Inelastic Electron Scattering in the Symmetric Quark Model*

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In this paper we use the nonrelativistic symmetric quark model to compute form factors for the process $e+p\rightarrow e+N^*$, where N^* is one of nine nucleon resonances. We assume a harmonic-oscillator potential and obtain Gaussian-type form factors. Center-of-mass motion is separated out, and $1/M_q^2$ corrections are discussed. Plots are made of the form factors and transition probabilities; the theoretical numbers generally agree with the existing experiment when $q^2<1$, but the predicted magnitudes are too small for $q^2>1$.

1. INTRODUCTION

THE nonrelativistic quark model[1] can be used to study many different aspects of high-energy physics. In particular, one can examine the production of nucleon resonances via inelastic electron scattering. In this paper we shall apply the symmetric quark model to the process

$$e+p\rightarrow e+N^*, \tag{1}$$

where e is an electron, p is a proton, and N^* denotes a nucleon resonance. Six $I=\frac{1}{2}$ and three $I=\frac{3}{2}$ resonant states will be considered.

The symmetric quark model gives us a picture of the nucleon as a bound state of three quarks in a potential well; a nucleon resonance is viewed as an excited state of the three-quark system. We shall adopt this model and shall choose the potential well to be that of a harmonic oscillator; our total wave functions will be made symmetric under exchange of any two quarks. We then calculate the form factors associated with the process of Eq. (1) and obtain predictions for differential cross sections in inelastic electron scattering.

Each of our form factors is proportional to an exponential $e^{-q^{*2}b^2/3}$, where q^{*2} is the square of the three-momentum transfer as seen from the isobar rest frame; b^2 is a parameter. This type of behavior is to be expected from our (Gaussian) harmonic-oscillator wave functions, since the Fourier transform of a Gaussian is just another Gaussian. Two of our three parameters (the harmonic-oscillator parameter b^2 and the ratio of quark g factor to quark mass) are determined by a comparison with elastic scattering (for small q^2); the third parameter M_q (quark mass) is free. In finding numerical values of the form factors we choose two values: $M_q=\frac{1}{3}m_{\rm proton}$ and $M_q=\infty$.

Spurious c.m. motion is separated out of the wave functions; it is seen that this can restrict the $SU(6)$ representations to which the particle can belong. The vanishing of various form factors is discussed. We also investigate $1/M_q^2$ corrections to the Coulomb form factors. The theoretical cross sections agree with the existing experimental values for $q^2<1$, but are too small for larger q^2.

The plan of this paper is as follows. In Sec. 2, we review the formalism needed to calculate form factors for the process of Eq. (1) in the symmetric quark model. Section 3 contains a discussion of our results. These results are listed in Tables II–V and are plotted in Figs. 2–16. In Sec. 4, we then compare our results with experiment. Section 5 discusses related theoretical work.

2. FORMALISM

In this section we shall review the formalism needed to calculate form factors for the process

$$e+p\rightarrow e+N^*$$

(p is a proton and N^* is a nucleon resonance) in the symmetric quark model. This process is assumed to proceed via the one-photon exchange diagram (shown in Fig. 1). The cross section has been derived by Bjorken and Walecka.[2] The result is

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm lab}=\frac{\alpha^2\cos^2\frac{1}{2}\theta}{4\epsilon^2\sin^4\frac{1}{2}\theta[1+(2\epsilon/m)\sin^2\frac{1}{2}\theta]}\times\left\{\frac{q^4}{q^{*4}}|f_c|^2+\left(\frac{q^2}{2q^{*2}}+\frac{M^2}{m^2}\tan^2\tfrac{1}{2}\theta\right)(|f_+|^2+|f_-|^2)\right\}, \tag{2}$$

where θ is the electron scattering angle, ϵ is the incident electron energy, and m and M are the nucleon and N^* masses. q^2 is the invariant four-momentum transfer, q^{*2} is the square of the three-momentum transfer in the

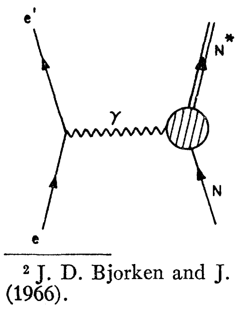

FIG. 1. Inelastic electron scattering.

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[1] G. Zweig, CERN Reports Nos. Th. 401, 1964, and Th. 412, 1964 (unpublished); M. Gell-Mann, Phys. Letters 8, 214 (1964); G. Morpurgo, Physics 2, 95 (1965): Y. Nambu, in *Proceedings of the Second Coral Gables Conference on Symmetry Principles at High Energy, 1964*, edited by B. Kursunoglu, A. Perlmutter, and I. Sakmar (W. H. Freeman and Co., San Francisco, 1965).

[2] J. D. Bjorken and J. D. Walecka, Ann. Phys. (N. Y.) 38, 35 (1966).

c.m. system of the N^* resonance, and f_c, f_+, and f_- are form factors. The quantities $|f_c|^2$ and $|f_+|^2+|f_-|^2$ will be calculated using the symmetric quark model.

In this model the nucleon is pictured as a system of three noninteracting quarks bound in a potential well (which we take to be a harmonic oscillator), and each N^* resonance is viewed as an excited state of the three-quark system. Since the calculation is done in a nonrelativistic framework, we will have wave functions for the proton and the N^* resonance. We require that each total wave function (with space, spin, and isospin variables) be totally symmetric under exchange of any two quarks, and we assign the proton to the **56** representation of $SU(6)$. The N^* resonances considered are the $1525(\frac{3}{2}^-)$, $1570(\frac{1}{2}^-)$, $1670(\frac{5}{2}^-)$, $1688(\frac{5}{2}^+)$, $1700(\frac{1}{2}^-)$, and $2190(\frac{7}{2}^-)$ states (all with $I=\frac{1}{2}$), and the $1236(\frac{3}{2}^+)$, $1670(\frac{1}{2}^-)$, and $1920(\frac{7}{2}^+)$ states (with $I=\frac{3}{2}$), as listed by Rosenfeld *et al.*[3] These resonances are assigned the quark quantum numbers and $SU(6)$ representations listed in Table I. The assignments of the resonant states agree with those of Dalitz,[4] Greenberg,[5] and Moorhouse.[6]

Since for fixed S_z there are two ways of forming total quark spin $S=\frac{1}{2}$ (from three spin-$\frac{1}{2}$ quarks), there are two linearly independent $|\frac{1}{2}+\frac{1}{2}\rangle$ quark spin functions:

$$u_a=(\sqrt{\tfrac{1}{2}})[\beta(1)\alpha(2)\alpha(3)-\alpha(1)\beta(2)\alpha(3)] \qquad (3)$$

and

$$u_b=(\sqrt{\tfrac{1}{6}})[\beta(1)\alpha(2)\alpha(3)+\alpha(1)\beta(2)\alpha(3)-2\alpha(1)\alpha(2)\beta(3)]$$

(α denotes spin up and β denotes spin down). Similarly, there are two analogous mutually perpendicular $|\frac{1}{2}+\frac{1}{2}\rangle$ quark isospin functions,

$$f_a=(\sqrt{\tfrac{1}{2}})[\beta'(1)\alpha'(2)\alpha'(3)-\alpha'(1)\beta'(2)\alpha'(3)] \qquad (4)$$

and

$$f_b=(\sqrt{\tfrac{1}{6}})[\beta'(1)\alpha'(2)\alpha'(3)+\alpha'(1)\beta'(2)\alpha'(3)-2\alpha'(1)\alpha'(2)\beta'(3)].$$

Thus the **56** spin-isospin function (totally symmetric in the interchange of any two quarks), for example, is given by

$$(\sqrt{\tfrac{1}{2}})(u_af_a+u_bf_b), \qquad (5)$$

when $|SS_z\rangle=|\frac{1}{2}+\frac{1}{2}\rangle$ and $|II_z\rangle=|\frac{1}{2}+\frac{1}{2}\rangle$ for the three-quark system.

