Equivalence Principle for Massive Bodies. II. Theory

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(Received 16 October 1967; revised manuscript received 18 January 1968)

The acceleration of a massive body in an external field for general space-time geometrical gravitational theories is obtained. The condition on the metric is such that $m_g/m_i=1$ is obtained, and we reobtain the result that $m_g/m_i=1$ in Einstein's theory for massive objects with time-independent internal structure. But it is shown that a measurement of m_g/m_i for astronomical bodies would measure space-time metric components which have not been measured in other gravitational experiments. In the scalar-tensor gravitational theory due to Brans and Dicke, it is shown that m_g/m_i differs from 1 by a term of the order of the massive body's gravitational self-energy divided by its total energy.

I. INTRODUCTION

In another paper,[1] it was shown that the experiments of Eotvos[2] and Dicke[3] which measure the equality of gravitational and inertial masses of bodies to be within a part of 10^{11} indicate nothing about whether the gravitational self-energy of bodies contributes equally to both the gravitational and inertial masses. If the ratio of gravitational to inertial mass for a body is assumed to be

$$\frac{m_g}{m_i}=1+\eta\frac{G}{c^2}\int\rho(x)\rho(x')\frac{d^3xd^3x'}{|\mathbf{x}-\mathbf{x}'|}\Big/\int\rho(x)d^3x, \quad (1)$$

where η is a dimensionless constant of order 1, G is the gravitational constant, c is the velocity of light, and $\rho(x)$ is the mass density of the body, then the correction term in (1) is of order 10^{-25} for the bodies used by Eotvos[2] and Dicke[3] in their experiments.

For astronomical bodies, the correction term in (1) becomes much larger (10^{-8} for the planet Jupiter, 10^{-5} for the Sun). In I, several experiments were proposed to measure m_g/m_i for astronomical bodies and thereby measure η.

In this paper, gravitational theories will be examined with the purpose of determining what a measurement of η in (1) would reveal about the gravitational theories. Our consideration will be restricted to gravitational theories which can be expressed as geometrical theories, that is, as curved Riemannian space-time geometries in which "test particles" move along geodesics of the geometry. The equivalence principle (EP) is therefore immediately valid for "test particles" which follow geodesics of the geometry, but this paper is concerned with the movement of massive bodies and whether $m_g/m_i=1$ for them also.

Massive bodies will be placed at rest in a space-time also containing a distant external mass source M_e. The acceleration of the massive bodies, which is proportional to

$$\mathbf{g}=-(GM_e/R^3)\mathbf{R}, \quad (2)$$

where $\mathbf{R}$ is the vector from the external mass to the massive body, will be calculated. The ratio of m_g/m_i for the massive body will be obtained by making the identification

$$d^2\mathbf{x}/dt^2=(m_g/m_i)\mathbf{g}. \quad (3)$$

This approach to the problem fits the original domain of application of the EP—weak gravitational fields and slowly moving bodies. Also, by focusing on the Newtonian $1/R^2$ acceleration, several potential complications are bypassed.

(a) A massive body is by necessity an extended body which can sample higher multipoles of the external gravitational field. However, multipole acceleration terms go as R^{-n}, $n>2$, and will not contribute to (3).

(b) Relativistic gravitational theories are known to yield accelerations toward external bodies which deviate from the Newtonian R^{-2} acceleration, but these deviations go as R^{-3}, etc., and also will not contribute to (3).

In this paper we reobtain the results of Fock[4] and Papapetrou,[5] that in Einstein's gravitational theory $m_g/m_i=1$ for a *stationary*, stable massive body. However, this null result is shown to be due to the exact cancellation of several nonzero correction terms in (1), and here we explicitly express our result in terms of the several nonzero terms. Therefore, an experimental measure of η in (1), as proposed in I, would offer an experimental test of metric terms in Einstein's gravitational theory which have not been measured to date.

Also, it is shown that the Brans-Dicke gravitational theory[6] does not fulfill the EP. We obtain the result that $m_g/m_i\neq 1$ for that theory.

II. GENERAL METRIC EXPANSION (SINGLE STATIC SOURCE)

To illustrate the approach to the problem which will be employed in this paper, we review the metric analy-

[1] K. Nordtvedt, Jr., preceding paper, Phys. Rev. **169**, 1014 (1968); hereafter referred to as I.
[2] R. V. Eotvos, Ann. Physik **68**, 11 (1922).
[3] P. G. Roll, R. Krotkov, and R. H. Dicke, Ann. Phys. (N. Y.) **26**, 442 (1964).
[4] V. Fock, *The Thoery of Space, Time, and Gravitation* (The Macmillan Co., New York, 1964), 2nd ed., Chap. VI.
[5] A. Papapetrou, Proc. Phys. Soc. (London) **64A**, 57 (1951).
[6] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).

sis of Eddington,[7] Robertson,[8] and Schiff[9] which we will follow and extend. Given a spherically symmetric static source of gravitation of strength

$$m \equiv GM/c^2 \tag{4}$$

(through most of the rest of the work, we will use units in which $G=c=1$), the most general Reimannian space-time exterior geometry can be written as

$$ds^2 = g_{ij}dx^i dx^j, \tag{5}$$

with the metric components given by a general power-series expansion in the sole dimensionless constant of the problem m/r;

$$g_{00} = 1-2\alpha(m/r)+2\beta(m/r)^2+\cdots, \tag{6a}$$

$$g_{0k} = 0, \tag{6b}$$

$$g_{kk'} = -[1+2\gamma(m/r)]\delta_{kk'}+\cdots. \tag{6c}$$

Equations (6a)–(6c) are required to approach the Lorentz metric as $r\to\infty$, 0 indicates the time coordinate, $k=1, 2, 3$ are the three spatial coordinates, r is a radial variable, $r=(x^2+y^2+z^2)^{1/2}$, $\alpha, \beta, \gamma, \cdots$ are dimensionless constants of order 1 which are determined by the assumption of a particular gravitational theory and by field equations for the g_{ij}. The power series Eqs. (6a)–(6c) are assumed to be convergent for sufficiently small m/r. With (4) giving the connection between the source mass M and the parameter m, $\alpha\equiv 1$ in order to obtain Newtonian gravitation as a weak-field limit of the gravitational theory.

