of $|LL_z\rangle|SS_z\rangle$ states. If the resonance belongs to a **56** representation of $SU(6)$, we then symmetrize $|LL_z\rangle$. The **56** spin-isospin wave function is next constructed by taking linear combinations of product states [such as $\alpha(1)\beta(2)\alpha(3)\beta'(1)\alpha'(2)\alpha'(3)$] and requiring symmetry under exchange of any two quarks. The result for $S=S_z=+\frac{1}{2}$ and $I=I_z=+\frac{1}{2}$ has been given [see Eq. (5)]. This yields the total wave function for the **56** case.

If the resonance belongs to a **70** representation, then instead of completely symmetrizing $|LL_z\rangle$, we form the analogs of u_a and u_b:

$$\phi_a=(\sqrt{\tfrac{1}{2}})[\phi_{\text{ex}}(1)\phi_0(2)\phi_0(3)-\phi_0(1)\phi_{\text{ex}}(2)\phi_0(3)]$$

and

$$\phi_b=(\sqrt{\tfrac{1}{6}})[\phi_{\text{ex}}(1)\phi_0(2)\phi_0(3)+\phi_0(1)\phi_{\text{ex}}(2)\phi_0(3)-2\phi_0(1)\phi_0(2)\phi_{\text{ex}}(3)].$$

(ϕ_{ex} denotes an excited state.) These functions are then multiplied by appropriate **70** spin-isospin functions When $S=S_z=I=I_z=+\frac{1}{2}$, for example, the result is

$$\Psi=(\sqrt{\tfrac{1}{4}})[\phi_a(u_af_b+u_bf_a)+\phi_b(u_af_a-u_bf_b)].$$

(This function is totally symmetric under interchange of any two quarks.)

After obtaining the wave functions, we remove any c.m. motion, as described in the text.

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Momentum Transfer Dispersion Relations for Three-Particle Potential Scattering Amplitudes*

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The analytic properties in momentum transfer of a class of three-particle scattering amplitudes are investigated in this paper. The amplitudes considered are those in which there is a two-particle bound state in both the initial and final state. The following results are obtained: (1) The amplitudes are analytic inside a Lehmann ellipse in the scattering angle for all real energies. (2) For real energies below the three-free-particle threshold, the amplitudes are analytic in the momentum transfer plane except for real left- and right-hand cuts.

I. INTRODUCTION

A KNOWLEDGE of the analytic properties of multiparticle scattering amplitudes is central to a complete S-matrix theory calculation of hadron parameters and also to the extended phenomenological analysis of their reaction processes. While no dispersion relations for relativistic multiparticle scattering amplitudes have been rigorously established, some progress towards this goal has been made in the laboratory of potential scattering. In particular, dispersion relations in the total energy for fixed directions of the individual momenta have been proved for nonrelativistic three-particle scattering amplitudes.[1]

In this paper, we investigate the analytic properties in the momentum transfer variable of a class of three-particle scattering amplitudes. The class of amplitudes we consider are those which describe the elastic scattering of a single particle from a bound state and those which describe rearrangement collisions in which two of the initial and final particles are in a bound state. We will refer to these as "bound-state amplitudes." This class of processes is distinguished by having a single well-defined momentum transfer. The present study is restricted to the study of spinless, nonrelativistic particles which interact via two-body central potentials, which can be written as a superposition of Yukawa potentials.

In Sec. II we extend a result of Immirzi[2] to show that the bound-state amplitudes are analytic functions of the scattering angle inside a Lehmann ellipse for all real values of the energy E. In Sec. III we study the analytic properties in the scattering angle of all perturbation-theory diagrams. For real energies below the three-free-particle threshold we find that each of the perturbation-theory diagrams is analytic in the entire scattering-angle plane with the exception of cuts along the positive and negative real axes. In the remaining sections we show that these analytic properties are enjoyed by the full amplitude as well. In Sec. IV we prove this result for the individual terms in the Fredholm expansion of the Faddeev equations, and in Sec. V we show that this expansion converges uniformly in the domain of analyticity. Thus, the result holds for the full scattering amplitude. The conclusion of this paper is that the nonrelativistic scattering amplitudes with a

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[1] M. H. Rubin, R. L. Sugar, and G. Tiktopoulos, Phys. Rev. **146**, 1130 (1966); **159**, 1348 (1967); **162**, 1555 (1967). These papers will be referred to as I, II, and III, respectively.

[2] G. Immirzi, Nuovo Cimento **34**, 1361 (1964).

bound state in the initial and final states are analytic in the cut momentum transfer plane for all real energies below the three-free-particle threshold and inside a Lehmann ellipse for all real energies.

II. LEHMANN ELLIPSE

In this section, we show that the bound-state scattering amplitude is analytic inside an ellipse in the scattering-angle plane. This result was obtained by Immirzi[2] for the particular case when the initial and final bound states are spinless. The result is generalized here to bound states of arbitrary spin following his method of proof closely. The representation of the scattering amplitude obtained in the course of the demonstration is useful for the later proof of analyticity in a larger domain.

For simplicity in the following discussion, we consider only the scattering of particles of equal mass M and choose a system of units in which $\hbar^2/2M=1$. The generalization to the case of unequal masses is straightforward. We shall also take the two-body potentials between the particles to be the same, but this is not essential to our argument. Since we are dealing with a superposition of Yukawa potentials, we have the momentum-space representation

$$\langle \mathbf{k}'|V|\mathbf{k}\rangle=\int_{\mu_0}^{\infty} d\mu\,\alpha(\mu)[(\mathbf{k}'-\mathbf{k})^2+\mu^2]^{-1}. \qquad (2.1)$$

The letters $\alpha\beta\gamma\cdots$ will label the three scattering particles and B, B', B'', $\cdots$ the two-body bound states of the potential V. If the bound state is between particles β and γ we will denote it by B_α, where α is the particle not in the bound state. The bound-state scattering amplitudes are thus written

$$\langle B_\alpha'\mathbf{k}'|T(E)|B_\beta\mathbf{k}\rangle, \qquad (2.2)$$

where $\mathbf{k}$ is the initial momentum of the free particle β, and $\mathbf{k}'$ the final momentum of the free particle α. Often we will abbreviate this to

$$T^{(\alpha\beta)}(\mathbf{k}',\mathbf{k};E). \qquad (2.2')$$

The Faddeev equation may be written in the notation of Ref. 3 as

$$T_{\alpha\delta}=\hat{t}_\alpha\delta_{\alpha\delta}+\hat{t}_\alpha G_0(T_{\beta\delta}+T_{\gamma\delta}). \qquad (2.3)$$

Here, G_0 is the free three-particle Green's function and $\hat{t}_\alpha$ is the off-energy-shell two-body amplitude in which particle α does *not* participate in the scattering. The amplitudes $T_{\alpha\beta}$ are those for the scattering of three free particles in which particle α does *not* interact first (in the sense of the diagrams corresponding to an individual term in the iteration of the Faddeev equations) and particle β does not interact last. The bound-state scattering amplitudes may then be written in a form of which the following expressions are typical:

$$T^{(33)}(\mathbf{k}',\mathbf{k},E)=\langle B_3'\mathbf{k}'|\sum_{\alpha,\beta\neq 3} T_{\alpha\beta}|B_3\mathbf{k}\rangle, \qquad (2.4)$$

$$T^{(23)}(\mathbf{k}',\mathbf{k},E)=\langle B_2'\mathbf{k}'|V_3+\sum_{\alpha\neq 3,\beta\neq 2} T_{\alpha\beta}|B_3\mathbf{k}\rangle, \qquad (2.5)$$

where V_3 is the potential between particles 1 and 2. It is convenient to write

$$T_{\alpha\beta}=V_\alpha\delta_{\alpha\beta}+V_\alpha g_{\alpha\beta}V_\beta. \qquad (2.6)$$

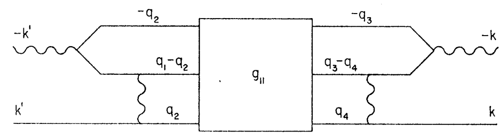

FIG. 1. The contribution to the amplitude $T^{(33)}(\mathbf{k}',\mathbf{k},E)$ arising from the Green's function g_{11}. The low-order perturbation-theory diagrams which together with this diagram make up $T^{(33)}$ are shown in Fig. 8.