In constructing the spatial wave functions, we separate out any c.m. excitation. For a harmonic-oscillator potential, the conversion from c.m. and relative coordinates (denoted by $\mathbf{R}$ and $\mathbf{r}$, $\boldsymbol{\rho}$, respectively) to individual quark coordinates (denoted by $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$) in the wave functions is simple and straightforward.

TABLE I. Quantum numbers of states used in symmetric quark model. (L is the total quark orbital angular momentum, S is the total quark spin, and I is the total isospin.)

State	J^P	L	S	I	$SU(6)$ representation
940	$\frac{1}{2}^+$	0	$\frac{1}{2}$	$\frac{1}{2}$	56
1525	$\frac{3}{2}^-$	1	$\frac{1}{2}$	$\frac{1}{2}$	70
1570	$\frac{1}{2}^-$	1	$\frac{1}{2}$	$\frac{1}{2}$	70
1670	$\frac{5}{2}^-$	1	$\frac{3}{2}$	$\frac{1}{2}$	70
1688	$\frac{5}{2}^+$	2	$\frac{1}{2}$	$\frac{1}{2}$	56
1700	$\frac{1}{2}^-$	1	$\frac{3}{2}$	$\frac{1}{2}$	70
2190	$\frac{7}{2}^-$	3	$\frac{1}{2}$	$\frac{1}{2}$	70
1236	$\frac{3}{2}^+$	0	$\frac{3}{2}$	$\frac{3}{2}$	56
1670	$\frac{1}{2}^-$	1	$\frac{1}{2}$	$\frac{3}{2}$	70
1920	$\frac{7}{2}^+$	2	$\frac{3}{2}$	$\frac{3}{2}$	56

The Hamiltonian of the system can be written as the sum of three individual quark Hamiltonians or equivalently as the sum of three Hamiltonians associated with fictitious noninteracting "particles," again each in a harmonic-oscillator potential (the latter particles have coordinates $\mathbf{R}$, $\mathbf{r}$, and $\boldsymbol{\rho}$). The absence of c.m. excitation means that only the ground-state wave function may be used for the $\mathbf{R}$ coordinate. Letting

$$\mathbf{R}=\tfrac{1}{3}(\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3),$$
$$\mathbf{r}=\tfrac{1}{2}(\mathbf{x}_1+\mathbf{x}_2-2\mathbf{x}_3),$$
$$\boldsymbol{\rho}=\mathbf{x}_1-\mathbf{x}_2,$$

and writing the Hamiltonian in these coordinates, we see that $M_R=3M_q$, $M_r=\frac{2}{3}M_q$, and $M_\rho=\frac{1}{2}M_q$. A bit of algebra shows that the exponentials in the harmonic-oscillator wave functions can be written equally well in $(\mathbf{R},\mathbf{r},\boldsymbol{\rho})$ or in $(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$ coordinates. Thus the only pieces of the wave function requiring any work are the Hermite polynomials, and the conversion from c.m. and relative coordinates to individual quark coordinates is rather simple. We note that an $L=1$ resonance cannot have a purely symmetric spatial wave function (and hence does not belong to a **56** representation in our model), since this would correspond to pure c.m. excitation.

One then uses the total wave functions to compute the form factors $|f_c|^2$ and $|f_+|^2+|f_-|^2$ of Eq. (1); expressions for these form factors in terms of reduced multipole matrix elements[7] have been derived by deForest and Walecka[8] and by Walecka.[9] The idea is that the incident electron emits a virtual photon (Fig. 1) which then excites one of the quarks in the nucleon to a higher state; the three quark amplitudes are added

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[4] R. H. Dalitz, in *Proceedings of the Oxford International Conference on Elementary Particles, 1965* (Rutherford High-Energy Laboratory, Berkshire, England, 1966), p. 157.

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[9] J. D. Walecka, in *International School of Physics "Enrico Fermi," Italian Physical Society Course 38*, edited by T. E. O. Ericson (Academic Press Inc., New York, 1967), p. 17.

coherently. The explicit formulas are[8,9]

$$|f_c|^2=2\pi\sum_{J=0}^{\infty}|\langle J_f\|\hat{M}_J{}^{\mathrm{Coul}}(q^*)\|J_i\rangle|^2 \tag{6}$$

and

$$|f_+|^2+|f_-|^2=2\pi\sum_{J=1}^{\infty}\{|\langle J_f\|\hat{T}_J{}^{\mathrm{el}}(q^*)\|J_i\rangle|^2 + |\langle J_f\|\hat{T}_J{}^{\mathrm{mag}}(q^*)\|J_i\rangle|^2\},$$

where

$$\hat{M}_{JM}{}^{\mathrm{Coul}}(q^*)=\int d^3x\; j_J(q^*x)Y_{JM}(\Omega_x)\hat{\rho}(\mathbf{x}),$$

$$\hat{T}_{JM}{}^{\mathrm{el}}(q^*)=\frac{1}{q^*}\int d^3x[\nabla\times j_J(q^*x)\mathbf{Y}_{JJ1}{}^M(\Omega_x)]\cdot\hat{\mathbf{J}}(\mathbf{x}),$$

$$\hat{T}_{JM}{}^{\mathrm{mag}}(q^*)=\int d^3x\; j_J(q^*x)\mathbf{Y}_{JJ1}{}^M(\Omega_x)\cdot\hat{\mathbf{J}}(\mathbf{x}). \tag{7}$$

Here $\mathbf{Y}_{JJ1}{}^M(\Omega_x)$ is a vector spherical harmonic, and

$$\hat{\rho}(\mathbf{x})=\sum_{i=1}^{3}\hat{Q}(i)\delta(\mathbf{x}-\mathbf{r}_i),$$

$$\hat{\mathbf{J}}(\mathbf{x})=\mathbf{j}(\mathbf{x})+\nabla\times\boldsymbol{\mu}(\mathbf{x}) =\sum_{j=1}^{3}\frac{\hat{Q}(j)}{2iM_q}\{\delta(\mathbf{x}-\mathbf{r}_j)\nabla_x\}_{\mathrm{sym}}+\nabla \times\sum_{i=1}^{3}\frac{\hat{Q}(i)}{2M_q}\delta(\mathbf{x}-\mathbf{r}_i)\boldsymbol{\sigma}(i)g_q. \tag{8}$$

These expressions for $\hat{\rho}$ and $\hat{\mathbf{J}}$ are nonrelativistic. We note that they satisfy current conservation, i.e., matrix elements of

$$\nabla\cdot\hat{\mathbf{J}}(\mathbf{x})+\partial\hat{\rho}(\mathbf{x})/\partial t$$

vanish. Letting $\mu_q=g_q/2M_q$, it is known that $\mu_q=\mu_p\cong-\frac{3}{2}\mu_n$.

The calculation of the form factors is now straightforward. We take our quark-model wave functions for the initial and final states and use the above expressions to find $|f_c|^2$ and $|f_+|^2+|f_-|^2$. The resulting "shell-model" form factors are then multiplied by an appropriate factor to eliminate the contribution coming from the ground-state wave function of the c.m. coordinate (as described in Ref. 8).

