Equivalence Principle for Massive Bodies. I. Phenomenology

KENNETH NORDTVEDT, JR.
Department of Physics, Montana State University, Bozeman, Montana
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The experimentally measured equality of inertial mass (m_i) and gravitational mass (m_g) of a body indicates nothing about the m_g/m_i ratio to the order of gravitational self-energy/total energy of the body. Experiments are discussed, using astronomical bodies, which measure m_g/m_i to the order of the gravitational self-energy of the bodies. The configuration of the stable three-body problem of Lagrange is shown to be particularly sensitive to the possible difference $\Delta \equiv m_g/m_i - 1$ of the Sun.

I. INTRODUCTION

THE equality (in appropriate units) of the gravitational mass to the inertial mass of small laboratory-size bodies has been experimentally checked to a part in 10^9 by Eotvos[1] and to a part in 10^{11} by the more recent work of Dicke.[2] This equality of inertial and gravitational mass leads to all bodies accelerating at the same rate in an external gravitational field.

The generalization of the above experimental results into the equivalence principle (EP) was made by Einstein,[3] the EP stating that a uniform gravitational field is indistinguishable (locally) from an accelerated coordinate system.

It is important to point out that the laboratory-size bodies used by Eotvos[1] and Dicke[2] contain an infinitesimal fraction of gravitational self-energy. For a body of mass m, characteristic size a, the ratio of gravitational self-energy to total energy is

$$\frac{Gm^2/a}{mc^2} = \frac{Gm}{c^2 a}, \tag{1}$$

which is of order 10^{-25} for laboratory-size bodies. The experiments of Eotvos[1] and Dicke[2] indicate nothing, therefore, about whether the gravitational self-energy of a body makes a contribution to both the inertial and gravitational masses of the body and, more important, whether the gravitational mass is equal to the inertial mass including contributions of order (1).

Without in any way contradicting the experimental results of Eotvos[1] and Dicke,[2] the ratio of gravitational to inertial mass for a general body can be assumed to be

$$m_g/m_i = 1 + \eta \frac{G}{c^2} \int \rho(x)\rho(x') \frac{d^3x\, d^3x'}{|x-x'|} \Big/ \int \rho(x) d^3x \tag{2}$$

$$\equiv 1 + \Delta,$$

where η is an unknown coefficient of order of magnitude 1. $\rho(x)$ is the mass density of the body, G is the universal gravitational constant, and c is the velocity of light.

$\eta \neq 0$ would mean a violation of the EP which would become observable only for very massive bodies. The purpose of this paper is to examine possible experimental measures of the coefficient η.

It will be shown that Kepler's third law is not fully sensitive to possible anomalies of the form (2) for the Sun, a result which has been mentioned before by Dicke.[4] However, the stable three-body configuration of Lagrange is shown to be fully sensitive to a possible $\Delta \neq 0$ for the Sun.

II. KEPLER'S THIRD LAW

The correction term Δ in (2) grows with the mass of a body. It is worthwhile to consider, then, the most massive bodies for which accurate dynamical knowledge is available—solar-system astronomical bodies. For a uniform-density sphere of mass m, radius a,

$$\Delta = (6/5)\eta(Gm/c^2 a). \tag{3}$$

With $\eta \simeq 1$, the planet Jupiter yields $\Delta_J \simeq 10^{-18}$. For the Sun, the correction term Δ is substantially larger. The Sun is not a uniform density sphere, rather its mass is strongly concentrated in its center. A solar density distribution of

$$\rho(x) = \rho_0 \exp(-4x/a), \tag{4}$$

where a is the solar radius, fits the standard stellar models well.[5] Then

$$\Delta_S = (15/2)\eta(Gm/c^2 a), \tag{5}$$

which yields for the Sun $\Delta_S \simeq 1.5 \times 10^{-5}$ when $\eta \simeq 1$.

Consider the Sun-Jupiter system. If separated by a distance R, Jupiter will experience an acceleration

$$a_J = -(Gm_S/R^2)(1+\Delta_J), \tag{6a}$$

while the Sun's acceleration is

$$a_S = -(Gm_J/R^2)(1+\Delta_S), \tag{6b}$$

where (2) has been used for the m_g/m_i ratio of the bodies. Equations (6a) and (6b) yield a modified third law of Kepler,

$$4\pi^2 R^3/T^2 = G(m_S + m_J + m_J \Delta_S + m_S \Delta_J). \tag{7}$$

[1] R. V. Eotvos, Ann. Physik **68**, 11 (1922).
[2] P. G. Roll, R. Krotkov, and R. H. Dicke, Ann. Phys. (N. Y.) **26**, 442 (1964).
[3] A. Einstein, Ann. Physik **36** (1911).
[4] R. H. Dicke, in *Gravitation and Relativity*, edited by H. Chin and W. Hoffmann (W. A. Benjamin, Inc., New York, 1964).
[5] M. Schwarzschild, *Structure and Evolution of the Stars* (Princeton University Press, Princeton, N. J., 1958), Chap. VIII.

