum_{\text{spins}}|M^{\text{rad}}|^2$$
$$\cong\left[\frac{2p_1\cdot p_2}{(p_2+p_2)\cdot lp_2\cdot l} - \frac{m^2}{(p_2\cdot l)^2}\right]|M(E_1,\,E_2+l_0)|^2$$
$$+\left[\frac{2p_1\cdot p_2}{(p_1+p_2)\cdot lp_1\cdot l} - \frac{m^2}{(p_1\cdot l)^2}\right]|M(E_1-l_0,\,E_2)|^2$$
$$+O\left[\frac{1}{(p_1+p_2)\cdot l}\right]|M(p_1,p_2)|^2. \quad (11)$$

The third term indicates the order of magnitude of the ignored terms. We have assumed implicitly that $1/[(p_1+p_2)\cdot l]$ does *not* peak. Upon performing the angular integration in Eq. (4), using the approximate expression of Eq. (11), we get

$$d\bar{\sigma}^{\text{rad}}\,(\text{peaking}) = \frac{\alpha}{\pi}\left(\ln\left|\frac{2p_1\cdot p_2}{m^2}\right| - 1\right)$$
$$\times\left\{\int_{\delta}^{l_0(\max 1)}\frac{dl_0}{l_0}d\sigma_0(E_1-l_0,\,E_2)(E_1-l_0)/E_1\right.$$
$$\left.+\int_{\delta}^{l_0(\max 2)}\frac{dl_0}{l_0}d\sigma_0(E_1,\,E_2+l_0)E_2/(E_2+l_0)\right\}. \quad (12)$$

The notation $l_{0(\max 1)}$ or $l_{0(\max 2)}$ indicates that, in general, the maximum energy of the photon depends on its direction of motion, which has been taken here to be dominantly parallel to $\mathbf{p}_1$ or $\mathbf{p}_2$. The factors $(E_1-l_0)/E_1$ and $E_2/(E_2+l_0)$ are included because the substitutions $E_1\to E_1-l_0$ and $E_2\to E_2+l_0$ in the basic cross section $d\sigma_0(E_1,E_2)$ change the factors (kin). [See Eq. (4).] The change is cancelled out by the above factors, as it should be. Upon integration of the third term of Eq. (11), we find finally that a necessary condition for the peaking approximation to be valid is

$$\ln\frac{2p_1\cdot p_2}{m^2} \gg \ln\frac{4(E_1+E_2)^2}{2p_1\cdot p_2}. \quad (13)$$

This condition is rather restrictive. However, the peaking approximation is really only applied to terms in Eq. (3) of order l_0^{-1} or higher, thereby somewhat relaxing the condition of Eq. (13). To see this, let $l_0\to 0$ in Eq. (12) except in the factor $1/l_0$. Then Eq. (12) reduces to the soft-photon approximation. [See Eqs. (5), (7), and (8).] Thus we have also shown that the soft-photon approximation depends essentially on $d\sigma_0(E_1-l_0,\,E_2)$ and $d\sigma_0(E_1,\,E_2+l_0)$] varying slowly with l_0.

Let us now proceed further in improving our approximation of $\sum|M^{\text{rad}}|^2$ by noting that A_{ν} may be written as follows:

$$A_{\nu} = -\left[\frac{\partial M(p_1,p_2)}{\partial p_1^{\nu}} + \frac{\partial M(p_1,p_2)}{\partial p_2^{\nu}}\right] + O(l_0). \quad (14)$$