An analysis of past and future experimental tests of relativity in terms of the general metric above yields their dependence on the parameters $\alpha, \beta, \gamma, \cdots$.

(a) The frequency shift of spectral lines in a gravitational potential ϕ is[9]

$$\delta\nu/\nu = -\alpha(\phi/c^2). \tag{7}$$

(b) The deflection of light passing at distance d from a source m is[9]

$$\delta\theta = 2(m/d)(\alpha+\gamma). \tag{8}$$

(c) The angular advance of the perihelion position of a planetary orbit of semimajor axis a, period T, and eccentricity e is (per revolution)[9]

$$\theta = [2\alpha(\alpha+\gamma)-\beta]\frac{8\pi^3 a^2}{c^2T^2}\frac{1}{1-e^2}. \tag{9}$$

(d) The change in round-trip radar time between two planets (in circular orbits of radius r_1 and r_2) when the radar path passes close by the Sun at distance d is[10]

$$\delta t = [2(\alpha+\gamma)\ln(4r_1r_2/d^2)-\tfrac{4}{3}(\gamma+2\beta)](m/c). \tag{10}$$

[To obtain (10), it is important to state that the zero-order time must be defined in terms of measurables, i.e., orbital periods of the two planets, not their radial distances r_1 and r_2 which are coordinate-system-dependent.][11]

(e) The geodetic precession of a gyroscope spin axis when the gyroscope is in a circular orbit of angular frequency ω about a central body m, with the gyroscope spin axis in the orbital plane, is[11]

$$\Omega = -[(\alpha+2\gamma)/2](m/r)\omega. \tag{11}$$

For Einstein's theory of gravitation $\alpha=\beta=\gamma=1$, but the value of the general analysis above is that it allows a simple determination of the expected experimental results for any space-time metric. Also the use of the coefficients yields the sensitivity of any experiment on each of the metric components.

III. GENERAL METRIC EXPANSION (SEVERAL MOVING SOURCES)

For the purposes of this paper, Eqs. (6a)–(6c) must be generalized to give the metric for several moving sources. Only g_{00} will be needed beyond the linear order in the source strengths, and g_{00} will only be needed to second order.

Immediately, (6c) generalizes to

$$g_{kk'} = -\left[1+2\gamma\left(\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|}+\frac{m_2}{|\mathbf{r}-\mathbf{r}_2|}\right)\right]\delta_{kk'}+\cdots \tag{12a}$$

for two sources, where correction terms due to motion of the sources are not required to be kept in $g_{kk'}$. Equation (12a) is uniquely determined to this approximation by imposing the condition that a two-source metric must become the one-source metric in the limit as either of the source strengths vanishes.

For moving sources, the mixed space-time components of the metric are nonzero and to lowest order in the velocity of the sources are

$$g_{0k} = 4\Delta\left(\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|}\frac{dx_1^k}{dt}+\frac{m_2}{|\mathbf{r}-\mathbf{r}_2|}\frac{dx_2^k}{dt}\right) +4\Delta'\left(\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|^3}(\mathbf{r}-\mathbf{r}_1)\cdot\frac{d\mathbf{x}_1}{dt}(r-r_1)^k +\frac{m_2}{|\mathbf{r}-\mathbf{r}_2|^3}(\mathbf{r}-\mathbf{r}_2)\cdot\frac{d\mathbf{x}_2}{dt}(r-r_2)^k\right)+\cdots, \tag{12b}$$

[7] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, New York, 1957), p. 105.
[8] H. P. Robertson, in *Space Age Astronomy*, edited by A. J. Deutsch and W. E. Klemperer (Academic Press Inc., New York, 1962), p. 228.
[9] L. I. Schiff, in Proceedings of the 1965 Summer Seminar on Relativity and Astrophysics (unpublished).
[10] D. H. Ross and L. I. Schiff, Phys. Rev. **141**, 1215 (1966).
[11] L. I. Schiff, in Proceedings of International Conferences on Relativity and Gravitation, Warsaw, 1962 (unpublished).

where the new dimensionless constants Δ and Δ' are introduced. Equation (12b) is the most general expression which transforms like a spatial vector under spatial rotations, and which is linear in source strength and source velocity. (Einstein's theory gives $\Delta=1$, $\Delta'=0$, but the Δ' term cannot be ruled out in considering all possible geometries fulfilling our general conditions.)

The g_{00} metric component to lowest order for two sources is

$$g_{00}{}^{(1)}=1-2\alpha\left(\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|}+\frac{m_2}{|\mathbf{r}-\mathbf{r}_2|}\right). \tag{12c}$$

The next-order general terms which can contribute to g_{00} are given by

$$g_{00}{}^{(2)}=2\beta\left(\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|}+\frac{m_2}{|\mathbf{r}-\mathbf{r}_2|}\right)^2+2\alpha'\frac{m_1m_2}{|\mathbf{r}_1-\mathbf{r}_2|}\left(\frac{1}{|\mathbf{r}-\mathbf{r}_1|}+\frac{1}{|\mathbf{r}-\mathbf{r}_2|}\right)-\alpha''\left[\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|}\left(\frac{d\mathbf{x}_1}{dt}\right)^2+\frac{m_2}{|\mathbf{r}-\mathbf{r}_2|}\left(\frac{d\mathbf{x}_2}{dt}\right)^2\right]$$
$$+\chi\left[\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|}(\mathbf{r}-\mathbf{r}_1)\cdot\mathbf{a}_1+\frac{m_2}{|\mathbf{r}-\mathbf{r}_2|}(\mathbf{r}-\mathbf{r}_2)\cdot\mathbf{a}_2\right]+\alpha'''\left[\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|^3}\left((\mathbf{r}-\mathbf{r}_1)\cdot\frac{d\mathbf{x}_1}{dt}\right)^2+\frac{m_2}{|\mathbf{r}-\mathbf{r}_1|}\left((\mathbf{r}-\mathbf{r}_2)\cdot\frac{d\mathbf{x}_2}{dt}\right)^2\right]+\cdots, \tag{12d}$$

where $\mathbf{a}=d^2\mathbf{x}/dt^2$. Equation (12c) follows by the superposition principle for linear terms. Equation (12d) is unique (up to the magnitude of the dimensionless coefficients) with the imposition of the following conditions:

(a) g_{00} becomes (6a) as either mass is set equal to zero or the position of either mass goes to infinity and the other mass is at rest.

