Fluctuations in a Josephson Junction*

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The classical-field description of a long Josephson junction is reinterpreted as a quantum-field theory, which allows a unified treatment of fluctuation phenomena in both the Meissner state and the mixed state. For practical configurations, the mean square fluctuations in the field φ are negligible, and the zero-point corrections to the energy in the mixed state are less than 1%.

I. INTRODUCTION

THE electrodynamic behavior of a long Josephson junction[1–10] is similar to that of a bulk type-II superconductor.[11] In a weak external magnetic field ($H_0<H_{c1}$), the junction excludes all magnetic flux except in a small region of length λ_J (the Josephson penetration depth) near its ends. When the external magnetic field exceeds H_{c1}, quantized flux lines penetrate the junction, forming a regular one-dimensional lattice. The associated supercurrents are confined to a thin layer of thickness λ_J (the London penetration depth) on each side of the barrier. This mixed state has recently been studied in great detail with Josephson's classical field equations.[7,9,10] In many one- or two-dimensional systems,[12–15] however, a classical description is inadequate because fluctuations reduce or destroy the long-range order. For this reason, we here present a quantum field theory of a long Josephson junction. In Sec. II, the Hamiltonian operator is derived with a canonical transformation; the resulting excitation spectrum is discussed in Sec. III. The theory is then used to examine spatial correlations in the phase φ and in the order parameter (Sec. IV) and to compute the zero-point contribution to the energy (Sec. V).

II. FUNDAMENTAL EQUATIONS

Consider two identical semi-infinite superconductors separated by a thin dielectric sheet of thickness l, lying in the xy plane. Josephson's phenomenological description of the barrier[2] is based on a two-dimensional field $\varphi(x, y, t)$ that represents the increase in the phase of the order parameter on crossing the barrier from $z=0-$ to $z=0+$. The derivatives of φ are related to the *total* electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$ in the junction:

$$\partial\varphi/\partial t=2elE/\hbar=2eV/\hbar, \tag{1}$$

$$\nabla\varphi=(2ed/\hbar c)\mathbf{H}\times\hat{z}, \tag{2}$$

where V is the voltage across the dielectric, and $d=2\lambda_L+l\gg l$ is the effective thickness of the region containing the magnetic fields. The phase φ also determines the supercurrent $\mathbf{j}=j\hat{z}$ flowing across the barrier,[1–3]

$$j=j_1\sin\varphi, \tag{3}$$

where j_1 is a constant characteristic of the particular junction. Equations (1)–(3) may be combined with Maxwell's equation to yield the fundamental field equation[2,4]

$$\nabla^2\varphi-\bar{c}^{-2}(\partial^2\varphi/\partial t^2)=\lambda_J^{-2}\sin\varphi, \tag{4}$$

where

$$\bar{c}=c(l/\epsilon d)^{1/2}, \tag{5}$$

and

$$\lambda_J=(\hbar c^2/8\pi j_1 ed)^{1/2}. \tag{6}$$

Here, ϵ is the dielectric constant of the barrier, c is the velocity of light in vacuum, and ∇^2 is a two-dimensional Laplacian. Typical numerical values[7] are $\bar{c}\approx10^9$ cm sec^{-1}, $\lambda_J\approx10^{-2}$ cm, and $d\approx10^{-5}$ cm.

Equation (4) may be considered as the variational equation for the following Lagrangian:

$$\mathfrak{L}=(\hbar j_1/2e)\iint dx\,dy\,[\tfrac{1}{2}(\lambda_J/\bar{c})^2(\partial\varphi/\partial t)^2-\tfrac{1}{2}\lambda_J^2(\nabla\varphi)^2-1+\cos\varphi], \tag{7}$$

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where the additive constant is chosen to ensure that $\mathcal{L}$ vanishes if $\varphi=0$. The corresponding Hamiltonian $\mathcal{H}$ is

$$\mathcal{H}=\iint dx\,dy\,\{\tfrac{1}{2}(2e/\hbar j_1)(\bar{c}/\lambda_J)^2\pi^2 +(\hbar j_1/2e)[\tfrac{1}{2}\lambda_J^2(\nabla\varphi)^2+1-\cos\varphi]\},\quad (8)$$

with the canonical equations of motion

$$\delta\mathcal{H}/\delta\varphi=-\dot{\pi},\qquad \delta\mathcal{H}/\delta\pi=\dot{\varphi}.\quad (9)$$