This is a result of the Ward identity. The term $e^{\nu}A_{\nu}$ corresponds to the third diagram of Fig. 1(b); and, to order l_0^0, we may think of it as equal to the diagram of Fig. 1(a) with insertions of zero-energy photons into every internal propagator. There is no contribution from making such insertions into closed loops, and therefore only insertions along the pion line originating in the incoming pion and terminating in the outgoing pion are necessary. Every propagator along this line contains either p_1 or p_2. Therefore, by the Ward identity, the differentiation of Eq. (14) inserts a zero-energy photon into every one of these propagators. From Eqs. (3) and (14) we can now write $\sum|M^{\text{rad}}|^2$

as follows, good to order l_0^{-1}:

$$\sum_{\text{spins}} |M^{\text{rad}}|^2 = -\frac{m^2}{(p_1\cdot l)^2}|M(p_1-l,\,p_2)|^2 \times -\frac{m^2}{(p_2\cdot l)^2}|M(p_1,\,p_2+l)|^2 + \frac{2p_1\cdot p_2}{(p_1\cdot l)(p_2\cdot l)}\operatorname{Re}\left\{M(p_1-l,\,p_2)M^*(p_1,\,p_2+l) + \left(\frac{2p_2{}^\nu}{p_2\cdot l}-\frac{2p_1{}^\nu}{p_1\cdot l}\right)M^*(p_1,p_2) \times\left[\frac{\partial M(p_1,p_2)}{\partial p_1{}^\nu}+\frac{\partial M(p_1,p_2)}{\partial p_2{}^\nu}\right]\right\}+O(l_0{}^0). \quad (15)$$

Expanding $M(p_1-l,\,p_2)$ and $M(p_1,\,p_2+l)$ by means of the Taylor series and recombining terms, we get

$$\sum |M^{\text{rad}}|^2 = -\frac{m^2}{(p_1\cdot l)^2}|M(p_1-l,\,p_2)|^2 - \frac{m^2}{(p_2\cdot l)^2}|M(p_1,\,p_2+l)|^2 + \frac{2p_1\cdot p_2}{(p_1\cdot l)(p_2\cdot l)}|M(\bar{p}_1,\bar{p}_2)|^2+O(l_0{}^0), \quad (16)$$

where

$$\bar{p}_1 = p_1 - \tfrac{1}{2}l - \tfrac{1}{2}p_1(p_2\cdot l)/(p_1\cdot p_2) + \tfrac{1}{2}p_2(p_1\cdot l)/(p_1\cdot p_2),$$
$$\bar{p}_2 = p_2 + \tfrac{1}{2}l + \tfrac{1}{2}p_2(p_1\cdot l)/(p_1\cdot p_2) - \tfrac{1}{2}p_1(p_2\cdot l)/(p_1\cdot p_2). \quad (17)$$

Again we have expressed $\sum|M^{\text{rad}}|^2$ in terms of the known matrix element squared, $|M(p_1,p_2|^2$. There are two points to be made about the substitutions of Eq. (17). First, they reproduce the kinematics exactly:

$$\delta^4(p_1-p_2-k)\rightarrow\delta^4(\bar{p}_1-\bar{p}_2-k)=\delta^4(p_1-p_2-l-k). \quad (18)$$

Second, in the limits of $\mathbf{l}\|\mathbf{p}_1$ and $\mathbf{l}\|\mathbf{p}_2$, we have

$$\bar{p}_1\rightarrow p_1-l;\quad \bar{p}_2\rightarrow p_2 \quad \text{for } \mathbf{l}\|\mathbf{p}_1,$$
$$\bar{p}_2\rightarrow p_2+l;\quad \bar{p}_1\rightarrow p_1 \quad \text{for } \mathbf{l}\|\mathbf{p}_2. \quad (19)$$

Applying the peaking approximation to Eq. (16), we find, using Eq. (19), that the answer reduces to Eq. (11). From this we conclude that the terms of order $l_0{}^0$ ignored in Eq. (16) must be zero in the peaking approximation. Therefore, we have found a simple approximation which reproduces $\sum|M^{\text{rad}}|^2$ to order l_0^{-1} and automatically estimates the terms of higher orders in l_0 by means of the peaking approximation. This expression must be substituted into Eq. (4) and the required integrations performed. These integrations are, in general, difficult, and must sometimes be performed numerically. Apparently, we must pay for each improvement in the approximation by having to perform a more difficult final integration. This is hardly surprising. Nonetheless, Eq. (16) is still remarkably simple, considering the general complexity of calculating the radiative correction exactly.