(b) $g_{00}\to 1$ as one moves far from all sources. Before the limit 1 is reached, $g_{00}\to 1-2(M/R)+2\beta(M/R)^2$ [see (6a)], where $M=m_1+m_2$ to lowest order.

(c) The correction terms in g_{00} due to motion of the sources are second order in time derivatives of position. (This is part of a more general condition that g_{0k} be odd in time derivatives of source position, while the other metric components are even in the time derivatives.[12])

(d) g_{00} is symmetric under the interchange of source labels.

(e) g_{00} is a scalar under spatial rotations of the coordinate system.

IV. EQUATION OF MOTION

The massive bodies which will be studied are to be considered an assembly of mass elements, each of which is assumed to follow geodesics of the geometry produced by all the other matter in the space, i.e., each mass element moves in the geometry produced by any external masses plus the other mass elements in the massive body.

The result we possibly expect is that a massive body m of radius a could have an anomalous acceleration in a gravitational field $\mathbf{g}$ of order

$$\mathbf{a}\sim(m/a)\mathbf{g}. \tag{13}$$

If this massive body is divided up into N elements of mass

$$\delta m\sim m/N \tag{14a}$$

and size

$$\delta a\sim a/N^{1/3}, \tag{14b}$$

then the internal effects in each mass element would be expected to cause anomalous accelerations for each element of order

$$\delta\mathbf{a}\sim(\delta m/\delta a)\mathbf{g}\sim(1/N^{2/3})(m/a)\mathbf{g}, \tag{15}$$

which goes to zero as $N\to\infty$. We can then be justified in considering mass elements as following geodesic paths in the geometry produced by *all other matter*.

The equation of motion for a body with its position x^k given as a function of coordinate time t is desired, so we write the proper time integral

$$s=\int dt(g_{00}+2g_{0k}v^k+g_{kk'}v^kv^{k'})^{1/2}, \tag{16}$$

with $v^k=dx^k/dt$. The proper time s is required to be an extremum for the actual trajectory $x^k(t)$. We write

$$g_{00}=1+h_{00}{}^{(1)}+h_{00}{}^{(2)}, \tag{17a}$$

$$g_{0k}=h_{0k}, \tag{17b}$$

$$g_{kk'}=-(1-h_{ss})\delta_{kk'}, \tag{17c}$$

and expand (16) to sufficient approximation to obtain

$$s=\int dt[1-\tfrac{1}{2}v^2-\tfrac{1}{8}v^4+\tfrac{1}{2}h_{00}{}^{(1)}-\tfrac{1}{8}h_{00}{}^{(1)2}+\tfrac{1}{2}h_{00}{}^{(2)}+h_{0k}v^k+v^2(\tfrac{1}{2}h_{ss}+\tfrac{1}{4}h_{00}{}^{(1)})]. \tag{18}$$

Under a variation of the trajectory, $x^k\to x^k+\delta x^k$ which vanishes on the end points, and integrating by parts, the equation of motion for the mass elements is obtained:

$$\frac{d\mathbf{v}}{dt}+\frac{1}{2}\frac{d}{dt}(v^2\mathbf{v})-\frac{d}{dt}(C\mathbf{v})-\frac{d\mathbf{B}}{dt}=-\nabla A-\nabla(\mathbf{B}\cdot\mathbf{v})-\tfrac{1}{2}v^2\nabla C, \tag{19}$$

where

$$A=\tfrac{1}{2}(h_{00}{}^{(1)}+h_{00}{}^{(2)})-\tfrac{1}{8}(h_{00}{}^{(1)})^2, \tag{20a}$$

$$B_{x,y,z}=h_{0x,0y,0z}, \tag{20b}$$

$$C=h_{ss}+\tfrac{1}{2}h_{00}{}^{(1)}. \tag{20c}$$

[12] This time-symmetry property imposed on the metric implies neglect of gravitational radiation terms. See A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. 39, 65 (1938).

V. TWO-BODY BOUND SYSTEM (CIRCULAR ORBIT)

Consider two masses m_i and m_j in a circular orbit about each other, and a third distant external mass m_e. This is the simplest massive system that we can construct, using only gravitational forces. We seek that part of the acceleration of the two-mass system toward m_e which is proportional to the inverse square of the distance to m_e.

Operationally, this acceleration can be measured by the following procedure. Any local clock being used by an experimenter is to be calibrated in terms of universal coordinate time t, i.e., the proper time of a clock which is at rest at a large distance from all the masses in the experiment. We place the massive system in a circular orbit about m_e. The radius of the orbit can be calibrated unambiguously by the round-trip time (t_0) for light to travel to the central mass m_e and return. The orbital frequency is measured in terms of t. Then the central acceleration toward m_e can be determined by using Kepler's third law. In particular, we seek the m_g/m_i ratio of massive objects, so

$$m_i\omega^2 r = m_g m_e/r^2$$

or

$$m_g/m_i = \omega^2 r^3/m_e,$$

with ω and r measured as described above and m_e measured by the $\omega^2 r^3$ value of a very small test particle. One can in principle do the experiment with very large ($r\to\infty$) orbits eliminating any order (m_e/r) corrections to the above equations.