The classical field theory will now be quantized by interpreting φ and π as quantum-mechanical operators satisfying the canonical equal-time commutation relations

$$[\varphi(x,y,t),\pi(x',y',t)]=i\hbar\delta(x-x')\delta(y-y').\quad (10)$$

These operators obey the Heisenberg equations of motion

$$i\hbar\dot{\varphi}=[\varphi,\mathcal{H}],\qquad i\hbar\dot{\pi}=[\pi,\mathcal{H}],\quad (11)$$

where $\mathcal{H}$ is the Hamiltonian operator [Eq. (8)]. In addition, it is convenient to perform a simple canonical transformation, defined by the unitary operator[16,17]

$$S=\exp\left(i\hbar^{-1}\iint dx\,dy\,\varphi_0\pi\right),\quad (12)$$

where φ_0 is a c-number function. The field variables undergo the following transformation:

$$S\varphi S^\dagger=\varphi_0+\varphi,$$
$$S\pi S^\dagger=\pi,\quad (13)$$

while the Hamiltonian separates into three distinct terms,

$$S\mathcal{H}S^\dagger=\mathcal{H}_0+\mathcal{H}_1+\mathcal{H}_2,\quad (14)$$

where

$$\mathcal{H}_0=(\hbar j_1/2e)\iint dx\,dy\,\{\tfrac{1}{2}\lambda_J^2(\nabla\varphi_0)^2+1-\cos\varphi_0\} +(\hbar j_1/2e)\iint dx\,dy\,\{[\sin\varphi_0-\lambda_J^2\nabla^2\varphi_0]\varphi\},\quad (15)$$

$$\mathcal{H}_1=\iint dx\,dy\,\{(\hbar j_1/2e)[\tfrac{1}{2}\lambda_J^2(\nabla\varphi)^2+\tfrac{1}{2}\cos\varphi_0\varphi^2] +\tfrac{1}{2}(2e/\hbar j_1)(\bar{c}/\lambda_J)^2\pi^2\},\quad (16)$$

$$\mathcal{H}_2=(\hbar j_1/2e)\iint dx\,dy\,\{\sin\varphi_0\,(\sin\varphi-\varphi) +\cos\varphi_0(1-\tfrac{1}{2}\varphi^2-\cos\varphi)\}.\quad (17)$$

If the function φ_0 is now chosen to satisfy the equation

$$\lambda_J^2\nabla^2\varphi_0=\sin\varphi_0,\quad (18)$$

then the second term of $\mathcal{H}_0$ vanishes, and Eq. (15) becomes a c-number Hamiltonian

$$\mathcal{H}_0=(\hbar j_1/2e)\iint dx\,dy\,\{\tfrac{1}{2}\lambda_J^2(\nabla\varphi_0)^2+1-\cos\varphi_0\}.\quad (19)$$

The classical field equation (18) has two solutions of particular interest.[2,6–10] The simplest choice $\varphi_0=0$ describes the Meissner state, which persists up to a critical magnetic field H_{c1}. Above H_{c1}, magnetic flux penetrates the junction, and the sample enters the mixed state. The corresponding solution φ_0 is given by

$$\sin[\tfrac{1}{2}(\varphi_0-\pi)]=\mathrm{sn}(x/\gamma\lambda_J,\gamma),\quad (20)$$

where sn is a Jacobian elliptic function.[18] For simplicity, we consider only the behavior in zero electric field, but a finite voltage can always be included by a suitable Lorentz transformation.[7,9] The modulus γ $(0\leq\gamma\leq1)$ fixes the lattice spacing a of the mixed state

$$a=2\lambda_J\gamma K(\gamma);\quad (21)$$

it also determines the constitutive relation between the induction $B\equiv\bar{H}$ and the applied field H_0 through the parametric equations

$$B=\bar{H}=(\hbar c/2ed\lambda_J)\pi[\gamma K(\gamma)]^{-1}=\Phi_0/ad,\quad (22)$$

$$H_0=(\hbar c/2ed\lambda_J)4\pi^{-1}\gamma^{-1}E(\gamma).\quad (23)$$

Here, K and E are the complete elliptic integrals of the first and second kind, and $\Phi_0=hc/2e$ is the quantum of magnetic flux. Equation (23) is a monotonically decreasing function of γ, and its minimum value at $\gamma=1$ defines the lower critical field

$$H_{c1}=2\Phi_0/\pi^2d\lambda_J.\quad (24)$$

The limiting forms of the constitutive relation may then be found explicitly:

$$B\approx(\Phi_0/d\lambda_J)\{\ln[8H_{c1}/(H_0-H_{c1})]\}^{-1},\qquad (H_0-H_{c1}\ll H_{c1})\quad (25)$$

$$B\approx H_0[1+O(H_{c1}^4/H_0^4)],\qquad (H_0\gg H_{c1}).\quad (26)$$

III. SPECTRUM OF SMALL OSCILLATIONS

The time dependence of the quantum fields is governed by the total Hamiltonian operator $\mathcal{H}_1+\mathcal{H}_2$, containing harmonic and anharmonic terms, respectively. In Sec. V, however, the anharmonic effects are shown to be negligible, so that the harmonic Hamiltonian $\mathcal{H}_1$ suffices to determine the physical properties of the system. If φ_1 denotes the harmonic field describing small oscillations about a given classical solution φ_0, then Eqs. (11) and (16) yield a linear field equation

$$\lambda_J^2[\nabla^2-\bar{c}^{-2}(\partial^2/\partial t^2)]\varphi_1(x,y,t)=\cos\varphi_0(x)\varphi_1(x,y,t).\quad (27)$$

[16] L. I. Schiff, Phys. Rev. **86**, 625 (1952).
[17] D. R. Yennie, Phys. Rev. **88**, 527 (1952).

[18] See, for example, H. B. Dwight, *Tables of Integrals and Other Mathematical Data* (The Macmillan Company, New York, 1957), 3rd ed., pp. 168–173.

The substitution[19]

$$\varphi_1(x, y, t) = b(q, x)e^{iky}e^{-i\omega t} \tag{28}$$

reduces this equation to

$$\lambda_J^2(d^2/dx^2 + \bar{c}^{-2}\omega^2 - k^2)b(q, x) = \cos\varphi_0(x)b(q, x), \tag{29}$$

which is analogous to a one-dimensional Schrödinger equation in a periodic potential. Kulik[7] has obtained a long-wavelength expansion for $b(q, x)$, while Lebwohl and Stephen[9] have given the exact solution for all q. As usual,[20] the Bloch function may be expressed as the product of a plane wave e^{iqx} and a periodic function $u_q(x)$ with period a,

$$b(q, x) = e^{iqx}u_q(x). \tag{30}$$

The Bloch functions obey an orthogonality relation

$$\int_0^{L_x} dx\, b^*(q, x)b(q', x) = \delta_{qq'}, \tag{31}$$

while time-reversal invariance implies

$$b(q, x) = b^*(-q, x). \tag{32}$$

An extended-zone scheme is used throughout this paper, so that band indices are unnecessary.

The field operators may be expanded in normal modes

$$\varphi_1(x, y) = L_y^{-1/2} \sum_{qk} e^{iky}b(q, x)\varphi_{qk},$$

$$\pi_1(x, y) = L_y^{-1/2} \sum_{qk} e^{-iky}b^*(q, x)\pi_{qk}, \tag{33}$$

and a straightforward calculation shows that

$$\mathcal{H}_1 = \tfrac{1}{2} \sum_{qk} [\mu^{-1}\pi_{qk}\pi_{-q-k} + \mu\omega_{qk}^2\varphi_{qk}\varphi_{-q-k}], \tag{34}$$

where

$$\mu = (\hbar j_1/2e)(\lambda_J/\bar{c})^2 \tag{35}$$

has the dimensions of a mass. The final diagonalization of $\mathcal{H}_1$ is now achieved with the linear transformation

$$a_{qk} = (2\hbar\mu\omega_{qk})^{-1/2}\pi_{qk} - i(\mu\omega_{qk}/2\hbar)^{1/2}\varphi_{-q-k}, \tag{36}$$

which yields the standard result

$$\mathcal{H}_1 = \sum_{qk} \hbar\omega_{qk}(a_{qk}^\dagger a_{qk} + \tfrac{1}{2}). \tag{37}$$

It is not difficult to verify that the operators $a^\dagger$ and a obey boson commutation relations

$$[a_{q'k'}, a_{qk}^\dagger] = \delta_{qq'}\delta_{kk'} \tag{38}$$

and are therefore identified as the creation and destruction operators for a vibration quantum with wave vector $q\hat{x} + k\hat{y}$ and frequency ω_{qk}.