II. SPIN-$\frac{1}{2}$ PARTICLES

In this section we extend our results to the case of lepton scattering. Consider again the diagrams of Fig. 1 where now the charged particles are leptons. The corresponding matrix elements are

$$M^{\text{rad}} = \bar{U}(p_2)\boldsymbol{e}(\boldsymbol{p}_2+\boldsymbol{l}-m)^{-1}\Gamma(p_1,\,p_2+l)U(p_1) + \bar{U}(p_2)\Gamma(p_1-l,\,p_2)(\boldsymbol{p}_1-\boldsymbol{l}-m)^{-1}\boldsymbol{e}U(p_1) + e^\nu\bar{U}(p_2)B_\nu U(p_1). \quad (20)$$

By the Ward identity

$$B_\nu = -\left[\frac{\partial\Gamma(p_1,p_2)}{\partial p_1{}^\nu}+\frac{\partial\Gamma(p_1,p_2)}{\partial p_2{}^\nu}\right]+O(l_0). \quad (21)$$

We now square M^{rad}, and sum over the polarizations of the radiated photon and over the spins of the leptons. Using Eq. (21) and the identities

$$(\boldsymbol{p}_1-\boldsymbol{l}-m)^{-1}\boldsymbol{e}U(p_1) = -(2p_1\cdot l)^{-1}(2p_1\cdot e-\boldsymbol{le})U(p_1)$$

and

$$\bar{U}(p_2)\boldsymbol{e}(\boldsymbol{p}_2+\boldsymbol{l}-m)^{-1} = (2p_2\cdot l)^{-1}\bar{U}(p_2)(2p_2\cdot e+\boldsymbol{el}).$$

we obtain, after some manipulation,

$$\sum_{\text{spins}}|M^{\text{rad}}|^2 = -[m^2/(p_1\cdot l)^2]\operatorname{Tr}\{(\boldsymbol{p}_2+m)\Gamma(p_1-l,\,p_2)[\boldsymbol{p}_1-\boldsymbol{l}+\surd(p_1-l)^2]\bar{\Gamma}(p_1-l,\,p_2)\}$$
$$-[m^2/(p_2\cdot l)^2]\operatorname{Tr}\{[\boldsymbol{p}_2+\boldsymbol{l}+\surd(p_2+l)^2]\Gamma(p_1,\,p_2+l)(\boldsymbol{p}_1+m)\bar{\Gamma}(p_1,\,p_2+l)\}$$
$$+\frac{2p_1\cdot p_2}{(p_1\cdot l)(p_2\cdot l)}\operatorname{Tr}[(\boldsymbol{p}_2+\surd\bar{p}_2{}^2)\Gamma(\bar{p}_1,\,\bar{p}_2)(\boldsymbol{p}_1+\surd\bar{p}_1{}^2)\bar{\Gamma}(\bar{p}_1,\,\bar{p}_2)]+O(l_0{}^0). \quad (22)$$

$\bar{p}_1$ and $\bar{p}_2$ are given by Eq. (17). Notice that

$$\bar{p}_1{}^2 = m^2\left(1-\frac{l\cdot p_2}{p_1\cdot p_2}\right)+O(l_0{}^2),\qquad \bar{p}_2{}^2 = m^2\left(1+\frac{l\cdot p_1}{p_1\cdot p_2}\right)+O(l_0{}^2). \quad (23)$$

Squaring the matrix element corresponding to diagram Fig. 1(a), and summing over spins, we get

$$\sum_{\text{spins}} |M|^2 = L(p_1,p_2) = \mathrm{Tr}[(\not{p}_2+\sqrt{p_2^2}) \times \Gamma(p_1,p_2)(\not{p}_1+\sqrt{p_1^2})\bar{\Gamma}(p_1,p_2)]. \quad (24)$$

Now, writing $\sum|M^{\mathrm{rad}}|^2$ in terms of the known function $L(p_1,p_2)$ we have

$$\sum_{\text{spins}} |M^{\mathrm{rad}}|^2 = -\frac{m^2}{(p_1\cdot l)^2}L(p_1-l,\, p_2) - \frac{m^2}{(p_2\cdot l)^2}L(p_1,\, p_2+l) + \frac{2p_1\cdot p_2}{(p_1\cdot l)(p_2\cdot l)}L(\bar{p}_1,\bar{p}_2)+O(l_0^0). \quad (25)$$