In (19), several of the terms contain v^2 of the test particle. We will need v^2 in those terms only to the lowest Newtonian order, so we can use for circular orbits

$$v_i^2 = m_j^2/Mr_{ij}, \tag{21a}$$

$$v_j^2 = m_i^2/Mr_{ij}, \tag{21b}$$

with $M=m_i+m_j$, $r_{ij}=|\mathbf{r}_i-\mathbf{r}_j|$. It is assumed that the two-body system has acquired no significant velocity toward m_e, and it is assumed that m_e is at rest. We now analyze each term in the equation of motion (19):

(a) $$-\frac{1}{2}\frac{d}{dt}(v^2\mathbf{v}) = -\tfrac{1}{2}v^2\mathbf{a}-\mathbf{v}\cdot\mathbf{a}\mathbf{v}.$$

But we divide the particle's acceleration into an internal ($\mathbf{a}_{\text{int}}$) and external ($\mathbf{a}_e$) part with $\mathbf{a}_e$ being proportional to m_e and directed toward m_e. Then this term divides into

$$(-\tfrac{1}{2}v^2\mathbf{a}_e-\mathbf{v}\cdot\mathbf{a}_e\mathbf{v})+(-\tfrac{1}{2}v^2\mathbf{a}_{\text{int}}-\mathbf{v}\cdot\mathbf{a}_{\text{int}}\mathbf{v}).$$

We are interested only in the external parts of our acceleration terms, so finally we keep

$$-[\tfrac{1}{2}v^2+v_{\parallel}^2]\mathbf{a}_e - v_{\parallel}a_e\mathbf{v}_{\perp}, \tag{22}$$

where the acceleration term $\sim\mathbf{v}$ has been divided into a part parallel to $\mathbf{a}_e$ and a part perpendicular to $\mathbf{a}_e$.

(b) $$\frac{d}{dt}(C\mathbf{v}) = C\mathbf{a}+\frac{\partial C}{\partial t}\mathbf{v}+\mathbf{v}\cdot\nabla C\mathbf{v}.$$

But from (20c), (12a), and (12c)

$$C=-(2\gamma+1)\left(\frac{m_e}{|\mathbf{r}-\mathbf{r}_e|}+\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\right). \tag{23}$$

Using our assumption that m_e is at rest, this term gives

$$-(2\gamma+1)\left[\frac{m_e}{|\mathbf{r}-\mathbf{r}_e|}\mathbf{a}_{\text{int}}+\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\mathbf{a}_e +(\mathbf{v}-\mathbf{v}_i)\cdot\nabla\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\mathbf{v}+\mathbf{v}_j\cdot\nabla\frac{m_e}{|\mathbf{r}-\mathbf{r}_e|}\mathbf{v}\right],$$

plus totally internal terms, plus terms proportional to m_e^2. Keeping only accelerations linear in m_e, we have

$$-(2\gamma+1)\left[\frac{m_e}{|\mathbf{r}-\mathbf{r}_e|}\mathbf{g}_{\text{int}}(r)+\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\mathbf{g}_e+\mathbf{v}\mathbf{v}\cdot\mathbf{g}_e\right], \tag{24}$$

where

$$\mathbf{g}_e=\nabla(m_e/|\mathbf{r}-\mathbf{r}_e|) \tag{25a}$$

and

$$\mathbf{g}_{\text{int}}(r)=\nabla(m_i/|\mathbf{r}-\mathbf{r}_i|). \tag{25b}$$

(c) $$\dot{B}_k=\frac{d}{dt}\left[4\Delta\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}v_i^k +4\Delta'\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|^3}(\mathbf{r}-\mathbf{r}_i)\cdot\mathbf{v}_i(r-r_i)^k\right],$$

where we have again used the assumption that m_e is at rest. Keeping only terms proportional to m_e, we have

$$4\Delta\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\mathbf{g}_e+4\Delta'\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|^3}(\mathbf{r}-\mathbf{r}_i)\cdot\mathbf{g}_e(\mathbf{r}-\mathbf{r}_i). \tag{26}$$

(d) This is the usual potential term:

$$-\nabla A=-\nabla\left[(\beta-\tfrac{1}{2})\left(\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}+\frac{m_e}{|\mathbf{r}-\mathbf{r}_e|}\right)^2-\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\frac{m_e}{|\mathbf{r}-\mathbf{r}_e|}+\alpha'\frac{m_im_e}{|\mathbf{r}_i-\mathbf{r}_e|}\left(\frac{1}{|\mathbf{r}-\mathbf{r}_i|}+\frac{1}{|\mathbf{r}-\mathbf{r}_e|}\right) +\tfrac{1}{2}\chi\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\mathbf{r}-\mathbf{r}_i\cdot\mathbf{a}_i+(\text{solely internal terms})\right].$$

Again dividing $\mathbf{a}_i$ into $\mathbf{a}_e+\mathbf{a}_{\text{int}}$ and keeping all terms above linear in m_e, we have

$$-\nabla A=\nabla\frac{m_e}{|\mathbf{r}-\mathbf{r}_e|}-(2\beta-1)\left(\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\nabla\frac{m_e}{|\mathbf{r}-\mathbf{r}_e|}+\frac{m_e}{|\mathbf{r}-\mathbf{r}_e|}\nabla\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\right)-\alpha'\frac{m_e}{|\mathbf{r}_i-\mathbf{r}_e|}\nabla\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}$$
$$-\tfrac{1}{2}\chi\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\mathbf{a}_e+\tfrac{1}{2}\chi\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|^3}(\mathbf{r}-\mathbf{r}_i)\cdot\mathbf{a}_e(\mathbf{r}-\mathbf{r}_i).$$

Regrouping the terms of interest above, then

$$-\nabla A=\mathbf{g}_e\left[1-(2\beta-1+\tfrac{1}{2}\chi)\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\right]-\mathbf{g}_{\mathrm{int}}(r)\left[\frac{m_e}{|\mathbf{r}-\mathbf{r}_e|}\alpha'+(2\beta-1)\frac{m_e}{|\mathbf{r}-\mathbf{r}_e|}\right]+\tfrac{1}{2}\chi\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|^3}\mathbf{r}-\mathbf{r}_i\cdot\mathbf{g}_e\mathbf{r}-\mathbf{r}_i. \quad (27)$$

(e)
$$\mathbf{B}\cdot\mathbf{v}=4\Delta\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\mathbf{v}_i\cdot\mathbf{v}+4\Delta'\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|^3}\mathbf{r}-\mathbf{r}_i\cdot\mathbf{v}\mathbf{r}-\mathbf{r}_i\cdot\mathbf{v}_i.$$

The divergence of $\mathbf{B}\cdot\mathbf{v}$ gives totally an internal acceleration.