The various types of terms which can occur are shown diagrammatically in Fig. 1. We will now prove analyticity in a Lehmann ellipse for each of the terms contributing to Eqs. (2.4) and (2.5). We consider first the contribution to $T^{(33)}$ of the form $V_1g_{11}V_1$ illustrated in Fig. 1(a) which we will here denote as J. To evaluate J we use the representation proved in Appendix A for the wave function of a bound state with binding energy $B>0$, angular momentum l, and angular momentum projection m.

$$\psi_{Blm}(\mathbf{p})=H_l^m(\mathbf{p})\int_B^\infty \frac{\sigma(\kappa)}{\kappa+2\mathbf{p}^2}d\kappa. \qquad (2.7)$$

Here, $H_l^m(\mathbf{p})=|\mathbf{p}|^lY_l^m(\hat{p})$ is a harmonic polynomial in the components of $\mathbf{p}$ of order l. The weight function has a δ-function contribution at $\kappa=B$ and a continuum beginning at $\kappa=(\mu_0+B^{1/2})^2$, where μ_0 is the minimum range of the potential. To simplify the derivation, we will consider only a single Yukawa potential of range μ and unit coupling. We then have for J,

$$J=\int d^3q_1\int d^3q_2\int d^3q_3\int d^3q_4\int_{B'}^\infty \sigma_3(\kappa')d\kappa'\int_B^\infty \sigma_3(\kappa)d\kappa$$
$$\times H_{l'}^{m'}(-\mathbf{q}_1+\tfrac{1}{2}\mathbf{k}')^*H_l^m(-\mathbf{q}_3+\tfrac{1}{2}\mathbf{k})g_{11}(\mathbf{q}_1,\mathbf{q}_2;\mathbf{q}_3,\mathbf{q}_4)$$
$$\times\{[\kappa'+2(\tfrac{1}{2}\mathbf{k}'-\mathbf{q}_1)^2]\cdot[(\mathbf{k}'-\mathbf{q}_2)^2+\mu^2]$$
$$\cdot[(\mathbf{k}-\mathbf{q}_4)^2+\mu^2][\kappa+2(\tfrac{1}{2}\mathbf{k}-\mathbf{q}_4)^2]\}^{-1}. \qquad (2.8)$$

In order to rewrite this integral for J, we employ the coordinate system originally used by Lehmann,

$$\mathbf{k}=k(0,0,1),$$
$$\mathbf{k}'=k'(0,\sin\theta,\cos\theta), \qquad (2.9)$$
$$\mathbf{q}_i=q_i(\cos\beta_i,\sin\beta_i\cos\alpha_i,\sin\beta_i\sin\alpha_i).$$

[3] This representation has been previously obtained for the case of S-wave bound states by A. Martin, Nuovo Cimento **14**, 403 (1959); R. Blankenbecler and L. F. Cook, Phys. Rev. **119**, 1745 (1960); and L. Bertocchi, C. Ceolin, and M. Tonin, Nuovo Cimento **18**, 770 (1960).

We may then write

$$J=\int\prod d^3q_i\int d\kappa\int d\kappa' H_{l'}{}^{m'}(-\mathbf{q}_1+\tfrac{1}{2}\mathbf{k}')^*$$
$$\times H_l{}^m(-\mathbf{q}_3+\tfrac{1}{2}\mathbf{k})g_{11}{}^\star[(\lambda_1-\cos\alpha_1)(\lambda_2-\cos\alpha_2)$$
$$\times(\lambda_3-\cos(\alpha_3+\theta))(\lambda_4-\cos(\alpha_4+\theta))]^{-1}, \quad (2.10)$$

where $g_{11}{}^\star$ is a new function of k, k', and the $\mathbf{q}_i$ but not of θ; and the λ_i are

$$\begin{aligned}\lambda_1&=(\kappa'+2q_1^2+\tfrac{1}{2}k'^2)/2q_1k'\sin\beta_1,\\ \lambda_2&=(\mu^2+q_2^2+k'^2)/2q_2k'\sin\beta_2,\\ \lambda_3&=(\kappa'^2+2q_3^2+\tfrac{1}{2}k^2)/2q_3k\sin\beta_3,\\ \lambda_4&=(\mu^2+q_4^2+k^2)/2q_4k\sin\beta_4.\end{aligned} \quad (2.11)$$

As a consequence of the rotational invariance of the full Green's function, a rotation of all vectors $\mathbf{q}_i$ about the z axis leaves $g_{11}{}^\star$ unchanged. Hence, $g_{11}{}^\star$ depends only on the differences of the α_i and one angular integration can be performed, say α_1. If one puts $W=\exp(i\alpha_1)$ and expresses $g_{11}{}^\star$ so that it is independent of α_1, then the integration over α_1 reduces to the evaluation of

$$I=\frac{1}{2\pi i}\oint dW\, W^a\prod_{i=1}^4 e^{-i\gamma_i}[(W-W_i^+)$$
$$\times(W-W_i^-)]^{-1}, \quad (2.12)$$

where the contour runs around the unit circle and

$$\begin{aligned}&W_i^\pm=[\lambda_i\pm(\lambda_i^2-1)^{1/2}]e^{-i\gamma_i},\\ &\gamma_1=0,\qquad \gamma_3=\theta+\alpha_3-\alpha_1,\\ &\gamma_2=\alpha_2-\alpha_1,\quad \gamma_4=\theta+\alpha_4-\alpha_1.\end{aligned} \quad (2.13)$$

It suffices to consider the case $a>0$. The entire θ dependence of the integral J is contained in the integral I. Noting that only the poles W_i^- lie inside the unit circle, the integral may be evaluated by the calculus of residues to give

$$I=\sum_{i=1}^4 (W_i^-)^a e^{-i\gamma_i}(W_i^--W_i^+)^{-1}$$
$$\times\prod_{j\neq i}[(W_i^--W_j^-)(W_i^--W_j^+)]^{-1}e^{-i\gamma_j}. \quad (2.14)$$

From Eq. (2.14) it is clear that there are no singularities in $z=\cos\theta$ arising from the vanishing of the denominator of the form $W_i^--W_j^-$ because a singularity of this form in one term of the sum is cancelled by that arising from $W_j^--W_i^-$ in another. If we clear fractions in the sum of Eq. (2.14), we have in the denominator

$$[\prod_{i<j}(W_i^--W_j^+)(W_j^--W_i^+)][\prod_{i<j}(W_i^--W_j^-)]$$
$$\times[\prod_{i=1}^4(W_i^--W_i^+)]. \quad (2.15)$$

We have already argued that the terms $W_i^--W_j^+$ produce no singularities when they vanish. Terms like $W_i^--W_i^+$ contribute factors of $e^{-i\gamma_i}$ which can be taken into the numerator. Therefore, the only singularities which arise from the vanishing of the denominators come from the term $\prod_i(W_i^--W_i^+)$.

When I is substituted into J, the numerator of J will have a number of factors of $e^{i\theta}$ coming from the denominator of I and also a number of factors of $\cos\theta$ and $\sin\theta$ coming from the harmonic polynomials. Thus, J can be written

$$J(\theta)=\prod_{i=1}^4\int d\lambda_i\prod_{j=2}^4\int d\alpha_j P(\sin\theta,\cos\theta)$$
$$\times\{\prod_{m\neq n}[\lambda_m\lambda_n+(\lambda_m^2-1)^{1/2}(\lambda_n^2-1)^{1/2}$$
$$-\cos(\theta+\chi_{mn})]\}^{-1}, \quad (2.16)$$

where χ_{mn} is linear in the α_j and P is a polynomial in $\sin\theta$ and $\cos\theta$ whose coefficients depend on the λ_i and α_i.