3. RESULTS

Using the formalism of Sec. 2, we have calculated $|f_c|^2$ and $|f_+|^2+|f_-|^2$ [see Eq. (2)] for the reaction

$$e+p\rightarrow e+N^*,$$

where N^* is one of the nine resonances listed in Table I. The results are listed in Tables II–V and are plotted in Figs. 2–16. This section is devoted to a discussion of our results.

The form factors $|f_c|^2$ and $|f_+|^2+|f_-|^2$ are listed in Tables II–IV; we note that each of our form factors is proportional to an exponential. In particular, the inelastic form factors are proportional to the elastic ones (trivially so for the electric form factors). We would expect this behavior from the Gaussian harmonic-oscillator wave functions, since the Fourier transform of a Gaussian is just another Gaussian. The variable appearing in the reduced matrix elements is not unique when one is not in the static limit. For example, one could take it to be q^2 instead of q^{*2}. q^* is the three-momentum transfer in the c.m. frame of the N^*;

$$q^{*2}=q^2+(1/4M^2)(q^2-M^2+m^2)^2;$$

the choice of q^{*2} seems perhaps the most natural to us.[2]

The Coulomb terms $|f_c|^2$ are listed in Table II and are plotted in Fig. 2. These terms are obtained by inserting a charge-density operator $\hat{\rho}(\mathbf{x})$ into the appropriate integral. The equation for this operator as given by Eq. (8) is correct to order $1/M_q$. One can find the $1/M_q^2$ corrections to $\hat{\rho}(\mathbf{x})$ by expanding the relativistic current operator for a spin-$\frac{1}{2}$ Dirac particle expressed in two-component form (see Ref. 8). The result is

$$\hat{\rho}(\mathbf{x})=\sum_{j=1}^{3}\hat{Q}(j)\delta(\mathbf{x}-\mathbf{r}_j)\left\{1-\left[\frac{q^{*2}}{8M_q^2}-\frac{\mathbf{q}^*\cdot(\boldsymbol{\sigma}\times\nabla)}{4M_q^2}\right]\right\}. \tag{9}$$

[The quark g factor g_q is set equal to 1; in general, the $1/M_q^2$ terms are multiplied by $(2g_q-1)$.] Using this equation for $\hat{\rho}(\mathbf{x})$ instead of Eq. (8), we could find the $1/M_q^2$ corrections to $|f_c|^2$. The first $1/M_q^2$ term contributes a factor of $-q^{*2}/4M_q^2$ times the lowest-order $|f_c|^2$ (denoted by $|f_{c0}|^2$), but for the second term the answer depends on the resonance. Taking as a sample case the $1525(\frac{3}{2}-)$ state, we obtain

$$|f_c{}^{1525}|^2=|f_{c0}{}^{1525}|^2\left(1-\frac{q^{*2}}{4M_q^2}+\frac{1}{2M_q^2b^2}\right).$$

Actually, we do not attach great importance to the

TABLE II. Form factors $|f_c|^2$ in the symmetric quark model; $\hat{\rho}$ does not include $1/M_q^2$ corrections. (a_L is a factor arising from center-of-mass elimination; $a_1=1$; $a_2=\sqrt{\frac{2}{3}}$; $a_3=\frac{1}{2}$.)

State	I	$\lvert f_c\rvert^2$
940	$\frac{1}{2}$	$e^{-q^{*2}b^2/3}$
1525	$\frac{1}{2}$	$(1/9)(q^*b)^2e^{-q^{*2}b^2/3}$
1570	$\frac{1}{2}$	$(1/18)(q^*b)^2e^{-q^{*2}b^2/3}$
1670	$\frac{1}{2}$	0
1688	$\frac{1}{2}$	$(1/90)(q^*b)^4e^{-q^{*2}b^2/3}$
1700	$\frac{1}{2}$	0
2190	$\frac{1}{2}$	$(1/2520)(q^*b)^6e^{-q^{*2}b^2/3}$
1236	$\frac{3}{2}$	0
1670	$\frac{3}{2}$	$(1/18)(q^*b)^2e^{-q^{*2}b^2/3}$
1920	$\frac{3}{2}$	0
$J_f=L+\frac{1}{2}$; $L>0$; $S=I=\frac{1}{2}$		$\dfrac{\lvert a_L\rvert^2(J+1)(q^*b)^{2J}e^{-q^{*2}b^2/3}}{3(2J+1)!!2^J}$

TABLE III. Form factors $(|f_+|^2+|f_-|^2)$ in the symmetric quark model: $(|f_+|^2+|f_-|^2)=2\pi|\langle J_f\|\hat{T}_J{}^{\text{mag}}\|J_i\rangle|^2+2\pi|\langle J_f\|\hat{T}_J{}'^{\text{el}}\|J_i\rangle|^2$.

State I	$2\pi\|\langle J_f\\|\hat{T}_J{}^{\text{mag}}\\|J_i\rangle\|^2$	$2\pi\|\langle J_f\\|\hat{T}_J{}^{\text{el}}\\|J_i\rangle\|^2$
940 $\frac{1}{2}$	$(2u_p{}^2/b^2)(q^*b)^2e^{-q^{*2}b^2/3}$	0
1525 $\frac{1}{2}$	$(2\mu_p{}^2/b^2)(1/12)(q^*b)^4e^{-q^{*2}b^2/3}$	$(1/2M_q{}^2b^2)(4/9)e^{-q^{*2}b^2/3}(1-g_qq^{*2}b^2/4)^2$
1570 $\frac{1}{2}$	0	$(1/2M_q{}^2b^2)(2/9)e^{-q^{*2}b^2/3}(1+g_qq^{*2}b^2/2)^2$
1670 $\frac{1}{2}$	0	0
1688 $\frac{1}{2}$	$(2\mu_p{}^2/b^2)(1/135)(q^*b)^6e^{-q^{*2}b^2/3}$	$(1/2M_q{}^2b^2)(2/15)(q^*b)^2e^{-q^{*2}b^2/3}(1-g_qq^{*2}b^2/6)^2$
1700 $\frac{1}{2}$	0	0
2190 $\frac{1}{2}$	$(2\mu_p{}^2/b^2)(1/4032)(q^*b)^8e^{-q^{*2}b^2/3}$	$(1/2M_q{}^2b^2)(2/105)(q^*b)^4e^{-q^{*2}b^2/3}(1-g_qq^{*2}b^2/8)^2$
1236 $\frac{3}{2}$	$(2\mu_p{}^2/b^2)(8/9)(q^*b)^2e^{-q^{*2}b^2/3}$	0
1670 $\frac{3}{2}$	0	$(1/2M_q{}^2b^2)(2/9)e^{-q^{*2}b^2/3}(1-g_qq^{*2}b^2/6)^2$
1920 $\frac{3}{2}$	$(2\mu_p{}^2/b^2)(32/19845)(q^*b)^6e^{-q^{*2}b^2/3}$	0
$J_f=L+\frac{1}{2}$; $L>0$; $S=I=\frac{1}{2}$	$\dfrac{(2\mu_p{}^2/b^2)\|a_L\|^2(J+1)(q^*b)^{2J}e^{-q^{*2}b^2/3}}{3(2J-1)!!}$	$\dfrac{\|a_L\|^2J(J+1)^2(q^*b)^{2J-2}e^{-q^{*2}b^2/3}}{2M_q{}^2b^2\ 3\ 2^{J-1}(2J+1)!!}\left(1-\dfrac{g_qq^{*2}b^2}{2(J+1)}\right)^2$

TABLE IV. Differential cross sections in inelastic electron scattering. ($d\sigma/d\Omega$ is measured in units of cm^2/sr; four-momenta are in BeV/c.)