(f)
$$-\tfrac{1}{2}v^2\nabla C$$

yields an external term

$$(\gamma+\tfrac{1}{2})v^2\mathbf{g}_e. \quad (28)$$

Combining all of the above results—(22), (24), and (26)–(28)—we arrive at the equation of motion for the particle m_j:

$$\mathbf{a}_j=\mathbf{g}_e\left[1+(4\Delta-2\beta-2\gamma-\tfrac{1}{2}\chi)\frac{m_i}{|\mathbf{r}_j-\mathbf{r}_i|}+\gamma v_j^2-(2\gamma+2)v_{\parallel j}^2\right]+(4\Delta'+\tfrac{1}{2}\chi)\frac{m_i}{|\mathbf{r}_j-\mathbf{r}_i|^3}(\mathbf{r}_j-\mathbf{r}_i)\cdot\mathbf{g}_e(\mathbf{r}_j-\mathbf{r}_i)$$
$$-\mathbf{g}_{\mathrm{int}}(r_j)\left[(2\gamma+2\beta)\frac{m_e}{|\mathbf{r}_j-\mathbf{r}_e|}+\alpha'\frac{m_e}{|\mathbf{r}_i-\mathbf{r}_e|}\right]-(2\gamma+2)v_{\parallel j}g_e\mathbf{v}_{\perp j}+(\text{solely internal accelerations}). \quad (29)$$

Equation (29) can be further simplified by setting

$$\mathbf{r}_j-\mathbf{r}_i=(\mathbf{r}_j-\mathbf{r}_i)_\parallel+(\mathbf{r}_j-\mathbf{r}_i)_\perp$$

(parallel and perpendicular always refer to the direction of $\mathbf{g}_e$). Also a center of mass is defined:

$$\mathbf{R}=(m_i\mathbf{r}_i+m_j\mathbf{r}_j)/M.$$

Then

$$\frac{1}{|\mathbf{r}_j-\mathbf{r}_e|}\cong\frac{1}{|\mathbf{R}-\mathbf{r}_e|}-\frac{(\mathbf{r}_j-\mathbf{R})\cdot(\mathbf{R}-\mathbf{r}_e)}{|\mathbf{R}-\mathbf{r}_e|^3} \quad (30a)$$

and

$$\frac{1}{|\mathbf{r}_i-\mathbf{r}_e|}\cong\frac{1}{|\mathbf{R}-\mathbf{r}_e|}-\frac{(\mathbf{r}_i-\mathbf{R})\cdot(\mathbf{R}-\mathbf{r}_e)}{|\mathbf{R}-\mathbf{r}_e|^3}. \quad (30b)$$

Equation (29) can then be written as

$$\mathbf{a}_j=\mathbf{g}_e\left[1+(4\Delta-2\beta-2\gamma-\tfrac{1}{2}\chi)\frac{m_i}{r_{ij}}+\gamma v_j^2-(2\gamma+2)v_{\parallel j}^2+\left(4\Delta'+\tfrac{1}{2}\chi+\frac{2(\gamma+\beta)m_i-\alpha' m_j}{M}\right)\frac{m_i}{r_{ij}^3}r_{ij\parallel}^2\right]$$
$$-(2\gamma+2)v_{\parallel j}g_e\mathbf{v}_{\perp j}+\left(4\Delta'+\tfrac{1}{2}\chi+\frac{2(\gamma+\beta)m_i-\alpha' m_j}{M}\right)\frac{m_i}{r_{ij}^3}r_{ji\parallel}g_e\mathbf{r}_{ji\perp}+(\text{internal accelerations}). \quad (31)$$

To obtain the acceleration of the m_i, m_j two-body system we take the combination

$$\mathbf{a}=(m_i\mathbf{a}_i+m_j\mathbf{a}_j)/M. \quad (32)$$

The acceleration $\mathbf{a}_i$ is obtained from (31) by the appropriate inversion of labels $i \leftrightarrow j$. The internal accelerations which we have not been interested in have the expected property

$$m_i \mathbf{g}(\mathbf{r}_i)_{\text{int}} + m_j \mathbf{g}(\mathbf{r}_j)_{\text{int}} = 0 . \tag{33}$$

So finally

$$\mathbf{a} = \mathbf{g}_e \left[1 + (8\Delta - 4\beta - 4\gamma - \chi) \frac{m_i m_j}{M r_{ij}} + \gamma \frac{m_i v_i^2 + m_j v_j^2}{M} - (2\gamma + 2) \frac{m_i v_{\parallel i}^2 + m_j v_{\parallel j}^2}{M} + (8\Delta' + \chi + 2\gamma + 2\beta - \alpha') \frac{m_i m_j}{M r_{ij}^3} r_{ij\parallel}^2 \right]$$
$$- (2\gamma + 2) g_e \left[\frac{(m v_\parallel \mathbf{v}_\perp)_i + (m v_\parallel \mathbf{v}_\perp)_j}{M} \right] + (8\Delta' + \chi + 2\gamma + 2\beta - \alpha') \frac{m_i m_j}{M r_{ij}^3} g_e r_{ji\parallel} \mathbf{r}_{ji\perp} . \tag{34}$$