Each denominator in Eq. (2.16) can not vanish if $z=\cos\theta$ is inside an ellipse whose semimajor axis lies along the real axis and is of length $\lambda_m\lambda_n+(\lambda_m^2-1)^{1/2}\times(\lambda_n^2-1)^{1/2}$. We then conclude that J may be written as

$$J(\theta)=J_1(z)+\sin\theta\, J_2(z), \quad (2.17)$$

where $J_1(z)$ and $J_2(z)$ are analytic in z inside an ellipse whose semimajor axis is

$$z_0=\min_{m\neq n}\min_{\mathbf{q}_i}[\lambda_m\lambda_n+(\lambda_n^2-1)^{1/2}(\lambda_m^2-1)^{1/2}]. \quad (2.18)$$

The minima of the λ_i are

$$\begin{aligned}\min\lambda_1&=\left(1+\frac{2B'}{k'^2}\right)^{1/2}\equiv\lambda_1{}^0,\\ \min\lambda_2&=\left(1+\frac{\mu^2}{k'^2}\right)^{1/2}\equiv\lambda_2{}^0,\\ \min\lambda_3&=\left(1+\frac{\mu^2}{k^2}\right)^{1/2}\equiv\lambda_3{}^0,\\ \min\lambda_4&=\left(1+\frac{2B}{k^2}\right)^{1/2}\equiv\lambda_4{}^0.\end{aligned} \quad (2.19)$$

The minimum is obtained by inserting the combination of $\lambda_i{}^0$ which minimizes the bracket in Eq. (2.18). For instance, for elastic scattering on the mass shell we have

$$z_0=1+4B/E. \quad (2.20)$$

We must still consider the number of factors of $\sin\theta$ in J. To do this we show the following: (1) For the amplitude $J_{m'm}$ in which the initial and final bound states have projection quantum numbers m and m', respectively,

$$J_{m'm}(-\theta)=(-1)^{|m'-m|}J_{m'm}(\theta). \quad (2.21)$$

(2) $J_{m'm}(\theta)\to \mathrm{const}\times\theta^{|m'-m|}$ as $\theta\to 0$. The properties (1) and (2) suffice to show that

$$J_{m'm}(\theta)=(\sin\theta)^{|m'-m|}\hat{J}_{m'm}(z), \quad (2.22)$$

where $\hat{J}_{m'm}(z)$ is analytic inside the Lehmann ellipse whose size we have just discussed.

To prove (1) we return to Eq. (2.8) and make the substitution $\theta\to-\theta$. From Eq. (2.9) this amounts to sending $k_y'\to -k_y'$. If at the same time we rotate the integration variables $\mathbf{q}_i$ by an angle π about the z axis (i.e., send $q_{ix}\to -q_{ix}$ and $q_{iy}\to -q_{iy}$), then the denominators in Eq. (2.8) are left unchanged. The factor g_{11} is also left unchanged by this rotation.

The $H_l{}^m$ may be written

$$H_l{}^m(\mathbf{v})=(\nu_+)^m\mathcal{H}_l{}^m(\mathbf{v}^2,\nu_z), \quad m\geq 0 \quad (2.23)$$

$$=(\nu_-)^m\mathcal{H}_l{}^m(\mathbf{v}^2,\nu_z), \quad m<0, \quad (2.24)$$

when $\nu_\pm=\nu_x\pm i\nu_y$. The numerator of Eq. (2.8) is, therefore, multiplied by $|m-m'|$ factors of -1, which demonstrates Eq. (2.21). To show (2) we write $\mathbf{k}'=(|\mathbf{k}'|/|\mathbf{k}|)\mathbf{k}+\mathbf{\Delta}$, where

$$\mathbf{\Delta}=k'(0,\sin\theta,\cos\theta-1). \quad (2.25)$$

For small θ, $\mathbf{\Delta}$ is a small vector. We may then write, in Eq. (2.8),

$$[\kappa+2(\tfrac{1}{2}\mathbf{k}'-\mathbf{q}_1)^2]^{-1}=[\kappa+2(\tfrac{1}{2}k'\hat{k}-\mathbf{q}_1)^2]^{-1}\sum_{n=0}^{\infty}\{4\mathbf{q}_1\cdot\mathbf{\Delta}[\kappa+2(\tfrac{1}{2}k'\hat{k}-\mathbf{q}_1)^2]^{-1}\}^n, \quad (2.26)$$

$$[\mu^2+(\mathbf{k}'-\mathbf{q}_2)^2]^{-1}=[\mu^2+(k'\hat{k}-\mathbf{q}_2)^2]^{-1}\sum_{n=0}^{\infty}\{2\mathbf{q}_2\cdot\mathbf{\Delta}[\mu^2+(k'\hat{k}-\mathbf{q}_2)^2]^{-1}\}^n.$$

For small θ, we have $\mathbf{q}\cdot\mathbf{\Delta}=\frac{1}{2}i\theta(q_+-q_-)$. Now make a rotation of the variables $\mathbf{q}_i$ about the z axis by an angle φ. The denominators remain unchanged. A term in the numerator which goes like $(q_+)^\mu(q_-)^\nu$ will be multiplied by $\exp[i(\mu-\nu)\varphi]$. All the integrals will vanish, therefore, except those in which the number of q_+ factors exactly equals the number of q_- factors. It is then easy to see that the lowest terms which contribute in the sum are proportional to $\theta^{|m-m'|}$ which proves (2).

It remains to be shown that the other terms which contribute to $T^{(\alpha\beta)}$ are analytic inside the Lehmann ellipse. The analysis of terms like $V_\alpha g_{\alpha\beta}V_\beta$ is exactly the same as the one given here and will not be repeated. The low-order perturbation-theory diagrams which are not included in these terms will be treated explicitly in Appendix B. When these are included we have shown that the bound-state amplitudes $T^{(\alpha\beta)}(k,k',z,E)$ are analytic inside a Lehmann ellipse whose major axis is given by Eqs. (2.18) and (2.19).

III. PERTURBATION-THEORY DIAGRAMS

In this section, we will show that the individual terms in the perturbation expansion of the bound-state amplitudes are analytic in the scattering-angle plane except for cuts along the positive and negative real axes, *provided the energy E is below the three-particle threshold* (i.e., $E=-K^2<0$). A term in the perturbation series is represented by a diagram typified by Fig. 2. If we let $\mathbf{q}_i$, $i=1\cdots N$ be the loop momenta with $\mathbf{q}_i$ the first and $\mathbf{q}_N$ the last, and denote by $I_{m'm}$ a general diagram for an amplitude, where the projection quantum numbers of the initial and final bound states are m and m', respectively, we have

$$I_{m'm}(k,k',z,K)=\prod_{i=1}^{N}\int d^3q_i\int d\kappa\int d\kappa'\sigma_\alpha(\kappa)\sigma_\beta(\kappa')\times H_{l'}{}^{m'}(\mathbf{q}_N+\tfrac{1}{2}\mathbf{k}')^*H_l{}^m(\mathbf{q}_1+\tfrac{1}{2}\mathbf{k})[A_1\cdots A_M]^{-1}. \quad (3.1)$$

The A_i are denominators which may be from a potential

$$A_i=(\mu^2+\mathbf{p}^2), \quad (3.2)$$

from a Green's function

$$A_i=K^2+\mathbf{p}^2, \quad (3.3)$$

or from a bound-state wave function

$$A_i=\kappa+2\mathbf{p}^2. \quad (3.4)$$

In each case $\mathbf{p}$ is some linear combination of the $\mathbf{q}_i$, $\mathbf{k}$, and $\mathbf{k}'$. Use may be made of the Feynman identity to combine the M denominators. After making a translation and rotation of the $\mathbf{q}_i$, the integral can be written in the form

$$I_{m'm}=\int d\kappa\,\sigma_\alpha(\kappa)\int d\kappa'\,\sigma_\beta(\kappa')\prod_{i=1}^{M}\int dx_j\prod_{i=1}^{N}d^3q_i\times H_{l'}{}^{m'*}(\mathbf{p}')H_l{}^m(\mathbf{p})D^{-M}, \quad (3.5)$$

where

$$D=\sum_i C_iq_i^2+\alpha k^2-2\beta kk'z+\alpha k'^2+aK^2+b\mu^2+cB+dB', \quad (3.6)$$

with $C_i,\alpha,\beta,\gamma,a,b,c,d$ being functions of the Feynman parameters x_i. The terms involving $\mathbf{q}_i$, $\mathbf{k}$, and $\mathbf{k}'$ arose from the momentum terms in the denominators. They

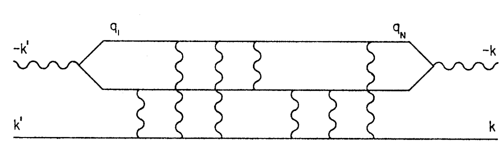

FIG. 2. Typical perturbation-theory contribution to the amplitude $T^{(33)}$. The horizontal wavy lines represent bound states. The vertical wavy lines represent potentials.