With $m_J/m_S \simeq 10^{-3}$, both correction terms in (7) which result from an EP violation are of order 10^{-8} times the leading m_S term. [Note that because Δ_S in (7) is multiplied by m_J, the part in 10^5 correction term is quenched.]

In order to detect the Δ_S and Δ_J correction terms, both R and T must be known to better than a part in 10^8. This is presently true for the T measurements; however, R measurements in the solar system are presently accurate to only about a part in 10^6. (R measurement accuracy should improve in the next several years by the use of radar range experiments between the planets.) Also in (7) the mass of Jupiter must be known to better than a part in 10^5.

Because the Sun is the "pivot of the solar system," so to speak, Kepler's third law corrections are not fully sensitive to the solar EP violation. This has previously been pointed out by Dicke.[4]

III. STABLE LIBRATION POINTS OF LAGRANGE

In this section it is shown that the stable three-body configuration of Lagrange is fully sensitive to Δ_S, the Sun's EP violation term.

Lagrange showed that a third body of negligible mass moves in a stable orbit in the presence of two massive orbiting bodies when placed in an orbital-plane equilateral-triangle configuration with the two massive bodies.[6] This arrangement is illustrated in Fig. 1.

For clarity, the circular-orbit case will be analyzed in this paper. The location of the stable libration point (SLP) in the case where the EP is not valid for the two massive bodies will now be obtained.

In Fig. 1, there are two massive bodies of masses M_1 and M_2 revolving about a common center at distances R_1 and R_2 with angular velocity ω.

Solving the two equations for the orbit,

$$(1+\Delta_2)GM_1/(R_1+R_2)^2=\omega^2R_2 \tag{8a}$$

and

$$(1+\Delta_1)GM_2/(R_1+R_2)^2=\omega^2R_1, \tag{8b}$$

gives

$$\omega^2(R_1+R_2)^3=G(M_1+M_2+M_1\Delta_2+M_2\Delta_1) \tag{9}$$

and

$$M_1R_1/M_2R_2=(1+\Delta_1)/(1+\Delta_2). \tag{10}$$

The location of the SLP is obtained by finding the point given by r_1 and r_2 in Fig. 1, where there are no accelerations tangential to r, and the acceleration along r is balanced by the centrifugal acceleration. These conditions yield

$$(GM_1/r_1^2)\sin\alpha_1=(GM_2/r_2^2)\sin\alpha_2 \tag{11}$$

and

$$(GM_1/r_1^2)\cos\alpha_1+(GM_2/r_2^2)\cos\alpha_2=\omega^2r. \tag{12}$$

[6] For a general discussion of the stable libration point including a proof of the stability of the equilibrium point see J. M. A. Danby, *Fundamentals of Celestial Mechanics* (The Macmillan Co., New York, 1962), Chap. 8.

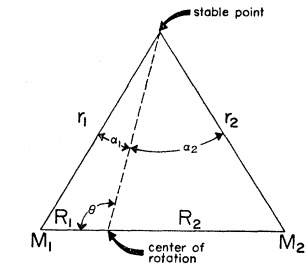

FIG. 1. Lagrange's stable three-body configuration.

There are no EP-violation terms in (11) and (12), because it is assumed that the body at the SLP is not massive, i.e., its Δ in (2) is negligible. This condition is true for manmade satellites and asteroids.

By the law of sines

$$(\sin\alpha_1)/R_1=(\sin\theta)/r_1 \tag{13a}$$

and

$$(\sin\alpha_2)/R_2=(\sin\theta)/r_2. \tag{13b}$$

Equation (11) then yields

$$M_1R_1/r_1^3=M_2R_2/r_2^3, \tag{14}$$

which by the use of (10) gives

$$(r_1/r_2)^3=(1+\Delta_1)/(1+\Delta_2). \tag{15}$$

Equation (15) indicates that if $\Delta_1\neq\Delta_2$, the SLP is no longer the third vertex of an equilateral triangle.