We now consider a circular orbit for m_i and m_j with the orbit plane normal vector making an angle θ with the direction toward m_e. This leads to the following time dependence of the velocity of m_i or m_j (z is the direction toward m_e):

$$v_x = v \sin\omega t \cos\theta , \tag{35a}$$

$$v_y = v \cos\omega t , \tag{35b}$$

$$v_z = v \sin\omega t \sin\theta , \tag{35c}$$

with the magnitude of v given by (21a) and (21b). The interparticle position vector is then

$$(r_j - r_i)_x = r_{ij} \cos\omega t \cos\theta , \tag{36a}$$

$$(r_j - r_i)_y = -r_{ij} \sin\omega t , \tag{36b}$$

$$(r_j - r_i)_z = r_{ij} \cos\omega t \sin\theta . \tag{36c}$$

All the terms in (34) can now be evaluated. The acceleration along $\mathbf{g}_e$ is given by

$$a_z = g_e \{ 1 + (m_i m_j / M r_{ij}) [(8\Delta - 4\beta - 3\gamma - \chi) + \tfrac{1}{2} \sin^2\theta (2\beta + \chi + 8\Delta' - \alpha' - 2)] \} \tag{37}$$

when averaged over the rotation period of m_i and m_j. There is no average acceleration in the y direction. In the x direction, however, there is a time-averaged acceleration

$$a_x = g_e (2\beta + \chi + 8\Delta' - \alpha' - 2)(m_i m_j / M r_{ij}) \times \tfrac{1}{2} \sin\theta \cos\theta . \tag{38}$$

Additional acceleration terms which oscillate as $\cos 2\omega t$ and $\sin 2\omega t$ but average to zero over a period of the orbital motion of m_i about m_j will be discussed in Sec. VII.

Equations (37) and (38) can be compared with (1) to give an expression for η, the EP violation coefficient, in terms of the general metric coefficients.

Demanding that $\eta = 0$ for arbitrary orientation of the two-body orbit gives two constraints on the metric expansion coefficients;

$$8\Delta - 4\beta - 3\gamma - \chi = 0 \tag{39a}$$

and

$$2\beta + \chi + 8\Delta' - \alpha' - 2 = 0 . \tag{39b}$$

Equation (39b) also guarantees the vanishing of the anomalous a_x acceleration given by (38).

In Einstein's gravitational theory $\gamma = \beta = \Delta = \alpha' = \chi = 1$, while $\Delta' = 0$,[12] so both (39a) and (39b) are fulfilled. Note that both (39a) and (39b) contain β, the coefficient of the nonlinear term in g_{00}. This result confirms the suggestion of the previous paper (I), in that the motion of a massive body depends on the gravitational acceleration of gravitational self-energy.

In an appendix to this paper, all of the coefficients above except α' are calculated in the scalar-tensor gravitational theory of Brans and Dicke. We obtain $\beta = \chi = 1$ ($\chi = 1$ is necessary for the metric to be properly retarded), $\gamma = (1 + w)/(2 + w)$, $\Delta = (3 + 2w)/(4 + 2w)$, and $\Delta' = 0$. w is a dimensionless parameter of the Brans-Dicke (BD) theory (as $w \to \infty$ Einstein's theory is reobtained). Equation (39a) is not fulfilled for the BD theory; using the above results,

$$(8\Delta - 4\beta - 3\gamma - \chi)_{\text{BD}} = -1/(2 + \omega) . \tag{40}$$

VI. MASSIVE GASEOUS SPHERE

The previous computation of the acceleration of a two-body system can be altered to give the acceleration of a massive gas sphere maintained in equilibrium by kinetic gas pressure. We can then apply the results of this work to the examination of the m_g/m_i ratio for normal stars (like the Sun) in which the equilibrium of the star is overwhelmingly produced by the balance of gravitational attraction and particle kinetic pressure.

The two-source metric expansion used previously is generalized to many sources by replacing in all the metric terms the single m_i contributions by a summation ($\sum_i$) over many m_i. Then (31) reads

$$\mathbf{a}_j = \mathbf{g}_e \left[1 + (4\Delta - 2\beta - 2\gamma - \tfrac{1}{2}\chi) \sum_i \frac{m_i}{r_{ij}} + \gamma v_j^2 - (2\gamma + 2) v_{\parallel j}^2 + \sum_i \left(4\Delta + \tfrac{1}{2}\chi + \frac{2(\gamma + \beta) m_i - \alpha' m_j}{M} \right) \frac{m_i}{r_{ij}^3} r_{ij\parallel}^2 \right]$$
$$- (2\gamma + 2) v_{\parallel j} g_e \mathbf{v}_{\perp j} + \sum_i \left[4\Delta + \tfrac{1}{2}\chi + \frac{2(\gamma + \beta) m_i - \alpha' m_j}{M} \right] r_{ji\parallel} g_e \mathbf{r}_{ji\perp} . \tag{41}$$

Taking a directional average by averaging over the kinetic motion and position of many particles in the gas sphere, we have the average quantities

$$\langle v_{ji\parallel}\mathbf{r}_{ji\perp}\rangle = 0\,, \tag{42a}$$

$$\langle v_{\parallel j}\mathbf{v}_{\perp j}\rangle = 0\,, \tag{42b}$$

$$\langle r_{ji\parallel}{}^2/r_{ij}{}^3\rangle = \tfrac{1}{3}(r_{ij})^{-1}\,, \tag{42c}$$

and

$$\langle v_\parallel{}^2\rangle = \tfrac{1}{3}v^2. \tag{42d}$$

Taking the sum

$$\mathbf{a} = \textstyle\sum_j m_j\mathbf{a}_j/M\,, \qquad M = \sum_j m_j\,, \tag{43}$$

we obtain the acceleration of the sphere

$$\mathbf{a} = \mathbf{g}_e\Big\{1 + [(4\Delta - 2\beta - 2\gamma - \tfrac{1}{2}\chi) + \tfrac{1}{6}(\beta + \gamma + 4\Delta' + \tfrac{1}{2}\chi - \tfrac{1}{2}\alpha')]\sum_{i,j}\frac{m_i m_j}{M r_{ij}} + \tfrac{1}{3}(\gamma - 2)\sum_j \frac{m_j v_j{}^2}{M}\Big\}. \tag{44}$$

The $v_j{}^2$ in (44) are needed only to the classical Newtonian order, so we can use the usual virial theorem for a system in equilibrium:

$$\sum_j m_j v_j{}^2 = \tfrac{1}{2}\sum_{i,j}\frac{m_i m_j}{r_{ij}}. \tag{45}$$

Equation (44) can finally be expressed as

$$\mathbf{a} = \mathbf{g}_e\Big\{1 + [\tfrac{1}{2}(8\Delta - 4\beta - 3\gamma - \chi) + \tfrac{1}{6}(2\beta + \chi + 8\Delta' - \alpha' - 2)]\sum_{i,j}\frac{m_i m_j}{M r_{ij}}\Big\} \tag{46}$$

for the acceleration of a massive gas sphere in an external field $\mathbf{g}_e$.