This is the same result as for pions, Eq. (16). If m^2 terms are important in $L(p_1,p_2)$ it is necessary, when calculating $L(p_1,p_2)$, to discriminate between p_1^2, p_2^2, and m^2 if the above substitutions are to give the correct answer. This is obviously true since

$$p_1^2 \to m^2\left(1-\frac{l\cdot p_2}{p_1\cdot p_2}\right),$$

while $m^2 \to m^2$. The same statement holds for pions. However, if m^2 terms in $L(p_1,p_2)$ are negligible, as they must be if the peaking approximation is to be used, then we need not concern ourselves with this problem.

Next we might hope to prove that the terms of order l_0^0 in Eq. (25) are zero in the peaking approximation just as in the pion case. Unfortunately, this is not so. In the Appendix we consider the peaking approximation for leptons and find the following, slightly modified form of Eq. (12):

$$d\bar{\sigma}^{\mathrm{rad}}\text{ (peaking)} = \frac{\alpha}{\pi}\left\{\int_\delta^{l_0(\max 1)} \frac{dl_0}{l_0} d\sigma_0(E_1-l_0,\, E_2)\frac{E_1-l_0}{E_1} \times\left[\left(1+\tfrac{1}{2}\frac{l_0^2}{E_1(E_1-l_0)}\right)\ln\left|\frac{2p_1\cdot p_2}{m^2}\right|-1\right] + \int_\delta^{l_0(\max 2)} \frac{dl_0}{l_0} d\sigma_0(E_1,\, E_2+l_0)\frac{E_2}{E_2+l_0} \times\left[\left(1+\tfrac{1}{2}\frac{l_0^2}{E_2(E_2+l_0)}\right)\ln\left|\frac{2p_1\cdot p_2}{m^2}\right|-1\right]\right\}, \quad (26)$$

where $d\sigma_0(p_1,p_2)$ is the basic cross section corresponding to Fig. 1(a). We may incorporate this modification into Eq. (25) as follows:

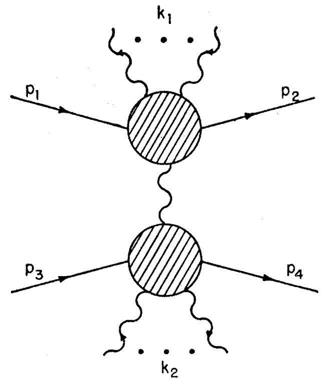

FIG. 2. A general scattering process involving four charged particles and any number of photons of total momentum k_1 and k_2 are shown.

$$\sum_{\text{spins}} |M^{\mathrm{rad}}|^2 = -\frac{m^2}{(p_1\cdot l)^2}L(p_1-l,\, p_2) - \frac{m^2}{(p_2\cdot l)^2}L(p_1,\, p_2+l) + \frac{2p_1\cdot p_2}{(p_1\cdot l)(p_2\cdot l)} \times\left(1+\tfrac{1}{2}\frac{(p_1\cdot l)^2+(p_2\cdot l)^2}{(p_1\cdot p_2)(\bar{p}_1\cdot\bar{p}_2)}\right)L(\bar{p}_1,\bar{p}_2)+O(l_0^0), \quad (27)$$

where now the ignored terms of order l_0^0 or higher are zero in the peaking approximation.

For an example of the application of Eq. (27), see Ref. 11.