The exact functions $b(q, x)$ and the dispersion relation ω_{qk} for the mixed state are given in the Appendix. Here, we shall list certain approximate dispersion relations that are useful in the following sections. In the extended-zone scheme, the spectrum in the q direction consists of two bands, the first lying inside the first Brillouin zone and the second lying outside. Thus, the spectrum has only a single gap, occurring at the edge of the first zone, with no gaps at the higher zone boundaries (Fig. 1). The edge of the first zone corresponds to

$$q = \pm q_m = \pm\pi/a = \pm\pi/2\lambda_J\gamma K(\gamma). \tag{39}$$

For definiteness, we shall refer to these two bands as vortex oscillations and plasma oscillations, respectively, but they are both transverse electromagnetic waves.

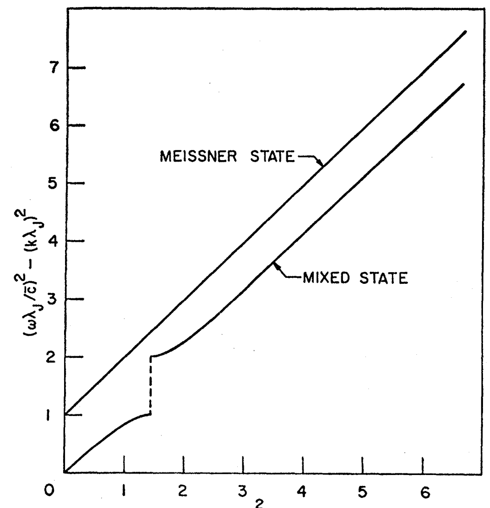

FIG. 1. The dispersion relation in the Meissner state and in the mixed state, where the following parameters have been used: $\gamma = \gamma' = 1/\sqrt{2}$, $a = 2.62\lambda_J$, and $H_0 = 1.91\, H_{c1}$.

A. Vortex Oscillations

In the long-wavelength limit $q \ll q_m$, the dispersion relation is

$$[\omega_v(q, k)]^2 = \bar{c}^{\,2}[k^2 + F^2(\gamma)q^2], \tag{40}$$

where

$$F^2(\gamma) = (\gamma' K/E)^2. \tag{41}$$

Actually, Eq. (40) is quite an accurate representation for all $|q| < q_m$. As shown in Ref. 9, the exact expression at $q = q_m$ is given by

$$[\omega_v(q_m, k)]^2 = \bar{c}^{\,2}[k^2 + (\gamma'/\gamma\lambda_J)^2], \tag{42}$$

while

$$F^2 q_m^2 = (\gamma'/\gamma\lambda_J)^2(\pi/2E)^2$$

differs from the last term in Eq. (42) by at most $\frac{1}{4}\pi^2$.

B. Plasma Oscillations

For $q - q_m \ll q_m$, the dispersion relation behaves like

$$[\omega_p(q, k)]^2 = \bar{c}^{\,2}[k^2 + (\gamma\lambda_J)^{-2} + G^2(\gamma)(q - q_m)^2], \tag{43}$$

[19] We use q and k to denote the wave numbers in the x and y directions, respectively, and γ and $\gamma' = (1-\gamma^2)^{1/2}$ to denote the modulus and complementary modulus of the elliptic functions. The junction is assumed to have an area $A = L_xL_y$.

[20] G. H. Wannier, *Elements of Solid State Theory* (Cambridge University Press, Cambridge, England, 1959), Chap. 5.

where

$$G^2(\gamma)=(\gamma K)^2(K-E)^{-2}. \quad (44)$$

Thus the gap at the edge of the first zone [Eqs. (42) and (43)] is

$$[\omega_p(q_m, k)]^2-[\omega_v(q_m, k)]^2=\bar{c}^{\,2}/\lambda_J^2, \quad (45)$$

independent of the magnetic field. In the opposite limit ($q\gg q_m$), the exact spectrum reduces to

$$[\omega_p(q, k)]^2=\bar{c}^{\,2}[k^2+q^2+(\gamma\lambda_J)^{-2}(2-\gamma^2-2E/K)]. \quad (46)$$

It is interesting to compare the mixed-state dispersion relation with that of the Meissner state

$$[\omega_M(q, k)]^2=\bar{c}^{\,2}[k^2+q^2+\lambda_J^{-2}]. \quad (47)$$

Equation (47) shows that the Meissner spectrum has the same gap as in Eq. (45), but the gap is now shifted to $q=k=0$. Indeed, Eq. (47) may be obtained as the limit of Eq. (46) for $\gamma\rightarrow 1(H_0\rightarrow H_{c1})$, since $q_m\rightarrow 0$ in this case.