The two contributions to η in (46) are the terms (39a) and (39b) which have already been shown to vanish in Einstein's theory. For a massive gaseous sphere we can now express the parameter which measures the violation of the EP [η in (1)] in terms of the expansion coefficients of the space-time metric:

$$\eta = 4\Delta - (5/3)\beta - \tfrac{3}{2}\gamma + \tfrac{4}{3}\Delta' - \tfrac{1}{6}\alpha' - \tfrac{1}{3}. \tag{47}$$

An experimental measurement of η as proposed in I is therefore seen to be a measurement of a combination of several terms in the space-time metric which have not been measured to date. Only γ and β have been measured, while Δ and Δ' will be measured in the orbiting-gyroscope experiment under development at Stanford University.[11]

VII. ANOMALOUS ACCELERATION OF NONSTATIONARY SYSTEMS

To obtain the results (37) and (38) for the acceleration of a two-body orbiting system in an external field, a time average was performed over a period of the orbiting motion of the two bodies around each other.

There are additional oscillatory terms in the exact expression for the acceleration of the two-body system; they are

$$a_x(\mathrm{osc}) = \xi\sin\theta\cos\theta\cos 2\omega t\,, \tag{48a}$$

$$a_y(\mathrm{osc}) = -\xi\sin\theta\sin 2\omega t\,, \tag{48b}$$

and

$$a_z(\mathrm{osc}) = \xi\sin^2\theta\cos 2\omega t\,, \tag{48c}$$

with

$$\xi = g_e(m_i m_j/2Mr_{ij})(8\Delta' + \chi + 4\gamma + 2\beta + 2 - \alpha').$$

The magnitude of $\mathbf{a}(\mathrm{osc})$ is

$$|\mathbf{a}(\mathrm{osc})| = \xi\sin\theta. \tag{49}$$

In this section we explore other configurations for massive bodies in order to see if the oscillatory anomalous acceleration found above exists in general.

Next consider a two-body orbit with $\sin\theta = 0$, so that the above effects (48a)–(48c) vanish. However, let the orbit be elliptical. Then the conditions (21a) and (21b) will not be valid at all times during the orbital motion, only on a time average over the orbital period.

Specializing (34) to $\sin\theta = 0$ gives the simpler expression

$$\mathbf{a} = \mathbf{g}_e\Big\{1 + (8\Delta - 4\beta - 2\gamma - \chi)\frac{m_i m_j}{M r_{ij}} + \gamma\Big[\frac{m_i v_i{}^2 + m_j v_j{}^2}{M}\Big]\Big\}. \tag{50}$$

Letting ϵ be the conserved Newtonian energy of the two-body orbit,

$$\epsilon = \tfrac{1}{2}(m_i v_i{}^2 + m_j v_j{}^2) - \frac{m_i m_j}{r_{ij}}\,, \tag{51}$$

Eq. (50) yields

$$\mathbf{a} = \mathbf{g}_e\Big[1 + (8\Delta - 4\beta - 2\gamma - \chi)\frac{m_i m_j}{M r_{ij}} + \frac{2\epsilon}{M}\gamma\Big]. \tag{52}$$

Using Einstein's theory's value for the coefficients gives

$$\mathbf{a} = \mathbf{g}_e\Big\{1 + \Big[\frac{m_i m_j}{r_{ij}} + 2\epsilon\Big]\Big/M\Big\}. \tag{53}$$

The anomalous acceleration term in (53) only vanishes when averaged over the orbital period. At orbital perigee, (53) gives

$$\mathbf{a}=\mathbf{g}_e\left(1+e\frac{m_im_j}{Mr_p}\right), \tag{54a}$$

while at orbital apogee it gives

$$\mathbf{a}=\mathbf{g}_e\left(1-e\frac{m_im_j}{Mr_a}\right), \tag{54b}$$

with e being the eccentricity of the orbit.

Finally we examine a pulsating gaseous sphere. Our previous results concerning the acceleration of the sphere depended on using an equilibrium condition for the sphere—the virial theorem relating mean kinetic energy to potential energy. Here we assume that the sphere radially pulsates about equilibrium.

When the pulsation is at the extreme condensed state, the kinetic energy of the gas will exceed the requirement of the virial theorem, i.e.,

$$\sum_j m_jv_j^2=\tfrac{1}{2}\sum_{i,j}\frac{m_im_j}{r_{ij}}+2\delta\epsilon_1. \tag{55}$$

But when the pulsation is at the extreme expanded state, the kinetic energy of the gas is less than required by the virial theorem;

$$\sum_j m_jv_j^2=\tfrac{1}{2}\sum_{i,j}\frac{m_im_j}{r_{ij}}-2\delta\epsilon_2. \tag{56}$$

Using Einstein's theory's value for the metric coefficients and evaluating expression (44) for the cases (55) and (56), we get, respectively, accelerations of the sphere

$$\mathbf{a}=\mathbf{g}_e\left(1-\frac{2}{3}\frac{\delta\epsilon_1}{M}\right) \tag{57}$$

and

$$\mathbf{a}=\mathbf{g}_e\left(1+\frac{2}{3}\frac{\delta\epsilon_2}{M}\right). \tag{58}$$

These anomalous accelerations will also vanish when time is averaged over the pulsation period.