III. FOUR CHARGED PARTICLES

Consider now the diagram of Fig. 2 involving four charged particles and any number of photons. To calculate the radiative correction to this diagram, we must consider the possibility of the photon radiating from any of the four external lines or from either of the two vertices. For simplicity, let the charged particles be pions. The extension to leptons is simple, from what has been said above. The calculation proceeds as in Sec. I; here we merely state the result. Define the following functions:

$$Y_i = \frac{m^2}{(p_i\cdot l)^2}|M(p_i\pm l)|^2,$$
$$X_{ij} = \frac{2p_i\cdot p_j}{(p_i\cdot l)(p_j\cdot l)}|M(\bar{p}_i,\bar{p}_j)|^2, \quad (28)$$

where

$$\bar{p}_i = p_i \pm \tfrac{1}{2}l \pm \tfrac{1}{2}p_i(p_j\cdot l)/(p_i\cdot p_j) \mp \tfrac{1}{2}p_j(p_i\cdot l)/(p_i\cdot p_j), \quad (29)$$
$$\bar{p}_j = p_j \pm \tfrac{1}{2}l \pm \tfrac{1}{2}p_j(p_i\cdot l)/(p_i\cdot p_j) \mp \tfrac{1}{2}p_i(p_j\cdot l)/(p_i\cdot p_j).$$

[11] B. Huld, Phys. Rev. 168, 1782 (1968).

Here $[(p_1-p_2-k_1)^{-2}]M(p_1,p_2,p_3,p_4)$ is the matrix element corresponding to Fig. 2. The substitutions of Eq. (29) always occur in pairs. In the expression for X_{ij}, we show only the pair of momenta in $M(p_1,p_2,p_3,p_4)$ which is modified. The choices of signs in Eqs. (28) and (29) depend on whether the momentum being modified belongs to an incoming or an outgoing particle, the top signs corresponding to the outgoing case. With these definitions we may write for the radiative correction to the process of Fig. 2.

$$\sum_{\text{spins}} |M^{\text{rad}}|^2 = -(Y_1+Y_2)[1/(p_1-p_2-k_1-l)^4]-(Y_3+Y_4) \times[1/(p_1-p_2-k_1)^4]+X_{12}[1/(p_1-p_2-k_1-l)^4] +X_{34}[1/(p_1-p_2-k_1)^4]+\{(X_{14}-X_{13}+X_{23}-X_{24}) \times[1/(p_1-p_2-k_1-l)^2(p_1-p_2-k_1)^2]\}+O(l_0{}^0). \quad (30)$$

In the peaking approximation, the term in the curly brackets goes to zero. The ignored terms are then zero in the peaking approximation for pions. For electrons, we would make the appropriate modification of X_{12} and X_{34}, while if p_1 and p_2 were electrons and p_3 and p_4 pions, we would modify only X_{12}. In the curly brackets of Eq. (30), the exact denominator $(p_1-p_2-k_1-l)^2 \times(p_1-p_2-k_1)^2$ is not the same as we would obtain by applying the substitutions of Eq. (29) to $(p_1-p_2-k_1)^4$ even to order l_0. Thus, in applying the work of this section to such complicated processes as trident production, care must be taken to treat the various momentum transfers correctly.

The fact that the problem of calculating the radiative corrections is separable into parts involving pairs of charged particles, facilitates a combined use of the various approximations discussed in this paper, depending on the behavior of the cross section with respect to each separate pair. This may help somewhat in doing complicated problems, such as trident production.

APPENDIX

In this Appendix we want to consider Eq. (20) in the peaking approximation, in order to demonstrate the validity of Eq. (26). Our proof here will follow a different path than the proof used for pions in Sec. I. An essential difference is the use of a different gauge for the radiated photon. Since the individual terms of Eq. (20) are not separately gauge invariant, a change of gauge will change the contribution from each separate term, though the total will, of course, remain the same. Thus, for transversely polarized photons used here, the main contribution will arise from the square of each of the first two terms of Eq. (20), rather than from the interference between the two as in Eq. (3).[12]

[12] D. R. Yennie, *Lectures on Strong and Electromagnetic Interactions. Brandeis Summer Institute, 1963* (Brandeis University, Waltham, Mass., 1963), Vol. 1.