State	ϵ	q^2	q^*	$d\sigma/d\Omega^a$	$d\sigma/d\Omega^b$	$d\sigma/d\Omega^c$	$d\sigma/d\Omega^d$	$d\sigma/d\Omega^e$
1238	2.358	0.99	1.01	0.15×10^{-32}		0.15×10^{-32}		0.92×10^{-32}
	2.988	1.55	1.30	0.004×10^{-32}		0.004×10^{-32}		0.312×10^{-32}
	4.874	3.62	2.25	0.4×10^{-42}		0.4×10^{-42}		0.0082×10^{-32}
1525		0	0.475	3.0×10^{-32}	4.9×10^{-32}	4.0×10^{-32}	4.8×10^{-32}	2×10^{-32}
	2.358	0.79	0.91	0.74×10^{-32}	1.2×10^{-32}	0.82×10^{-32}	1.2×10^{-32}	0.98×10^{-32}
	2.988	1.30	1.14	0.09×10^{-32}	0.13×10^{-32}	0.09×10^{-32}	0.13×10^{-32}	0.36×10^{-32}
	4.874	3.29	1.91	0.6×10^{-38}	0.9×10^{-38}	0.6×10^{-38}	0.9×10^{-38}	0.0069×10^{-32}
1688		0	0.58	1.16×10^{-32}	1.16×10^{-32}	1.7×10^{-32}	1.8×10^{-32}	1.5×10^{-32}
	2.358	0.65	0.90	0.85×10^{-32}	0.93×10^{-32}	1.0×10^{-32}	1.2×10^{-32}	0.89×10^{-32}
	2.988	1.14	1.10	0.23×10^{-32}	0.24×10^{-32}	0.27×10^{-32}	0.28×10^{-32}	0.33×10^{-32}
	4.874	3.05	1.78	0.4×10^{-36}	0.4×10^{-36}	0.4×10^{-36}	0.4×10^{-36}	0.0078×10^{-32}
1920	2.358	0.43	0.90	0.20×10^{-32}		0.20×10^{-32}		$\leq0.09\times10^{-32}$
	2.988	0.87	1.06	0.08×10^{-32}		0.08×10^{-32}		$\leq0.04\times10^{-32}$
	4.874	2.70	1.64	0.9×10^{-36}		0.9×10^{-36}		$\leq0.0112\times10^{-32}$

[a] Theory; $M_q=\frac{1}{3}m_p$; 1570, 1670, and 1700 states are omitted.
[b] Theory; $M_q=\frac{1}{3}m_p$; 1525+1570 and 1670+1688+1700 contributions.
[c] Theory; $M_q=\infty$; 1570, 1670, and 1700 states are omitted.
[d] Theory; $M_q=\infty$; 1525+1570 and 1670+1688+1700 contributions.
[e] Experimental values (Ref. 19).

$1/M_q{}^2$ corrections in $\hat{\rho}$ alone. An expansion in $1/M_q{}^2$ would give the exact relativistic charge-density operator, but since our wave functions are nonrelativistic and do not include recoil effects, we would still be left with possible errors of order $1/M_q{}^2$.

TABLE V. Photoproduction amplitudes: harmonic-oscillator potential.

State I	Ratio	Predictions of symmetric quark model $M_q=\frac{1}{3}M_p$	$M_q=\infty$	Walker[a]
1525 $\frac{1}{2}$	$M_2{}^-/E_2{}^-$	9.5	-1	0.53 ± 0.2
1570 $\frac{1}{2}$	$M_1{}^-/E_1{}^-$	0	0	
1670 $\frac{1}{2}$	$E_2{}^+/M_2{}^+$	0/0	0/0	-0.5 ± 0.5
1688 $\frac{1}{2}$	$M_3{}^-/E_3{}^-$	10.3	-1	0.5 ± 0.3
1700 $\frac{1}{2}$	$M_1{}^-/E_1{}^-$	0/0	0/0	
2190 $\frac{1}{2}$	$M_4{}^-/E_4{}^-$	-2.65	-1	
1236 $\frac{3}{2}$	$E_1{}^+/M_1{}^+$	0	0	-0.04 ± 0.08
1670 $\frac{3}{2}$	$M_1{}^-/E_1{}^-$	0	0	
1920 $\frac{3}{2}$	$E_3{}^+/M_3{}^+$	0	0	
Ratios of $\|f_+\|^2+\|f_-\|^2$				
$\frac{1}{2}^-(1570)/\frac{3}{2}^-(1525)$		1.6	0.50	0.15 ± 0.2
$\frac{5}{2}^-(1670)/\frac{5}{2}^+(1688)$		0	0	0.24 ± 0.3

[a] See Ref. 12.

The parameters b^2 and g_q/M_q are determined by a fit of the quark-model predictions to elastic scattering data ($g_q/2M_q=\mu_p\cong-\frac{3}{2}\mu_n$; $b^2\cong16.0$). The parameter M_q

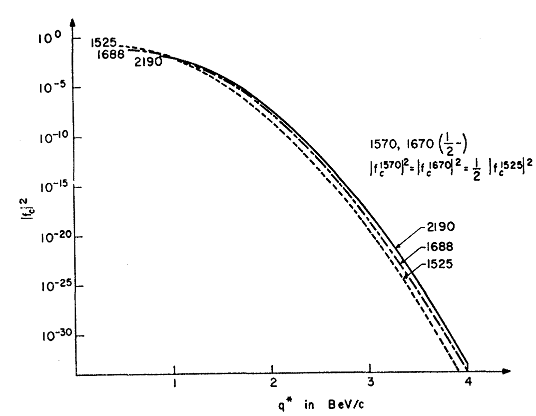

FIG. 2. $|f_c|^2$ in the symmetric quark model; $\hat{\rho}$ does not include $1/M_q{}^2$ corrections.

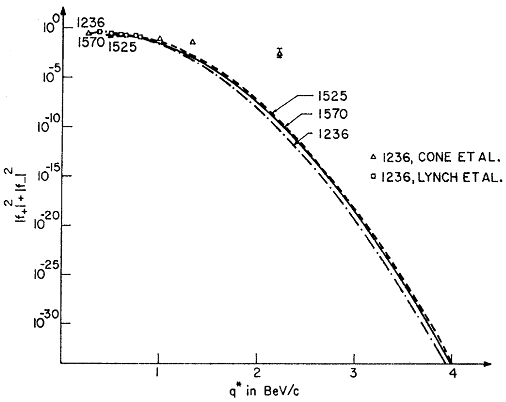

FIG. 3. $(|f_+|^2+|f_-|^2)$ in the symmetric quark model; $M_q=\frac{1}{3}m_{\text{proton}}$.

which appears in the convection part of the total current density operator [Eq. (8)] remains undetermined. We note that $\mathbf{j}(\mathbf{x})$ does not contribute to matrix elements of $\hat{T}_{J0}{}^{\text{mag}}$. This comes about as follows.[10] The relevant expression is (for one of the quarks)

$$\int d^3x\, j_J(q^*x)\mathbf{Y}_{JJ1}{}^0(\Omega_x)\cdot[(\nabla\psi_f{}^*(\mathbf{x}))\psi_i(\mathbf{x})-\psi_f{}^*(\mathbf{x})\nabla\psi_i(\mathbf{x})].$$