are therefore of the form

$$\sum_{j=1}^{M} x_j \mathbf{p}_j^2 = Q(q_i,\mathbf{k},\mathbf{k}',x_i)\,, \tag{3.7}$$

the $\mathbf{p}_j$ being some linear combination of $\mathbf{q}_i$, $\mathbf{k}$, and $\mathbf{k}'$. Since Q is a positive-definite quadratic form, one finds by setting $\mathbf{k}$ and $\mathbf{k}'$ equal to zero

$$\sum_i C_i \mathbf{q}_i^2 \geq 0 \tag{3.8}$$

for any values of the $\mathbf{q}_i$. Similarly, one also has

$$\alpha \geq 0\,, \quad \gamma \geq 0\,, \tag{3.9}$$

$$\alpha k^2 - 2\beta k k' z + \gamma k'^2 \geq 0 \quad \text{for} \quad -1 \leqslant z \leqslant 1\,. \tag{3.10}$$

The numbers a, b, c, and d are all positive, being of the form $x_{i_1}+x_{i_2}+\cdots+x_{i_M}$.

$$a\geq 0\,, \quad b\geq 0\,, \quad c\geq 0\,, \quad d\geq 0\,, \tag{3.11}$$

and finally

$$a+b+c+d=1\,. \tag{3.12}$$

The denominator D cannot vanish for physical z because for these values it is positive. The condition that it vanish is

$$z(\mathbf{q}_i,x_i) = \frac{1}{2\beta k k'}\left(\sum_i C_i \mathbf{q}_i^2 + \alpha k^2 + \gamma k'^2 + aK^2 + b\mu^2 + cB + dB'\right). \tag{3.13}$$

This is clearly satisfied only for real z. Since β can be positive or negative, D can vanish for positive or negative z. The vanishing of the denominator can, therefore, contribute cuts only along the positive and negative real axes. The position of the branch points is bounded below by the value z_0 given by

$$|z_0| = \min_{q_i,x_j} |z(q_i,x_j)|\,. \tag{3.14}$$

Since every term is positive in the above expression the minimum is assumed when all the $\mathbf{q}_i$ vanish. Further, from property (3.10) it follows that

$$(\alpha k^2+\gamma k'^2)/2|\beta| kk' \leq 1$$

and, therefore,

$$|z_0| \geqslant 1 + \min_{x_i}\left[\frac{aK^2+b\mu^2+cB+dB'}{2|\beta| kk'}\right]. \tag{3.15}$$

Making use of properties (3.11) and (3.12), one has

$$\min(K^2,\mu^2,B,B') = (a+b+c+d)\min(K^2,\mu^2,B,B') \leqslant aK^2+b\mu^2+cB+dB'\,, \tag{3.16}$$

so that

$$|z_0| \geq 1 + \frac{\min(K^2,\mu^2,B,B')}{2|\beta|_{\max} kk'}\,, \tag{3.17}$$

where $|\beta|_{\max}$ is the maximum value of $|\beta|$ over all the x_i. A crude bound on this quantity may be obtained in the following way. From Eq. (3.10) with $k=k'$ and $z=1$, one has $|\beta| \leq \frac{1}{2}(\alpha+\gamma)$. To obtain a bound on α and γ, we consider the quadratic form Q. The minimum of Q with respect to the $\mathbf{q}_i$ is clearly $\alpha k^2 - 2\beta k k' z + \gamma k'^2$. Setting $k'=0$, we have then for any choice of the $\mathbf{q}_i$,

$$Q(\mathbf{q}_i, k, k'=0, x_i) \geq \alpha k^2\,. \tag{3.18}$$

In order to get a crude bound on Q we can set all of the $\mathbf{q}_i$ equal to zero. One then has

$$Q = \tfrac{1}{2}x_1 k^2 + x_2 k^2 \geq \alpha k^2\,, \tag{3.19}$$

which implies $\alpha \leq 1$. Similarly, we have $\gamma \leq 1$. Thus $|\beta|_{\max} \leq 1$.

$$|z_0| \geq 1 + \frac{\min(K^2,\mu^2,B,B')}{kk'}\,. \tag{3.20}$$

The bound obtained here is sufficient to show that the right- and left-hand cuts begin a finite distance outside the physical region.

The kinematic singularities which arise from factors in the numerator of Eq. (3.1) must now be considered. Write the factors of $H_{l'}^{m'*}$ and H_l^m in the form given in Eqs. (2.22) and (2.23). If we consider for a moment only the integrations over q_{xi} and q_{yi}, then the integrals to be evaluated are of the general form

$$J = \left[\prod_{1=i}^{N} \int dq_{xi} dq_{yi} (q_{+i})^{n_i} (q_{-i})^{m_i}\right] D^{-M}\,. \tag{3.21}$$

Now make a rotation about the z axis by an angle φ of the integration vector $\mathbf{q}_i$. Because it depends only on the $\mathbf{q}_i^2$, the denominator is unchanged. One then has

$$J = e^{i(n_i - m_i)\varphi} J\,. \tag{3.22}$$

Thus, J is nonzero only if $n_i = m_i$, i.e., if the number of q_+ factors equals the number of q_- factors.

In order to see what this implies for the amplitude, we will argue through the example $m' \geq m \geq 0$; the results for other values of m' and m are obtained in the same way. We will work in the coordinate system given by Eq. (2.9) so that

$$\begin{aligned} k_\pm &= k_x \pm i k_y = 0\,, \\ k_\pm' &= \pm i k' \sin\theta\,. \end{aligned} \tag{3.23}$$

The factor of interest in the basic integral (3.5) is $(p_-')^{m'}(p_+)^m$, and we see that, in general,

$$\begin{aligned} (p_+)^m &= \prod_{i=1}^{N} (q_{+i})^{n_i}\,, \\ (p_-')^{m'} &= (-ik'\sin\theta)^{n_0'} \prod_{i=1}^{N} (q_{-i})^{n_i'}\,, \end{aligned} \tag{3.24}$$

where

$$\sum_{i=1}^{N} n_i = m\,, \quad \sum_{i=0}^{N} n_i' = m'\,. \tag{3.25}$$

Since the integral of Eq. (3.21) vanishes unless $n_i=n_i'$, we have $n_0=m'-m$. We therefore conclude that there are $m'-m$ factors of $\sin\theta$ multiplying an integral which is analytic in z except where the denominator D vanishes, i.e., except for the real axis outside of the limits given by Eq. (3.20). The argument for the other values of m and m' follows in the same way and we have for every diagram $I_{m'm}(\theta)$,

$$I_{m'm}(\theta)=(\sin\theta)^{|m'-m|}\hat{I}_{m'm}(z)\,, \tag{3.26}$$

where $\hat{I}_{m'm}(z)$ is analytic in the z plane cut along the real axis at least outside the limits of Eq. (3.20) for energies E below the three-particle threshold.

A few remarks on this result are in order. First, the bound obtained in Eq. (3.20) is not the best obtainable. However, we have already established analyticity in the Lehmann ellipse which intersects the real axis at a larger value of z than that given by the bound in Eq. (3.20).

Second, we have derived the analyticity properties of diagrams which contribute to amplitudes describing transitions between states in which the bound states have definite projections of angular momentum along some fixed z axis. From these properties we can easily derive the analytic properties of the helicity amplitudes. In the coordinate system given in Eq. (2.9), the helicity amplitudes arise by rotating the final z axis by an angle θ about the x axis. If $d_{m'm}{}^{l'}(\theta)$ is the matrix for this rotation, the helicity amplitudes $T_{\lambda'\lambda}(\theta)$ are related to the fixed-axis amplitude $T_{m'm}(\theta)$ by

$$T_{\lambda'\lambda}(\theta)=\sum_{m=-l'}^{+l'} d_{\lambda'm}{}^{l'}(\theta)T_{m\lambda}\,. \tag{3.27}$$

We have shown that

$$T_{m\lambda}(\theta)=(\sin\theta)^{|m-\lambda|}\hat{T}_{m\lambda}(z)\,, \tag{3.28}$$

where $\hat{T}_{m\lambda}$ is analytic in the Lehmann ellipse, and that term of the perturbation series for $\hat{T}_{m\lambda}$ is analytic in the larger region discussed above. Now

$$d_{\lambda'm}{}^{l'}(\theta)=(\sin\tfrac{1}{2}\theta)^{|\lambda'-m|}(\cos\tfrac{1}{2}\theta)^{|\lambda'+m|}P_{\lambda'm}{}^{l}(z)\,, \tag{3.29}$$

where $P_{\lambda'm}{}^{l}(z)$ is a polynomial in z.[4] It then directly follows that

$$T_{\lambda'\lambda}(\theta)=(\cos\tfrac{1}{2}\theta)^{|\lambda+\lambda'|}(\sin\tfrac{1}{2}\theta)^{|\lambda-\lambda'|}\hat{T}_{\lambda'\lambda}(z)\,, \tag{3.30}$$

where $\hat{T}_{\lambda'\lambda}(z)$ is analytic in the same region of the z plane that $\hat{T}_{m\lambda}$ is.