The common feature of all three systems examined above which showed oscillating anomalous accelerations was that all systems presented a nonstationary, oscillating configuration to the external mass m_e.

In a future paper we will study these oscillating accelerations to see if they produce in principle measurable effects or whether they are simply coordinate-system-dependent anomalies.

APPENDIX

The coefficients which appear in (39a) and (39b) can be obtained for Einstein's theory from the EIH paper.[12] In this paper we calculate these coefficients for the Brans-Dicke (BD) gravitational theory.[6] All of the needed coefficients except α' are calculated, and we hope to obtain α' in a later paper.

The BD field equations for the space-time metric tensor and their scalar field are[6]

$$\Box^2\phi=\frac{8\pi}{c^4}\frac{T}{3+2w} \tag{A1}$$

and

$$R_{ij}=-\frac{8\pi}{\phi c^4}\left(T_{ij}-\frac{1+w}{3+2w}Tg_{ij}\right)-\phi_{i\|j}/\phi-w\phi_i\phi_j/\phi^2, \tag{A2}$$

which reduce to Einstein's field equations in the limit $w\to\infty$. To lowest order, (A1) yields

$$\phi=\phi_0+\frac{2}{c^2}\frac{1}{3+2w}\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|} \tag{A3}$$

for a point source.

In the BD paper, the coefficients γ and β are calculated in order to obtain the advance of planetary perihelion. They obtain[6]

$$\gamma=(1+w)/(2+w) \tag{A4}$$

and

$$\beta=1. \tag{A5}$$

To obtain χ, we need the R_{00} equation to linear order in the source strength:

$$R_{00}=-\frac{8\pi}{\phi_0c^4}\left(T_{00}-g_{00}T\frac{1+w}{3+2w}\right)-\frac{1}{\phi_0}\frac{d^2}{dt^2}\phi. \tag{A6}$$

Only the part of R_{00} linear in Christoffel symbols is required:

$$R_{00}\cong\Gamma_{k0|0}{}^k-\Gamma_{00|k}{}^k \tag{A7}$$

or

$$R_{00}\cong-\tfrac{1}{2}\nabla^2g_{00}+g_{0k|k0}-\tfrac{1}{2}g_{kk|00}. \tag{A8}$$

Equation (A6) then yields the equation

$$-\tfrac{1}{2}\nabla^2g_{00}=-\frac{4\pi}{\phi_0}\left(\frac{4+2w}{3+2w}\right)\rho-\frac{1}{\phi_0}\frac{d^2}{dt^2}\phi+\tfrac{1}{2}g_{kk|00}-g_{0k|k0}. \tag{A9}$$

Using (A3), and[6]

$$g_{kk'}=-\delta_{kk'}\left(1+\frac{2+2w}{2+w}\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|}\right), \tag{A10}$$

$$g_{0k}=4\Delta\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|}v_1{}^k+4\Delta'\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|^3}\mathbf{r}-\mathbf{r}_1\cdot\mathbf{v}_1(\mathbf{r}-\mathbf{r}_1)^k, \tag{A11}$$

and[6]

$$[(4+2w)/(3+2w)](1/\phi_0)=1 \quad \text{(in units } G=c=1\text{)}, \tag{A12}$$

Eq. (A9) becomes

$$\nabla^2 g_{00}=8\pi\rho+\left(\frac{8+6w}{2+w}\right)\frac{d^2}{dt^2}\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|}-(8\Delta-8\Delta')\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|}\mathbf{r}-\mathbf{r}_1\cdot\mathbf{a}_1, \tag{A13}$$

which has the solution

$$g_{00}=1-2\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|}+\left(4\Delta-4\Delta'-\frac{4+3w}{2+w}\right)\times\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|}\mathbf{r}-\mathbf{r}_1\cdot\mathbf{a}_1. \tag{A14}$$

Comparing (A14) with (12d) gives

$$\chi=4\Delta-4\Delta'-[(4+3w)/(2+w)]. \tag{A15}$$

In order to obtain Δ and Δ', we use the R_{0k} equation which to linear order is

$$R_{0k}=-\frac{8\pi}{\phi_0 c^4}T_{0k}-\frac{1}{\phi_0}\frac{\partial}{\partial t}\frac{\partial}{\partial x^k}\phi, \tag{A16}$$

with the linearized R_{0k} given by

$$R_{0k}=-\tfrac{1}{2}\nabla^2 g_{0k}+\tfrac{1}{2}g_{0s|sk}-\tfrac{1}{2}g_{ss|0k}+\tfrac{1}{2}g_{ks|0s}. \tag{A17}$$

Therefore we have, using (A12),

$$\nabla^2 g_{0k}-g_{0s|ks}=16\pi\left(\frac{3+2w}{4+2w}\right)T_{0k}+\frac{3+2w}{4+2w}\frac{\partial^2}{\partial t\partial x^k}\phi-g_{ss|0k}+g_{ks|s0} \tag{A18}$$

or

$$\nabla^2 g_{0k}-g_{0k|ks}=16\pi\frac{3+2w}{4+2w}T_{0k}+\frac{6+4w}{2+w}\times\left[\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|^3}v_1{}^k-3\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|^5}\mathbf{r}-\mathbf{r}_1\cdot\mathbf{v}_1(r-r_1)^k\right], \tag{A19}$$

which yields

$$g_{0k}=4\left(\frac{3+2\omega}{4+2\omega}\right)\frac{m_1}{|\mathbf{r}-\mathbf{r}_1|}v_1{}^k. \tag{A20}$$

Comparing (A20) with (12b) gives

$$\Delta=(3+2\omega)/(4+2\omega), \tag{A21a}$$

$$\Delta'=0, \tag{A21b}$$

which inserted into (A15) gives

$$\chi=1. \tag{A22}$$

The evaluation of (39a) with the BD coefficients does not yield zero:

$$(8\Delta-4\beta-3\gamma-\chi)_{\text{BD}}=-1/(2+w), \tag{A23}$$

indicating that a massive body in the BD theory will possess an anomalous $1/R^2$ acceleration toward an external mass.