Integrating by parts and noting that $\nabla\cdot j_J\mathbf{Y}_{JJ1}{}^0$ must equal zero, this expression reduces to

$$-2\int d^3x\, j_J(q^*x)\mathbf{Y}_{JJ1}{}^0(\Omega_x)\cdot\psi_f{}^*(\mathbf{x})\nabla\psi_i(\mathbf{x}).$$

Now $\mathbf{Y}_{JJ1}{}^0$ is proportional to

$$(1/i)(\mathbf{r}\times\nabla)Y_{J0}$$

and

$$(1/i)(\mathbf{r}\times\nabla)Y_{J0}\cdot\nabla\psi_1=-\nabla Y_{J0}\cdot(1/i)(\mathbf{r}\times\nabla)\psi_i=-\nabla Y_{J0}\cdot\mathbf{l}\psi_1.$$

But $\mathbf{l}\psi_i=0$, since ψ_i is the ground-state wave function, having $l=0$ for each quark. Hence the integral in question vanishes, and thus $\hat{T}_{J0}{}^{\text{mag}}$ receives no contribution from $\mathbf{j}(\mathbf{x})$.

Thus the parameter M_q can affect only $\hat{T}^{\text{el}}$, in the approximation of including no $1/M_q{}^2$ corrections to $\hat{\rho}(\mathbf{x})$. We have chosen two values of M_q: $M_q=\frac{1}{3}m_{\text{proton}}$ (then the quarks have no anomalous magnetic moments) and $M_q=\infty$. Table III lists the quantities $(|f_+|^2+|f_-|^2)$, and the results are plotted in Figs. 3 and 4 (with $M_q=\frac{1}{3}m_{\text{proton}}$) and 5 (with $M_q=\infty$).

Our form factors are expected to be less accurate for large q^*. For the sake of easy visualization, however, we have plotted the results up to $q^*=4$ BeV/c.

For the resonances with $J=L+\frac{1}{2}$ and total quark spin $S=\frac{1}{2}$, we discover that if one does not separate out any c.m. motion of the quarks, then (see Tables II and III) the reduced matrix elements do not depend on whether the particular resonance belongs to a **56** or to a **70** representation of $SU(6)$.

We also note that some of the matrix elements are zero. Those resonances having $S=\frac{3}{2}$ can only be excited by $\boldsymbol{\mu}(\mathbf{x})$, since the initial quark spin is $\frac{1}{2}$ and $\hat{\rho}(\mathbf{x})$ and $\mathbf{j}(\mathbf{x})$ do not depend on quark spin. Hence $\hat{M}_{JM}{}^{\text{Coul}}$ has zero matrix element for the $1670(\frac{5}{2}^-)$, $1700(\frac{1}{2}^-)$, $1236(\frac{3}{2}^+)$, and $1920(\frac{7}{2}^+)$ states. The vanishing matrix elements for $\hat{T}^{\text{mag}}[1570(\frac{1}{2}^-)$ and $1670(\frac{1}{2}^-)$ states] and for $\hat{T}^{\text{el}}[940(\frac{1}{2}^+)]$ are easily seen to arise, using angular momentum and parity conservation. The zero results associated with $\hat{T}^{\text{mag}}$ and $\hat{T}^{\text{el}}$ [for both the $1670(\frac{5}{2}^-)$ and $1700(\frac{1}{2}^-)$ resonances] will be explained if we can show that $\boldsymbol{\mu}$ does not contribute, since we have already seen that $\mathbf{j}$ cannot contribute for $S=\frac{3}{2}$ states. (The vanishing of these matrix elements has been previously noted by Moorhouse.[6]) The derivation is as follows.

Now $\boldsymbol{\mu}=\boldsymbol{\mu}(1)+\boldsymbol{\mu}(2)+\boldsymbol{\mu}(3)$ from Eq. (8). It suffices to show that $\boldsymbol{\mu}(3)$ has zero matrix element. The final-state wave function contains spin-isospin pieces of the type $|\frac{3}{2}S_z\rangle|f_a\rangle$ and $|\frac{3}{2}S_z\rangle|f_b\rangle$, where f_a and f_b are defined in (4). The subscripts a and b refer to the isospin

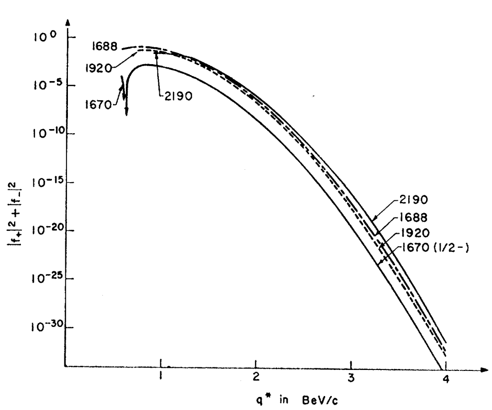

FIG. 4. Same as Fig. 3, but with different states.

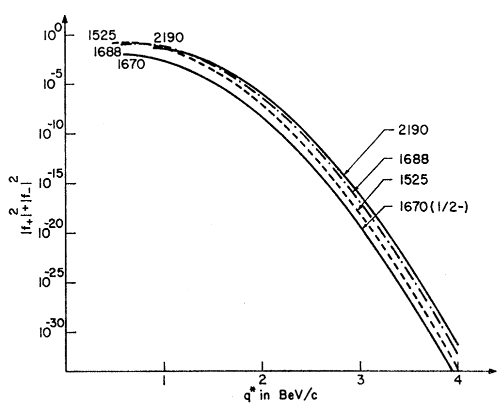

FIG. 5. $(|f_+|^2+|f_-|^2)$ in the symmetric quark model; $M_q=\infty$. The 1236 and 1920 states have the same $(|f_+|^2+|f_-|^2)$ as in Figs. 3 and 4. $(|f_+|^2+|f_-|^2)$ for the 1570 state is roughly $\frac{1}{3}$ that of the 1525 state.

[10] This proof is due to J. D. Walecka (private communication).

of particles one and two (a means $I_{12}=0$, and b means $I_{12}=1$). The initial state is a **56**, with wave function given by (5). Thus,

$$\langle \tfrac{3}{2}S_z|\langle f_a|\boldsymbol{\mu}(3)|(u_af_a+u_bf_b)\sqrt{\tfrac{1}{2}}\rangle$$

$$=\langle \tfrac{3}{2}S_z|\langle f_a|\boldsymbol{\mu}(3)|u_af_a\sqrt{\tfrac{1}{2}}\rangle$$ (this is true because the isospins I_{12} in f_a and f_b are perpendicular)

$$=0$$ (since u_a has spins 1 and 2 adding to give spin 0, whereas in $|\tfrac{3}{2}S_z\rangle$ these spins add to give spin 1).

Also,

$$\langle \tfrac{3}{2}S_z|\langle f_b|\boldsymbol{\mu}(3)|(u_af_a+u_bf_b)\sqrt{\tfrac{1}{2}}\rangle$$

$$=\langle \tfrac{3}{2}S_z|\langle f_b|\boldsymbol{\mu}(3)|u_bf_b\sqrt{\tfrac{1}{2}}\rangle$$ (here again, f_a and f_b have isospins I_{12} which are $\perp$)

$$=\left[\langle \tfrac{3}{2}S_z|\frac{\boldsymbol{\sigma}(3)\delta(\mathbf{x}-\mathbf{r}_3)}{2M_q}|u_b\sqrt{\tfrac{1}{2}}\rangle\right][\langle f_b|\hat{Q}(3)|f_b\rangle]$$

$$=0 \qquad [\text{since } \langle f_b|\hat{Q}(3)|f_b\rangle=0].$$

Hence $\boldsymbol{\mu}$ has zero matrix elements for the $1670(\frac{5}{2}^-)$ and $1700(\frac{1}{2}^-)$ cases; thus these resonances have no magnetic or electric matrix elements. In an analogous fashion one can show that $\hat{T}^{\rm el}$ has zero matrix elements for the $1236(\frac{3}{2}^+)$ and $1920(\frac{7}{2}^+)$ states (both of these states have $S=\frac{3}{2}$ and could thus only be excited by $\boldsymbol{\mu}$).