IV. FREDHOLM-SERIES DIAGRAMS

With this section we begin the proof that the analytic properties previously exhibited for the individual terms in the perturbation series hold for the full scattering amplitude. To do this, we will (1) write out the Fredholm solution to the Faddeev equation for the scattering amplitude, (2) show that each individual contribution to the Fredholm series has the desired analytic properties (since the Fredholm denominator is independent of z, we need only concern ourselves with the numerator series; we shall refer to it as the Fredholm series), and (3) show that the series converges uniformly in the scattering angle z. The proof of the convergence is given in the following section, while steps (1) and (2) are treated here.

A general term in the Fredholm numerator series is a finite sum of terms arising from the perturbation expansion of the Faddeev equation.[1] It is, therefore, sufficient to study the analytic properties of a general term in the perturbation expansion. A typical term is shown in Fig. 3. In order to study the analytic properties of the diagram shown in Fig. 3 we shall make use of the Fredholm solution for the two-particle t matrices and write each of them in the form $t=N/D$. We then write an integral representation for $1/D$ and expand each N in its Fredholm series. The diagram in Fig. 3 will then be expressed by a multiple sum. In this section, we shall show that each term in this sum has the desired analytic properties and in the next section we shall show that the sum converges uniformly in z.

It is not straightforward to follow this procedure directly, because the vanishing of D at the two-particle bound-state poles gives rise to denominators which are not easily handled by our previous techniques. We, therefore, first rewrite the two-body t matrices in a way which explicitly displays their bound-state poles. If there are N bound states with wave functions $|B_i\rangle$, then t can be written in the form

$$t=\bar{t}+\sum_{i,j=1}^{N} t|B_i\rangle B_{ij}\langle B_j|t\equiv\bar{t}+t_s\,. \tag{4.1}$$

Here, the matrix B_{ij} is the inverse of the $N\times N$ matrix $\langle B_i|t|B_j\rangle$, and $\bar{t}$ is the two-body t matrix arising from the potential

$$\bar{V}=V-\sum_{i,j=1}^{N} V|B_i\rangle A_{ij}\langle B_j|V\,, \tag{4.2}$$

A_{ij} being the inverse of the matrix $\langle B_i|V|B_j\rangle$. This decomposition is proved in Appendix C. It is clear from formula (4.1) that $\bar{t}$ has no bound-state poles, so the only singularity of $1/\bar{D}(E)$ is a cut running along the positive E axis. If we substitute Eq. (4.1) for each occurrence of t in Fig. 3, the resulting terms can be

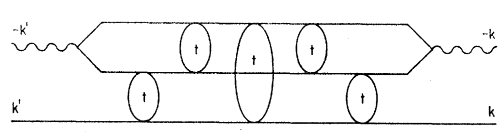

FIG. 3. An individual term in the Fredholm expansion of the Faddeev equations made up from two-body $\bar{t}$ matrices, t.

[4] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N. J., 1957).

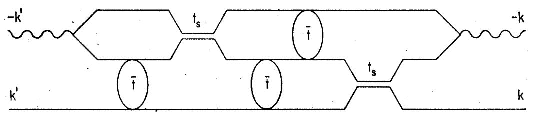

FIG. 4. An individual term in the Fredholm expansion of the Faddeev equation rewritten in terms of $\bar{t}$ and $t_s=t-\bar{t}$. An occurrence of t_s is shown by a double line to emphasize its separable character.

represented diagrammatically by Fig. 4. For each occurrence of t and $\bar{t}$ we now substitute the Fredholm expansion

$$t=\sum_i N_i/D\,,\quad \bar{t}=\sum_i \bar{N}_i/\bar{D}\,. \tag{4.3}$$

The diagram shown in Fig. 4 can now be written in the form

$$\int d^3k_1\int d^3k_2\cdots\int d^3k_M \sum_{j_1\cdots j_{M2}}{}' F_{ij_1}(\mathbf{k}',\mathbf{k}_1;E)$$
$$\times\bar{B}_{j_1j_2}(E-\tfrac{3}{2}\mathbf{k}_1{}^2)F_{j_2j_3}(\mathbf{k}_1,\mathbf{k}_2;E)$$
$$\times\bar{B}_{j_3j_4}(E-\tfrac{3}{2}\mathbf{k}_2{}^2)\cdots F_{j_{2M}i}(\mathbf{k}_M,\mathbf{k},E)\,. \tag{4.4}$$

Here, M is the number of occurrences of t_s in the diagram,

$$\bar{B}_{ij}=D(E)^{-2}B_{ij}(E)\,,$$

and $F_{ij}(\mathbf{k}',\mathbf{k},E)$ is a diagram of the type shown in Fig. 5. In Fig. 5 the circles labeled $\tilde{V}$ stand for the potential $\bar{V}$ multiplied by appropriate factors of $(\bar{D})^{-1}$ and $\bar{\sigma}_N=\mathrm{tr}[(G_0\bar{V})^N]$. To prove the analyticity of diagrams like that of Fig. 4, we now proceed in two steps. (1) We show that each of the diagrams F_{ij} have the desired analytic properties; (2) we combine them to show that these analytic properties are preserved when the diagrams are combined to form those of Fig. 4.

To prove the first of these assertions we note the following facts:

(a) The quantities $\bar{\sigma}_i(E)$ and $1/\bar{D}(E)$ satisfy the dispersion relations (see I, II, III)

$$\bar{\sigma}_i(E)=\int_0^\infty \frac{\mathrm{Im}\bar{\sigma}_i(E')}{E'-E}dE'\,, \tag{4.5}$$

$$\frac{1}{\bar{D}(E)}=1+\int_0^\infty \frac{\mathrm{Im}[1/\bar{D}(E')]}{E'-E}dE'\,. \tag{4.6}$$

(Recall that the potential $\bar{V}$ has no bound states.) Whenever one of these factors appears in a diagram, F_{ij}, it contributes, therefore, a denominator of exactly the same form as an energy denominator arising from the Green's functions.

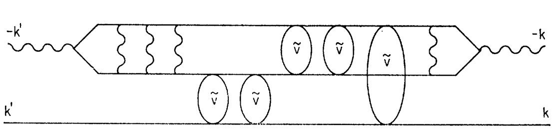

FIG. 5. The diagram corresponding to $F_{ij}(\mathbf{k}',\mathbf{k},E)$. $\tilde{V}$ denotes an occurrence of $\bar{V}$ multiplied by a number of factors of $\bar{D}(E)^{-1}$ and $\sigma_N(E)=\mathrm{tr}(G_0V)^N$.

(b) The potential $\bar{V}$ consists of V plus a separable part V_s. The matrix element of V_s between free states is

$$\langle\mathbf{p}'|V_s|\mathbf{p}\rangle=\sum_{ij}\langle\mathbf{p}'|V|B_i\rangle A_{ij}\langle B_j|V|\mathbf{p}\rangle\,. \tag{4.7}$$

Now,

$$\langle\mathbf{p}|V|B_i\rangle=-(2\mathbf{p}^2+B_i)\langle\mathbf{p}|B_i\rangle\,, \tag{4.8}$$

so that the analytic properties of $\langle\mathbf{p}|V|B_i\rangle$ in $\mathbf{p}^2$ are the same as those of the bound-state wave function. In particular, we can write the representation

$$\langle\mathbf{p}|V|B_i\rangle=H_l{}^m(\mathbf{p})\int_{\mu^2}^\infty \frac{\nu(\kappa)d\kappa}{\kappa+2\mathbf{p}^2}\,. \tag{4.9}$$

The matrix A_{ij} is diagonal in the angular momentum quantum numbers l, m, so that we can write

$$\langle\mathbf{p}'|V_s|\mathbf{p}\rangle=\sum_l (pp')^l P_l(\hat{p}'\cdot\hat{p})\sum_{ij}{}^l\int_{\mu^2}^\infty \frac{\nu(\kappa)d\kappa}{2\mathbf{p}^2+\kappa}\times\int_{\mu^2}^\infty \frac{\nu(\kappa')d\kappa'}{2\mathbf{p}'^2+\kappa'}\,, \tag{4.10}$$

where $\sum^l$ indicates the sum over bound states with a given angular momentum l. From Eq. (4.10) it is clear that as far as the denominators are concerned (which are the factors significant for the analytic properties) an occurrence of V_s contributes denominators of exactly the same type as a bound-state wave function.