Consider the first term of Eq. (20), and write it as follows:

$$M_1{}^{\text{rad}}=\bar{U}^{(2)}(p_2)e\{[p_2+l+m'+(m-m')]/2p_2\cdot l\} \times\Gamma(p_1,p_2+l)U^{(1)}(p_1), \quad \text{(A1)}$$

where $(m')^2=(p_2+l)^2$. Let p_1 and p_2 be highly relativistic. Then we may ignore the mass of the leptons in Eq. (A1), which implies that $\bar{U}^{(2)}(p_2)$ and $U^{(1)}(p_1)$ must have definite relative helicities, either the same helicities or opposite helicities, depending on the number of γ matrices in $\Gamma(p_1,p_2)$. Then we know, for example, that

$$\bar{U}^{(2)}(p_2)e\Gamma U^{(1)}(p_1)\sim m. \quad \text{(A2)}$$

Now expand p_2+l+m' in terms of spinors, and write

$$M_1{}^{\text{rad}}\cong[1/(2p_2\cdot l)]\bar{U}^{(2)}(p_2)eU^{(2)}(p_2+l)\bar{U}'^{(2)}(p_2+l) \times\Gamma(p_1,\, p_2+l)U^{(1)}(p_1), \quad \text{(A3)}$$

where we have ignored only terms of order $m^2/(p_2\cdot l)$ and $mm'/(p_2\cdot l)$. Here $\bar{U}^{(2)}(p_2+l)$ has the same helicity as $\bar{U}^{(2)}(p_2)$. Eq. (A3) may be rewritten as

$$M_1{}^{\text{rad}}\cong f_1(p_1,p_2)M(p_1,\, p_2+l), \quad \text{(A4)}$$

where $M(p_1,p_2)$ is the matrix element corresponding to scattering without radiation, and $f_1(p_1,p_2)$ is a known factor. In order to calculate f_1, we write it as follows:

$$f_1(p_1,p_2)=\bar{U}^{(2)}(p_2)eU^{(2)}(p_2+l)\bar{U}^{(2)}(p_2+l) \times\gamma_0U^{(2)}(p_2)\{2p_2\cdot l[\bar{U}^{(2)}(p_2+l)\gamma_0U^{(2)}(p_2)]\}^{-1}. \quad \text{(A5)}$$

We are interested in $f_1(p_1,p_2)$ only in the peaking approximation. Therefore we expand f_1 in θ, the angle between $\mathbf{p}_2$ and $\mathbf{l}$, and keep only first-order terms in θ. Using the expressions $e\cdot l=0$, $e\cdot p_2\cong -E_2\theta$, and $2p_2\cdot l \cong E_2l_0[\theta^2+m^2/E_2{}^2]$ we get (up to an over-all phase)

$$f_1(p_1,p_2)\cong\frac{E_2\theta}{2p_2\cdot l}\left\{\frac{(2E_2+l_0)-il_0}{[E_2(E_2+l_0)]^{1/2}}\right\}; \quad \text{(A6)}$$

and squaring,

$$|f_1(p_1,p_2)|^2\cong\frac{2E_2}{l_0(p_2\cdot l)}\left[1+\tfrac{1}{2}\frac{l_0{}^2}{E_2(E_2+l_0)}\right]. \quad \text{(A7)}$$

It remains to be shown that no other terms enter for $\mathbf{l}\|\mathbf{p}_2$ as long as the condition of Eq. (13) holds. $|f_1(p_1,p_2)|^2$ goes like $[\theta^2+(m^2/E_2{}^2)]^{-1}$ near $\mathbf{l}\|\mathbf{p}_2$, and, when integrated over the direction of $\mathbf{l}$ [see Eq. (4)], it goes like $\ln(E_2/m)$. The terms neglected above go like $m^2/(\theta^2+m^2/E_2{}^2)$ and $mm'/(\theta^2+m^2/E_2{}^2)$. These terms can indeed be ignored in the peaking approximation Eq. (13). The interference terms between $M_1{}^{\text{rad}}$ and other terms in Eq. (20), go like $\theta/(\theta^2+m^2/E_2{}^2)$ near $\mathbf{l}\|\mathbf{p}_2$, and can also be neglected.

A similar argument applies near $\mathbf{l}\|\mathbf{p}_1$. Then Eq. (26) is proven.

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