For those matrix elements which do not vanish, we note that the power of q^* multiplying the exponential is just what one must have to give the correct behavior as $q^*\to 0$.[2] The powers of q^* are relatively small, and hence on a log plot of form factor versus q^*, we see that the dominant behavior is simply a parabola arising from the exponential. This explains the similar appearance of the form factors plotted in Figs. 2–5. The exception to the rule is the $1670(\frac{1}{2}^-)$ resonance. As noted above, this resonance has no $\hat{T}^{\rm mag}$ matrix elements. Since the $\hat{T}^{\rm el}$ matrix element has a zero, we see that $|f_+|^2+|f_-|^2$ must also have a zero, and hence $\ln(|f_+|^2+|f_-|^2)$ has a singular point for the $1670(\frac{1}{2}^-)$ case.

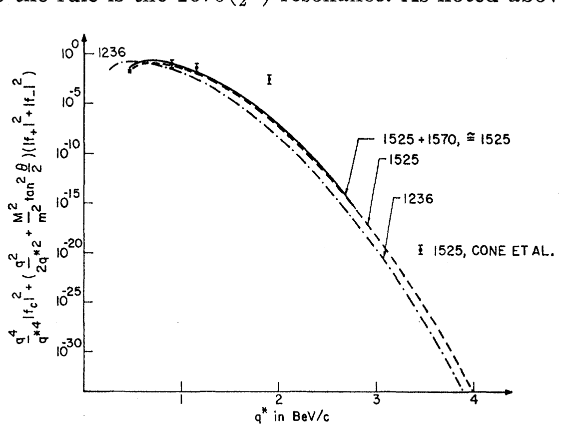

FIG. 6. Transition probabilities in the symmetric quark model; $\theta_{\rm lab}=31°$ and $M_q=\frac{1}{3}m_{\rm proton}$. $\hat{p}$ does not include $1/M_q^2$ corrections.

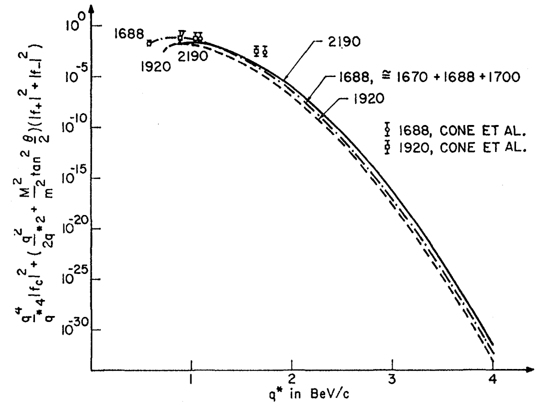

FIG. 7. Same as Fig. 6, but with different states.

The squares of the transition probabilities are plotted in Figs. 6 and 7 ($M_q=\frac{1}{3}m_p$) and 8 ($M_q=\infty$).[11] They exhibit the same general features as the form factors. The 1525 and 1688 resonance contributions are computed both separately and with the 1570, 1670, and 1700 contributions added, respectively. Adding these contributions (with masses taken at 1525 and 1688 MeV) increases the theoretical prediction for the differential cross sections. Table IV shows a comparison of the theoretical predictions with inelastic electron scattering data. (This will be discussed in Sec. 4.)

Figures 8–15 give plots of form factors and transition probabilities, all divided by the common factors

$$e^{-q^{*2}b^2/3}.$$

Figure 16 gives a comparison of the elastic prediction

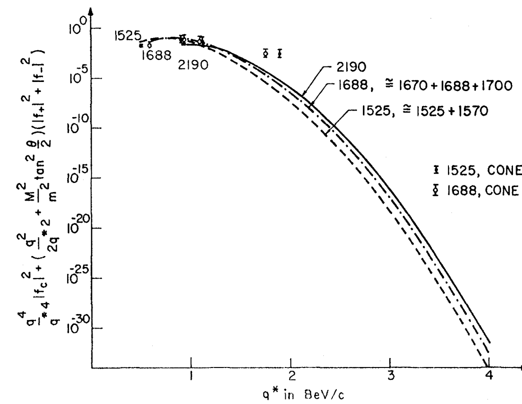

FIG. 8. Transition probabilities in the symmetric quark model; $\theta_{\rm lab}=31°$, and $M_q=\infty$. The 1236 and 1920 states have the same transition probabilities as in Figs. 6 and 7.

[11] In doing the numerical evaluations for the tables and the graphs of this paper, the factor M^2/m^2 (which multiplies $\tan^2\frac{1}{2}\theta$) in $d\sigma/d\Omega$ was set equal to 1 for states with $J_f\neq L+\frac{1}{2}$.

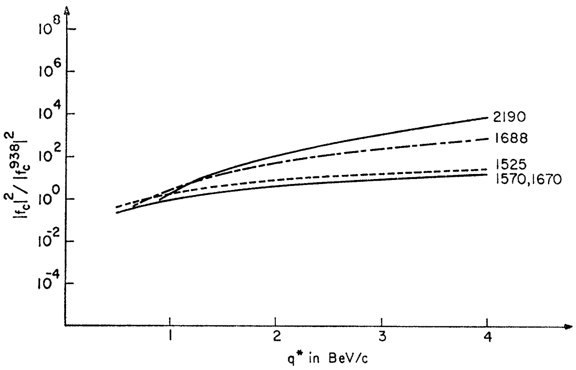

FIG. 9. $|f_c|^2/|f_c^{938}|^2$ in the symmetric quark model; $\hat{p}$ does not include $1/M_q^2$ corrections.

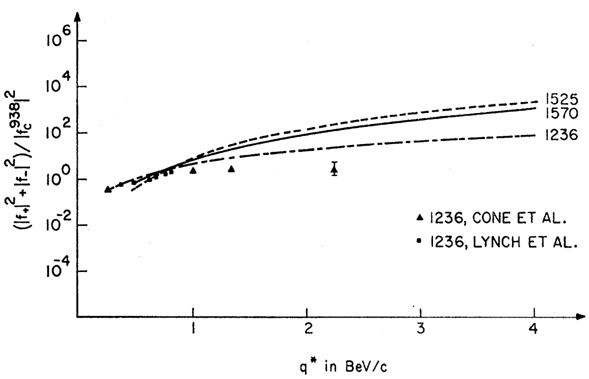

FIG. 10. $(|f_+|^2+|f_-|^2)/|f_c^{938}|^2$ in the symmetric quark model; $M_q=\frac{1}{3}m_{\text{proton}}$.