Properties (a) and (b) discussed above show that the denominators which occur in a diagram F_{ij} are no different in character from those which occur in a perturbation-theory diagram. Since the considerations of Sec. III in no way depended on how these denominators occurred or in what numbers, it follows that the analytic properties of the diagrams F_{ij} are the same as those of the perturbation-theory diagrams.

The second step in the argument leading to the analytic properties of a general term in the Fredholm solution for the Faddeev equations is to combine the diagrams F_{ij} through Eq. (4.4). To do this we first write the diagrams in their Feynman parametrized form

$$F_{ij}=\prod_{m,n}\int d^3q_m\int dx_n f(x,\mathbf{q})H_{l'}{}^{m'}(\mathbf{p}')^*H_l{}^m(\mathbf{p})\times[z_0(x,\mathbf{q})-z]^{-M}\,, \tag{4.11}$$

where f and z_0 are rotationally invariant functions of the x_i and $\mathbf{q}_i$, and $\mathbf{p}$ and $\mathbf{p}'$ are linear combinations of the $\mathbf{q}_i$, $\mathbf{k}$, and $\mathbf{k}'$ with coefficients depending on x. The basic integral to be used in combining two F's is

$$I_{MN}{}^{\alpha\beta}=\int d\Omega_{\mathbf{k}''}\frac{(k_+{}'')^\alpha(k_-{}'')^\beta}{[z_0'-\hat{k}'\cdot\hat{k}'']^M[z_0-\hat{k}''\cdot\hat{k}]^N}\,. \tag{4.12}$$

We can obtain $I_{MN}^{\alpha\beta}$ from $I_{11}^{\alpha\beta}$ by differentiation with respect to z_0 and z_0'. If α and β were zero, this would just be the integral relevant to the proof of the Lehmann ellipse in two-particle potential scattering.[5] The argument given in Ref. 5 can be generalized to the case when spin is present by the techniques given in Sec. III. Indeed, the method is so similar to that of Sec. III that we will only quote the answer here:

$$I_{MN}^{\alpha\beta}=\int d\eta \frac{H_{l'}^{m'}(\mathbf{p}')^* H_l^m(\mathbf{p})}{[\eta-z]^{M+N-1}} P(z,\eta). \quad (4.13)$$

Here, $z=\hat{k}\cdot\hat{k}'$, $\mathbf{p}$ and $\mathbf{p}'$ are linear combinations of $\mathbf{k}$ and $\mathbf{k}'$ with η-dependent coefficients, and P is a polynomial in z with η-dependent coefficients. The factors of H_l^m follow from the transformation property of the original integral under rotation.

The result given in Eq. (4.13) can be used to show that the amplitude resulting from combining two F's according to Eq. (4.4) has the same analyticity domain as the perturbation-theory graphs. The arguments for determining the number of factors of $\sin\theta$ are identical with those previously given in the discussion of the Lehmann ellipse.

Equation (4.13) gives the result of combining two of the F's according to Eq. (4.4). This result, however, has the same general form as the original integral for one F given in Eq. (4.11). We can, therefore, proceed by induction to show that the combination of an arbitrary number of F's has the analytic properties of the perturbation-theory graphs. This completes step (2) of the proof.

V. PROOF OF CONVERGENCE

In the previous section, we obtained the analyticity domain for each term in a series expansion of the scattering amplitude. In this section, we will show that the series converges uniformly in the domain of analyticity, so that the amplitude itself has the analyticity properties of the individual terms in the series.

The outline of the proof is as follows. We write the amplitude in the form used in deducing the Lehmann ellipse [Eq. (2.8)], so that it is expressed as an integral over g_{ij}. We next expand g_{ij} in its Fredholm series. (By Fredholm series we again mean the Fredholm numerator series. The Fredholm determinant is independent of the scattering angle, so it never really enters the problem.) Each term in the Fredholm series is expressible as an integral over two-particle t matrices,[1] and we expand each of these t matrices in its Fredholm series as was discussed in Sec. IV. g_{ij} is now given by a multiple sum. When we substitute it into Eq. (2.8) the on-energy-shell scattering amplitude is given by a series of diagrams of which Fig. 5 is typical. We have proved the analyticity of these diagrams in Sec. IV, so it only remains to show that their sum is uniformly convergent with respect to the scattering angle z.

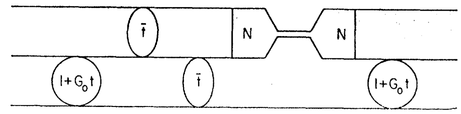

FIG. 6. A typical term in the Fredholm expansion of g_{11}. The initial and final interactions are $(1+G_0t)$ because one potential rung has already been subtracted out in the definition of g_{11}. The box labeled N denotes the numerator in the Fredholm expansion of t which is part of t_s.

A series of functions of momenta will be said to have the property S if the following conditions are satisfied:

(a) Every term in the series is bounded by a constant and is separately square integrable in all the momenta which are its arguments.

(b) The sum of the functions converges to a function which is separately square integrable in all of the momenta.

(c) The derivative of the series with respect to any momentum component p_i satisfies properties (a) and (b).

The proof now proceeds in three steps: We show (1) that each term in the Fredholm series for g_{ij} is itself given by an infinite series which has property S; (2) that the Fredholm series for g_{ij} has property S; and (3) that (1) and (2) imply that the series for the on-energy-shell amplitude converges uniformly with respect to z.

We start with part (1) of the proof. A typical term in the Fredholm series for g_{ij} is shown in Fig. 6. We wish to show that after replacing each $\bar{t}$ and N by its Fredholm series, the diagram in Fig. 6 is given by a series which has property S. The proof is made by induction. Let us focus our attention on a particular $\bar{t}$ (or N), for example, the one in the dashed box in Fig. 7. We assume that all of the $\bar{t}$'s and N's to the left of the one we are studying have been expanded in their Fredholm series, and we denote their contribution to Fig. 7 by $J'(\mathbf{q}_1',\mathbf{q}_3';\mathbf{q}_1',\mathbf{p}_3')$. The $\bar{t}$'s and N's to the right of the one we are studying have not yet been expanded. We denote their contribution to Fig. 7 by $J(\mathbf{q}_1'',\mathbf{p}_3;\mathbf{q}_1,\mathbf{q}_3)$. The whole of the diagram in Fig. 7 is

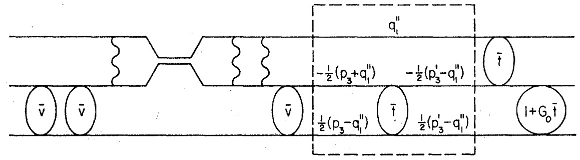

FIG. 7. One term in an expansion of a diagram like that shown in Fig. 6. Everything to the left of the dashed box has been expanded while everything to the right has not.

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[5] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) 10, 62 (1960).

denoted by H and given by

$$H(\mathbf{q}_1',\mathbf{q}_3';\mathbf{q}_1,\mathbf{q}_3)=\sum_{i=0}^{\infty}H_i(\mathbf{q}_1',\mathbf{q}_3';\mathbf{q}_1,\mathbf{q}_3)$$
$$=\sum_{i=0}^{\infty}\int d^3q_1''d^3p_3d^3p_3'J'(\mathbf{q}_1,'\mathbf{q}_3';\mathbf{q}_1'',\mathbf{p}_3')\times\langle\mathbf{p}_3'|\bar{t}_iG_0|\mathbf{p}_3\rangle J(\mathbf{q}_1'',\mathbf{p}_3;\mathbf{q}_1,\mathbf{q}_3)\,,\quad(5.1)$$

where $\bar{t}_i$ is the ith term in the Fredholm series for the $\bar{t}$ under consideration.

It follows directly from the results of II that J and J' are bounded and square integrable in each of their arguments. Because of the simple form of the derivative of the Yukawa potential with respect to momentum, it is straightforward to use the results of II to show that the derivatives of J and J' are also bounded and square integrable.