The law of cosines gives the relations

$$\cos\alpha_1=(r/r_1)-(R_1/r_1)\cos\theta \tag{16a}$$

and

$$\cos\alpha_2=(r/r_2)+(R_2/r_2)\cos\theta. \tag{16b}$$

Equations (16a), (16b), and (14) inserted into (12) give

$$(GM_1/r_1^3)+(GM_2/r_2^3)=\omega^2. \tag{17}$$

Equations (9) and (15) can then be used with (17) to give explicit solutions for r_1 and r_2:

$$[r_1/(R_1+R_2)]^3=1/(1+\Delta_2) \tag{18a}$$

and

$$[r_2/(R_1+R_2)]^3=1/(1+\Delta_1). \tag{18b}$$

Let M_1 be the Sun and M_2 a planet, then $\Delta_2\ll\Delta_1$ and will be neglected. Equation (18b) indicates a movement of the SLP toward the planet by an amount of

$$\delta r_2=-\tfrac{1}{3}\Delta_S(R_1+R_2). \tag{19}$$

Equation (19) gives an effect fully sensitive to Δ_S. This result suggests two possible experiments:

(1) It is known that several asteroids (the Trojan asteroids) are captured in the SLP of the Sun-Jupiter system.[7] These asteroids wander in closed orbits about the SLP of the Sun-Jupiter system. As viewed from Earth, (19) yields an angular movement of the SLP

[7] Astronomical Society of the Pacific Leaflet No. 381, 1961 (unpublished).

toward Jupiter by an amount of approximately

$$\delta\theta \simeq \delta r_2/(R_1+R_2) = \tfrac{1}{3}\Delta_S \simeq 1'' \qquad (20)$$

for $\eta \simeq 1$.

(2) Consider a satellite placed at the Sun-Earth SLP. If a radar signal is used to determine the range to the SLP from Earth, (19) gives a decrease in round-trip radar time of

$$\delta t = -\tfrac{2}{3}\Delta_S(R_1+R_2)/c. \qquad (21)$$

For $\eta \sim 1$, Eq. (21) represents a change in path length of 1000 miles, or a change in time of 5×10^{-3} sec.

Equation (21) can be compared with the proposed Venus radar range experiment of Shapiro.[8] When the Earth-Venus line of sight passes close by the Sun, Shapiro expects (on the basis of Einstein's theory of relativity) a change in radar round-trip time of

$$\delta t \simeq 4(GM_S/c^2)\ln(4r_E r_V/a^2), \qquad (22)$$

with r_E and r_V the Earth and Venus orbital radii, and a the Sun's radius. Equation (22) gives a light-path length change of 35 miles corresponding to a time of about 0.2×10^{-3} sec. Equation (21) is therefore about 30 times larger than (22).

There are several comments appropriate to the experimental observation of the effects in (20) and (21). No body can be inserted precisely at the SLP with the precise velocity necessary for the body to remain at rest relative to the SLP. Bodies in general move in bounded orbits about the SLP. An actual experiment must consist of tracking a body about the SLP [either angular tracking for the case of (20) or range, and perhaps range rate, tracking for the case of (21)].

By solving the classical equations of motion for a body's motion about the SLP and by data reduction, the SLP must be calculated from observational data.

It is also clear that other gravitational bodies in the solar system will perturb the location of the SLP. These effects must first be calculated, but again this is a classical correction which can be obtained by straightforward, though tedious, calculation. The movement of the SLP given by (20) and (21) represent additional effects produced by a possble EP violation for massive bodies.

It has been shown in (21) and (22) that the SLP shift due to corrections to Newtonian gravitation can be two orders of magnitude larger than Shapiro's[8] expected shift. However, Shapiro's expected shift has properties which make it easier to detect experimentally. The effect represented in Eq. (22) decreases rapidly as the Earth-Venus line of sight passes the Sun at a greater distance. When tracking Venus during its approach toward the Sun, Eq. (22) represents a rapidly changing shift for the radar round-trip time, which is added onto a slowly changing Newtonian radar round-trip time. Therefore the effect (22) is relatively easily extracted from the data.

Equation (21) represents a constant time shift, and there is no way to extract (22) from the classical round-trip time. However, (21) represents a part in 10^5 correction to the total radar round-trip time and should therefore be absolutely detectable above the present part in 10^6 uncertainty of solar-system distances.

IV. IMPLICATIONS FOR GRAVITATIONAL THEORIES

The detection of a violation of the EP for massive bodies would be interesting in its own right, but an analysis of the implications of $\eta \neq 0$ for gravitational theories would be desirable. In a separate paper,[9] gravitational theories will be examined. However, several comments are pertinent here in this more phenomenological paper.

First, it is reasonable that the rate at which a massive body falls in an external gravitational field is influenced by the nonlinear properties of the gravitational theory, since this question depends on the coupling of the gravitational self-energy of a body to the external gravitational field (a gravity-gravity interaction).

Second, even a null experimental result ($\eta = 0$) would have interesting significance for gravitational theories. Whereas for test particles the ratio $m_g/m_i = 1$ follows immediately from the assumption that test particles follow geodesics of space-time in the geometrical theories of gravitation, it is nontrivial to guarantee the same for massive test particles. In other words, an experimental observation of $\eta = 0$ would be a new test of gravitational theories.

[8] I. I. Shapiro, Phys. Rev. Letters **13**, 789 (1964).

[9] Kenneth Nordtvedt, Jr., following paper, Phys. Rev. **169**, 1017 (1968).