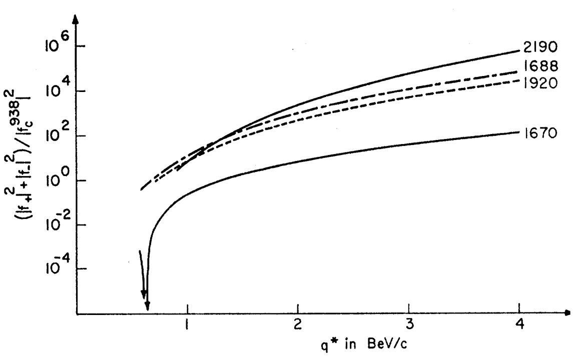

FIG. 11. Same as Fig. 10, but with different states.

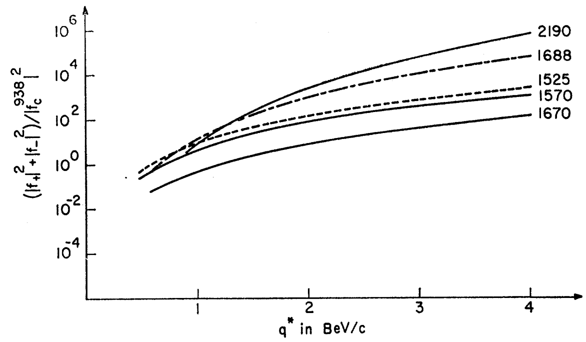

FIG. 12. $(|f_+|^2+|f_-|)^2/|f_c^{938}|^2$ in the symmetric quark model; $M_q=\infty$. The 1236 and 1920 states have the same $(|f_+|^2+|f_-|^2)/|f_c^{938}|^2$ as in Figs. 10 and 11.

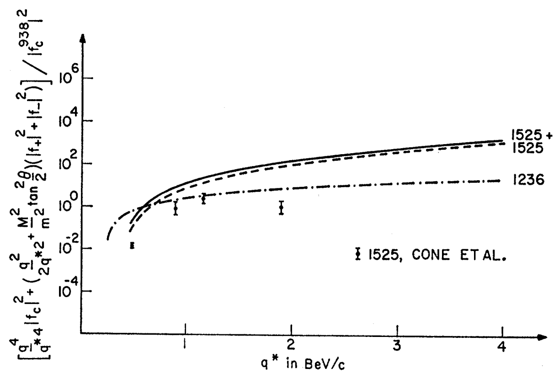

FIG. 13. Transition probabilities divided by $|f_c^{938}|^2$ in the symmetric quark model; $\theta_{\text{lab}}=31°$, and $M_q=\frac{1}{3}m_{\text{proton}}$.

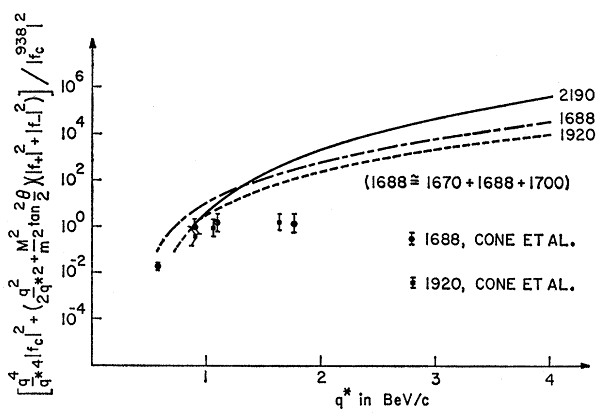

FIG. 14. Same as Fig. 13, but with different states.

with the dipole fit to $|G_{ep}|^2$. Table V shows the predictions for various photoproduction amplitudes.[12]

4. COMPARISON WITH EXPERIMENT

Inelastic electron scattering has been studied in several experiments. Panofsky and Allton,[13] Ohlsen,[14] and Hand[15] studied $N^*(1236)$ excitation. More recently, experiments have been performed by Baba *et al.*,[16] Brasse *et al.*,[17] Cone *et al.*,[18] and Lynch *et al.*[19] An experi-

[12] P. L. Pritchett and J. D. Walecka, Phys. Rev. (to be published).
[13] W. K. H. Panofsky and E. Allton, Phys. Rev. **110**, 1155 (1958).
[14] G. G. Ohlsen, Phys. Rev. **120**, 584 (1960).
[15] L. N. Hand, Phys. Rev. **129**, 1834 (1963).
[16] K. Baba, N. Kajiura, S. Kaneko, K. Huke, R. Kikuchi, Y. Kobayashi, and T. Yamakawa, Report of Hiroshima University, Japan, 1967 (unpublished).
[17] F. W. Brasse, J. Engler, E. Ganssauge, and M. Schweitzer, DESY Report, 1967 (unpublished).
[18] A. A. Cone, K. W. Chen, J. R. Dunning, Jr., G. Hartwig, Norman F. Ramsey, J. K. Walker, and Richard Wilson, Phys. Rev. **156**, 1490 (1967).
[19] H. L. Lynch, J. V. Allaby, and D. M. Ritson, Phys. Rev. **164**, 1635 (1967).

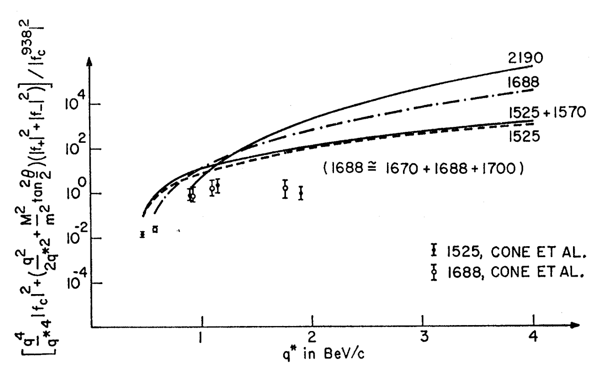

FIG. 15. Transition probabilities divided by $|f_c^{938}|^2$ in the symmetric quark model; $\theta_{\rm lab}=31°$, and $M_q=\infty$. The 1236 and 1920 states have the same curves as in Figs. 13 and 14.

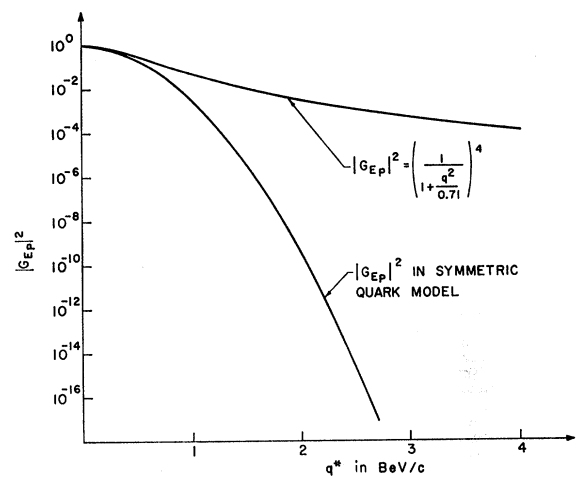

FIG. 16. $|f_c|^2/(1+q^2/4m^2)=|G_{Ep}^{\rm theor}|^2$ for elastic scattering. Also plotted is the dipole expression for $|G_{Ep}|^2$.

ment on inelastic electron scattering is also currently being performed at SLAC.

The data which lend themselves most readily to a comparison with our predictions are those of Cone *et al.*,[18] since these authors make a fit to obtain the total area under the resonance bumps. In other words, they give values of $d\sigma/d\Omega$ in addition to $d^2\sigma/d\Omega dE_f$. Since the separation of resonant and nonresonant parts of the cross section is difficult, the errors introduced in such a procedure could be from a factor of $1\frac{1}{4}$ to a factor of 4.[18] In Table IV, we list the values of $d\sigma/d\Omega$ obtained by Cone *et al.*[18] compared with the predictions of the symmetric quark model. (The experimental points at $q^2=0$ are obtained from photoproduction.[20–22]) The theoretical values are in general too small when $q^2>1$, but agree for smaller q^2.