We can obtain a bound on the general term in the series for H by making use of the Schwartz inequality. We find

$$|H_i|\le\left[\int d^3q_1''d^3q_3''|J'(\mathbf{q}_1,\mathbf{q}_3;\mathbf{q}_1'',\mathbf{q}_3'')|^2\right]^{1/2}\times\left[\int d^3q_1'''d^3q_3'''|J(\mathbf{q}_1,\mathbf{q}_3;\mathbf{q}_1''',\mathbf{q}_3''')|^2\right]^{1/2}\times\left[\int d^3p_3d^3p_3'|\langle p_3(\bar{t}_iG_0)_{\mathbf{q}_1''=0}|p_3'\rangle|^2\right]^{1/2}.\quad(5.2)$$

In obtaining Eq. (5.2) we have made use of that fact that $\bar{t}_iG_0$ depends on $\mathbf{q}_1''$ only through Green's functions and the two-particle Fredholm determinant. Since the total three-particle energy is negative, we can obtain a bound by setting $\mathbf{q}_1''=0$ everywhere in $\bar{t}_iG_0$. (See II for the details of obtaining such bounds.) Since it was shown in II that the series $\sum_{i=0}^{\infty}\|\bar{t}_iG_0\|$ converges, it follows at once that the series for H satisfies property S. By induction it also follows that the series obtained by replacing each $\bar{t}$ and N in Fig. 6 by its Fredholm series satisfies property S.

Part (2) of the proof, the fact that the Fredholm series for g_{ij} satisfies property S, follows immediately from the results of II and our discussion of the properties of J above.

We have thus reached the result that g_{ij} can be expanded in a series which has property S and which, when substituted into Eq. (2.8), generates a series for the on-energy-shell amplitude each term of which is a diagram of the type discussed in Sec. IV with a known domain of analyticity. We now complete the proof by showing step (3), that the series converges uniformly in the joint domain of analyticity of the individual diagrams. We do this by a generalization of the method used by Blankenbecler, Goldberger, Khuri, and Treiman for a similar problem in the two-body case.[5]

Equation (2.8) is a representation for the scattering amplitude in terms of g_{ij} and low-order perturbation-theory diagrams whose analyticity is explicitly discussed in Appendix B. Let us use the coordinate system for $\mathbf{k}$ and $\mathbf{k}'$ discussed in Sec. II and fix on a particular complex z in the domain of analyticity. Further, let us imagine that the integrations over $\mathbf{q}_1$ and $\mathbf{q}_2$ have been performed. The amplitude is then a polynomial in $\sin\theta$ and $\cos\theta$ with coefficients which have the general form

$$K=\sum_n\int d^3q_3d^3q_4d\kappa'\bar{g}^n(\mathbf{q}_3\mathbf{q}_4)[2k'^2q_3q_4\sin\beta_3\sin\beta_4]^{-1}\times\{[\lambda_3(q_3)-\cos(\theta+\alpha_3)]\times[\lambda_4(q_4)-\cos(\theta+\alpha_4)]\}^{-1}.\quad(5.3)$$

The series of the $\bar{g}^n$ has property S because to obtain the $\bar{g}^n$ one only has to integrate the $g_{ij}{}^n$ with a potential or a bound-state wave function, either of which is a square integrable function. We now show that the fact that the series for $\bar{g}^n$ has property S implies that the series for K converges uniformly in z.

For fixed complex θ the denominator in Eq. (5.3) can vanish only for a limited range of $|\mathbf{q}_3|$ or $|\mathbf{q}_4|$. Divide the $\mathbf{q}_3$ integration into time regions: S_3, a sphere enclosing the region of possible singularity, and $\bar{S}_3$, the region outside. Similarly, introduce regions S_4 and $\bar{S}_4$ for the $\mathbf{q}_4$ integration. We now divide the six-dimensional space of integration into four regions:

$$R_1=S_3\cap S_4,\quad R_3=S_3\cap\bar{S}_4,$$
$$R_2=\bar{S}_3\cap S_4,\quad R_4=\bar{S}_3\cap\bar{S}_4.\quad(5.4)$$

In the region R_4 we use the Schwartz inequality to write

$$|K|\le\sum_n\int d\kappa'\left[\int_{R_4}d^3q_3d^3q_4|\bar{g}^n(\mathbf{q}_3,\mathbf{q}_4)|^2\right]^{1/2}\times\left\{\int_{R_4}d^3q_3d^3q_4[(\mathbf{q}_3-\mathbf{k}')^2+\mu^2]^{-2}\times[(\mathbf{q}_4+\tfrac{1}{2}\mathbf{k}')^2+\kappa'^2]^{-2}\right\}^{1/2}.\quad(5.5)$$

The second integral exists because the integration region does not contain the parts where the denominators could vanish. The first series of integrals exists and converges because of property S.

Consider next the region R_3. The singularity of the denominator with q_3 is a curve described by $q_3=q_3{}^0(\beta_3,\alpha_3)$. Imagine now that the $\mathbf{q}_4$ integration over this region is done. By the same arguments as before, this leads to a series

$$I=\sum_n\int_{S_3}d^3q_3\frac{\tilde{g}^n(\mathbf{q}_3)}{kq_3\sin\beta_3[\lambda_3(\mathbf{q}_3)-\cos(\theta+\alpha_3)]},\quad(5.6)$$

where the series $\tilde{g}^n(\mathbf{q}_3)$ satisfies property S. Now

$$\tilde{g}^n(q_3\beta_3\alpha_3)=\tilde{g}^n(q_3\beta_3\alpha_3)-\tilde{g}^n(q_3{}^0\beta_3\alpha_3)+\tilde{g}^n(q_3{}^0\beta_3\alpha_3).\quad(5.7)$$

So using the fact that

$$|f(x)-f(y)|\leq|\max f'(x)||x-y|,$$

we can write a bound for the series given in Eq. (5.6).

$$I\leq\sum_n \max_{S_3}\left[\left|\frac{\partial \tilde{g}^n}{\partial q_3}\right|\int_{S_3} d^3q_3 \times\left|\frac{q_3-q_3^0(\beta_3,\alpha_3)}{kq_3\sin\beta_3[\lambda_3-\cos(\theta+\alpha_3)]}\right|\right] +\int_{S_3} d\beta_3 d\alpha_3|\tilde{g}^n(q_3^0)|\left|\int_{S_3}\frac{dq_3q_3}{k[\lambda_3-\cos(\theta+\alpha_3)]}\right|. \quad (5.8)$$

In Eq. (5.8) the first series converges because of property S. The integral exists because the numerator vanishes when the denominator does. In the second term the first series converges because of property S [properties (1) and (2) imply that $\tilde{g}^n$ can be bounded by a series of convergent constants and S_3 is finite]. The second integral converges because the series is integrable.

The arguments for the regions R_1 and R_2 are based on the same principles. We will not give them here. The conclusion is then that for any fixed value of z the series of diagrams can be bounded by a converging series of constants. Inside of an arbitrarily large circle we can take the maximum of these constants to show that the series of diagrams of Sec. IV converges uniformly in their mutual domain of analyticity. This completes the proof.

APPENDIX A: REPRESENTATION OF THE BOUND-STATE WAVE FUNCTION

In this section we prove the representation of the bound-state wave function given in Sec. II.

The wave function for a bound state of binding energy B and angular momentum l is given in momentum space by

$$\langle\mathbf{p}|\psi_{Blm}\rangle=(2\mathbf{p}^2+B)^{-1}\langle\mathbf{p}|V|\psi_{Blm}\rangle \equiv(2\mathbf{p}^2+B)^{-1}H_l^m(\mathbf{p})\Phi_B(\mathbf{p}^2), \quad (A1)$$

where $H_l^m(\mathbf{p})$ is the harmonic polynomial

$$H_l^m(\mathbf{p})=|\mathbf{p}|^l Y_l^m(\hat{p}). \quad (A2)$$

$\langle\mathbf{p}|V|\psi_{Blm}\rangle$ is related to the off-energy-shell two-particle t matrix $t(\mathbf{p},\mathbf{p}';E)$ by

$$\lim_{E\to-B}(E+B)t(\mathbf{p},\mathbf{p}';E)=\langle\mathbf{p}|V|\psi_{Blm}\rangle\langle\psi_{Blm}|V|\mathbf{p}'\rangle. \quad (A3)$$

If we write the Fredholm solution for $t(\mathbf{p},\mathbf{p}';E)$ in the form

$$t(\mathbf{p},\mathbf{p}';E)=N(\mathbf{p},\mathbf{p}';E)/D(E), \quad (A4)$$

then the pole in t at $E=-B$ arises from the zero of $D(E)$ at that point. As a result, one can determine the analyticity properties of $\Phi_B(\mathbf{p}^2)$ by studying those of $N(\mathbf{p},\mathbf{0};-B)$. To this end we expand $N(\mathbf{p},\mathbf{0};-B)$ in its Fredholm series. The nth term in the series is a finite sum over terms of the form[1]

$$\langle\mathbf{p}|[VG_0(-B)]^m V|\mathbf{0}\rangle=\int d^3q_1\cdots d^3q_m \times[[(\mathbf{p}-\mathbf{q}_1)^2+\mu^2][2\mathbf{q}_1^2+B][(\mathbf{q}_1-\mathbf{q}_2)^2+\mu^2]\cdots \times[2\mathbf{q}_m^2+B][\mathbf{q}_m^2+\mu^2]]^{-1}, \quad (A5)$$

where for simplicity we have taken the case of a single Yukawa potential of inverse range μ. None of the denominators in the integrand of Eq. (A5) vanish for values of $p=|\mathbf{p}|$ in the strip

$$|\mathrm{Im}p|<\mu. \quad (A6)$$

It is then clear that each term in the Fredholm series for $N(\mathbf{p},\mathbf{0};-B)$ is analytic in this strip. It is shown in II that the Fredholm series converges uniformly with respect to p in the strip, so $N(\mathbf{p},\mathbf{0};-B)$ is also analytic here.