In Figs. 3, 5, 6, 7, 10, and 13–15 we plot the experimental points (obtained from Ref. 18) due to Cone *et al.* and Lynch *et al.* The theoretical predictions for the form factors and transition are again too small for large q^2, but tend to agree when $q^2<1$.

[20] J. D. Walecka, Phys. Rev. **162**, 1462 (1967).
[21] J. D. Walecka, in Proceedings of the 1967 International Symposium on Electron and Photon Interactions at High Energies (unpublished).
[22] The experimental points in Figs. 2–14 are obtained from Refs. 20 and 21.

5. RELATED THEORETICAL WORK

Other authors have also investigated inelastic electron scattering to predict form factors for the nucleon resonances. The most closely related work is by Fujimura *et al.*,[23] who also use the quark model. Their basic wave functions are also those of a harmonic oscillator, but there are several important differences between the two approaches. The set of resonances examined overlaps but is not the same. Fujimura *et al.* also investigate the 2360, 2645, and 2825 states (which we do not), but they do not calculate the $1570(\frac{1}{2}-)$, $1670(\frac{5}{2}-)$, $1700(\frac{1}{2}-)$, and $1670(\frac{3}{2}-)$ states (included in the present paper). Different assignments to $SU(6)$ representations are made; Fujimura *et al.* place each N^* in a **56** representation, and of the five resonances calculated in both cases, we place two in a **70** representation. The harmonic-oscillator parameter b^2 is determined differently in the two cases. Fujimura *et al.* take pion cloud effects into account (which we do not). We briefly investigate effects of $1/M_q^2$ corrections in the charge-density operator $\hat{\rho}(\mathbf{x})$, and we separate out c.m. motion from the wave functions. (If one removes c.m. motion, then the 1525 cannot be placed in a **56** representation, as was done by Fujimura *et al.*)

Other theoretical studies of inelastic electron scattering have been made; a summary of $N^*(1236)$ work can be found in Ref. 17. Predictions for all the states considered in the present paper have been obtained by Walecka[20,21] and Walecka *et al.*,[12,24] using a model of oscillations in the meson field in the nucleon,[12,20] and a model employing N/D formalism.[21,24]

ACKNOWLEDGMENTS

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APPENDIX

In this Appendix we show in more detail how to construct wave functions for the resonances. (This material is not new, but is presented for the sake of completeness.) A given resonance has fixed total spin. Given the quantum number assignments of Table I, we use Clebsch-Gordan coefficients to write $|JJ_z\rangle$ in terms

[23] K. Fujimura, T. Kobayashi, T. Kobayashi, and M. Namiki, Progr. Theoret. Phys. (Kyoto) **37**, 916 (1967); **38**, 210 (1967).
[24] J. D. Walecka and P. A. Zucker, Phys. Rev. **167**, 1479 (1968).

of $|LL_z\rangle|SS_z\rangle$ states. If the resonance belongs to a **56** representation of $SU(6)$, we then symmetrize $|LL_z\rangle$. The **56** spin-isospin wave function is next constructed by taking linear combinations of product states [such as $\alpha(1)\beta(2)\alpha(3)\beta'(1)\alpha'(2)\alpha'(3)$] and requiring symmetry under exchange of any two quarks. The result for $S=S_z=+\frac{1}{2}$ and $I=I_z=+\frac{1}{2}$ has been given [see Eq. (5)]. This yields the total wave function for the **56** case.

If the resonance belongs to a **70** representation, then instead of completely symmetrizing $|LL_z\rangle$, we form the analogs of u_a and u_b:

$$\phi_a=(\sqrt{\tfrac{1}{2}})[\phi_{\rm ex}(1)\phi_0(2)\phi_0(3)-\phi_0(1)\phi_{\rm ex}(2)\phi_0(3)]$$

and

$$\phi_b=(\sqrt{\tfrac{1}{6}})[\phi_{\rm ex}(1)\phi_0(2)\phi_0(3)+\phi_0(1)\phi_{\rm ex}(2)\phi_0(3)-2\phi_0(1)\phi_0(2)\phi_{\rm ex}(3)].$$

($\phi_{\rm ex}$ denotes an excited state.) These functions are then multiplied by appropriate **70** spin-isospin functions When $S=S_z=I=I_z=+\frac{1}{2}$, for example, the result is

$$\Psi=(\sqrt{\tfrac{1}{4}})[\phi_a(u_af_b+u_bf_a)+\phi_b(u_af_a-u_bf_b)].$$

(This function is totally symmetric under interchange of any two quarks.)

After obtaining the wave functions, we remove any c.m. motion, as described in the text.

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Momentum Transfer Dispersion Relations for Three-Particle Potential Scattering Amplitudes*

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The analytic properties in momentum transfer of a class of three-particle scattering amplitudes are investigated in this paper. The amplitudes considered are those in which there is a two-particle bound state in both the initial and final state. The following results are obtained: (1) The amplitudes are analytic inside a Lehmann ellipse in the scattering angle for all real energies. (2) For real energies below the three-free-particle threshold, the amplitudes are analytic in the momentum transfer plane except for real left- and right-hand cuts.

I. INTRODUCTION

A KNOWLEDGE of the analytic properties of multiparticle scattering amplitudes is central to a complete S-matrix theory calculation of hadron parameters and also to the extended phenomenological analysis of their reaction processes. While no dispersion relations for relativistic multiparticle scattering amplitudes have been rigorously established, some progress towards this goal has been made in the laboratory of potential scattering. In particular, dispersion relations in the total energy for fixed directions of the individual momenta have been proved for nonrelativistic three-particle scattering amplitudes.[1]

In this paper, we investigate the analytic properties in the momentum transfer variable of a class of three-particle scattering amplitudes. The class of amplitudes we consider are those which describe the elastic scattering of a single particle from a bound state and those which describe rearrangement collisions in which two of the initial and final particles are in a bound state. We will refer to these as "bound-state amplitudes." This class of processes is distinguished by having a single well-defined momentum transfer. The present study is restricted to the study of spinless, nonrelativistic particles which interact via two-body central potentials, which can be written as a superposition of Yukawa potentials.

In Sec. II we extend a result of Immirzi[2] to show that the bound-state amplitudes are analytic functions of the scattering angle inside a Lehmann ellipse for all real values of the energy E. In Sec. III we study the analytic properties in the scattering angle of all perturbation-theory diagrams. For real energies below the three-free-particle threshold we find that each of the perturbation-theory diagrams is analytic in the entire scattering-angle plane with the exception of cuts along the positive and negative real axes. In the remaining sections we show that these analytic properties are enjoyed by the full amplitude as well. In Sec. IV we prove this result for the individual terms in the Fredholm expansion of the Faddeev equations, and in Sec. V we show that this expansion converges uniformly in the domain of analyticity. Thus, the result holds for the full scattering amplitude. The conclusion of this paper is that the nonrelativistic scattering amplitudes with a

* Supported in part by the National Science Foundation.

[1] M. H. Rubin, R. L. Sugar, and G. Tiktopoulos, Phys. Rev. **146**, 1130 (1966); **159**, 1348 (1967); **162**, 1555 (1967). These papers will be referred to as I, II, and III, respectively.

[2] G. Immirzi, Nuovo Cimento **34**, 1361 (1964).