In order to study $N(\mathbf{p},\mathbf{0};-B)$ in the rest of the upper-half p plane we use the rotation of contours argument introduced in II. Let us consider the integral of Eq. (A5) for a fixed value of p on the imaginary axis between the points $\pm i\mu$. We can simultaneously rotate the contours of integration of all of the $\mathbf{q}$ variables through an angle θ, $|\theta|<\frac{1}{2}\pi$, without crossing any singularities of the integrand. The resulting integral defines a function of p which is analytic in the strip

$$|\mathrm{Im}(pe^{i\theta})|<\mu\cos\theta. \quad (A7)$$

Since all of the strips of Eq. (A7) include the imaginary axis between $\pm i\mu$, the integral on the rotated contour defines an analytic continuation of the original integral. By letting θ vary from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$, we see that the integral of Eq. (A5) is in fact analytic in the entire p plane with the possible exception of the imaginary axis above the point $i\mu$ and below the point $-i\mu$. It is shown in II that the Fredholm series defined by the integrals on the rotated contours converges uniformly in p, so $N(\mathbf{p},\mathbf{0};-B)$ has the same analyticity domain as the integral of Eq. (A5).

From the above discussion we see that the function $p^l\Phi_B(p^2)$ is analytic in the upper-half p plane with the possible exception of a cut running from $i\mu$ to $i\infty$. In order to complete our proof of the integral representation for $\Phi_B(p^2)$, we must show that (1) $\Phi_B(p)$ has no pole at $p=0$ and (2) $\Phi_B(-p)=\Phi_B(p)$. These results follow at once from the Schrödinger equation for the radial wave function in momentum space:

$$\psi_{Bl}(p)=(2p^2+B)^{-1}\int_0^\infty dq\frac{1}{2pq} \times Q_l\left(\frac{p^2+q^2+\mu^2}{2pq}\right)\psi_{Bl}(q) \quad (A8)$$

or, using Eq. (A1),

$$\Phi_B(p)=p^{-l-1}\int_0^\infty \frac{dq}{2q}Q_l\left(\frac{p^2+q^2+\mu^2}{2pq}\right)\frac{\Phi_B(q)q^l}{2q^2+B}. \qquad \text{(A9)}$$

Thus $\Phi_B(p^2)$ is analytic in the p^2 plane with the exception of a cut from $-\mu^2$ to $-\infty$ [actually the cut starts at $-(\mu+\sqrt{B})^2$], and our proof of the integral representation of Eq. (2.7) is completed.

APPENDIX B: LOWEST-ORDER DIAGRAMS

In this section, we calculate the bounds on the domain of analyticity for the lowest-order perturbation-theory diagrams. In Sec. III it was shown that all diagrams are analytic except on the real axis. It is therefore only necessary to establish the minimum value of $|z|$ at which the Feynman parametrized denominator can vanish.

There are three types of lowest-order diagrams shown in Fig. 8. There is only one type (A) contributing to the elastic amplitude and two types (B) and (C) contributing to the amplitude for rearrangement collisions. The denominators for the three diagrams are

$$\begin{aligned} D_A&=[\mu^2+(\mathbf{k}-\mathbf{k}')^2][\kappa'+2(\mathbf{q}+\tfrac{1}{2}\mathbf{k}')^2]\\ &\qquad\times[\kappa+2(\mathbf{q}+\tfrac{1}{2}\mathbf{k})^2],\\ D_B&=[\mu^2+(\mathbf{q}-\mathbf{k})^2][\kappa'+2(\mathbf{k}+\tfrac{1}{2}\mathbf{k}')^2]\\ &\qquad\times[\kappa^2+2(\mathbf{q}+\tfrac{1}{2}\mathbf{k})^2],\\ D_C&=[\mu^2+(\mathbf{q}+\mathbf{k}+\mathbf{k}')^2][\kappa'+2(\mathbf{q}+\tfrac{1}{2}\mathbf{k}')^2]\\ &\qquad\times[\kappa+2(\mathbf{q}+\tfrac{1}{2}\mathbf{k})^2]. \end{aligned} \qquad \text{(B1)}$$

Diagram (A) requires only one Feynman parameter, and the minimum value of $|z|$ is easily established as

$$|z|_{\min}=\frac{k^2+k'^2}{2kk'}+\frac{(B^{1/2}+B'^{\,1/2})^2}{kk'} \qquad \text{(B2)}$$

since the minimum value of the κ integration is B. The diagram (B) is of the same form as (A) if we make the substitution $\mathbf{k}\rightarrow -\frac{1}{2}\mathbf{k}'$, $\mu^2\rightarrow\kappa'$, $\kappa'\rightarrow\mu^2$ and so the minimum is expressed by the above formula with these substitutions.

Diagram (C) can be Feynman-parametrized by three parameters whose sum is restricted to unity. The minimum value of z will eventually be expressed as the minimum of a certain expression of these three parameters subject to this restriction. A bound on the minimum value of $|z|$ can be obtained by setting the Feynman parameter multiplying the first bracket equal to zero. The resulting bound is then identical with Eq. (B2).

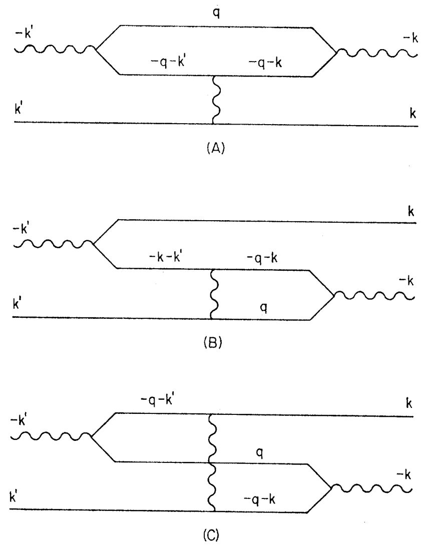

FIG. 8. The low-order perturbation-theory diagrams which together with the amplitude from Fig. 1 make up $T^{(33)}(\mathbf{k}',\mathbf{k},E)$.

APPENDIX C: DECOMPOSITION OF THE TWO-BODY t MATRIX

To prove the decomposition given in Eqs. (4.1) and (4.2), we denote $V-\bar{V}$ by V_s and write

$$t=V+VG_0t=\bar{t}+\bar{R}V_s+\bar{R}V_sG_0t. \qquad \text{(C1)}$$

Here G_0 is the two-particle free Green's function and $\bar{R}$ is the resolvent for $\bar{V}$:

$$\bar{R}=(1-\bar{V}G_0)^{-1}=R-RV_sG_0\bar{R}, \qquad \text{(C2)}$$

with R the resolvent for V. Multiplying Eq. (C2) on the right by $\langle B_k|VG_0$, one finds after some rearrangement

$$\langle B_k|VG_0\bar{R}=\sum_{i=1}^{n}(B^{-1}A)_{ki}\langle B_i|VG_0R, \qquad \text{(C3)}$$

where the matrices A and B are defined in Sec. IV. From this relation and Eq. (C2), one then has

$$\bar{R}V|B_k\rangle=t\sum_{ij}|B_i\rangle B_{ij}\langle B_j|V|B_k\rangle. \qquad \text{(C4)}$$

Writing out Eq. (C1), the decomposition follows.