IV. FLUCTUATIONS IN THE FIELD VARIABLES

Equation (37) defines the eigenvectors and eigenvalues of the harmonic Hamiltonian, and it is now possible to compute physical quantities of interest. As a first example, we consider the correlation function $\langle\varphi(x, y)\varphi(x', y')\rangle$, where the angular brackets denote an ensemble average at a temperature $T=(k_B\beta)^{-1}$. An elementary calculation yields

$$\langle\varphi(x, y)\varphi(x', y')\rangle=L_y^{-1}\sum_{qk}(\hbar/2\mu\omega_{qk})$$
$$\times\coth(\tfrac{1}{2}\beta\hbar\omega_{qk})b(q, x)b^*(q, x')e^{ik(y-y')}. \quad (48)$$

In particular, the mean square fluctuation is given by

$$\langle\varphi^2\rangle_{\text{av}}=A^{-1}\iint dx\, dy\, \langle\varphi(x, y)\varphi(x, y)\rangle$$
$$=(\hbar/2\mu A)\sum_{qk}\omega_{qk}^{-1}\coth(\tfrac{1}{2}\beta\hbar\omega_{qk}). \quad (49)$$

Equation (49) is easily evaluated in the Meissner state

$$\langle\varphi_M{}^2\rangle_{\text{av}}=(2\pi\mu\bar{c}^{\,2}\beta)^{-1}\ln[\sinh(\tfrac{1}{2}\hbar\bar{c}\beta k_m)/\sinh(\tfrac{1}{2}\hbar\bar{c}\beta\lambda_J^{-1})], \quad (50)$$

where an upper cutoff $k_m=2\pi/d\approx 6\times 10^5$ cm^{-1} has been introduced.[21] Typical orders of magnitude are $\mu\bar{c}^{\,2}\approx 10^{-11}$ erg, $\hbar\bar{c}k_m\approx 6\times 10^{-13}$ erg, $\langle\varphi_M{}^2\rangle_{\text{av}}\approx\hbar\bar{c}k_m/4\pi\mu\bar{c}^{\,2}\approx \frac{1}{2}\times 10^{-2}$, so that the field fluctuations are always small in the Meissner state.

For the mixed state, the approximate spectrum [Eq. (40)] may be used to evaluate the contribution of the vortex oscillations to the mean square fluctuations. When the sum is replaced by an integral, it is necessary to introduce a lower cutoff proportional to $A^{-1/2}$. The importance of a lower cutoff has been discussed by DeWames *et al.*[22] An approximate calculation gives

$$\langle\varphi_v{}^2\rangle_{\text{av}}=(\hbar q_m/2\mu\bar{c}\pi^2)[\ln(2k_m/Fq_m)+1]$$
$$+(2\mu\bar{c}^{\,2}\beta\pi F)^{-1}\ln(2A^{1/2}/\hbar\bar{c}\beta). \quad (51)$$

The first term, which represents the zero-point fluctuations, is again small. The second term represents the thermal fluctuations and is also small except for a narrow range of magnetic field near H_{c1}, where F behaves like

$$F\approx[(H_0-H_{c1})/2H_{c1}]^{1/2}\ln[8H_{c1}/(H_0-H_{c1})].$$

Strictly speaking, the mean square thermal fluctuations (51) diverge in the limit of a large system ($A\rightarrow\infty$), which means that the vortex lattice would be unstable with respect to thermal fluctuations. In practice, however, the dimensionless ratio $k_BT/\mu\bar{c}^2$ is $\approx 10^{-5}$ and the unit of length $\hbar\bar{c}/k_BT$ is $\approx 10^{-2}$ cm at 1°K, so that the fluctuations are small for all experimental systems. This divergent behavior may be understood qualitatively by observing that the frequency spectrum [Eq. (40)] is analogous to that of a two-dimensional lattice of point masses, where a similar divergence occurs.[12]

In order to investigate the long-range phase coherence between the two superconductors that make up the junction, we examine the correlation function for the order parameter

$$\langle e^{i\varphi(x,y)}e^{-i\varphi(x',y')}\rangle=\exp\{-\tfrac{1}{2}\langle[\varphi(x, y)-\varphi(x', y')]^2\rangle\}. \quad (52)$$

The expectation value on the right-hand side is readily evaluated in the Meissner state

$$\langle[\varphi(x, y)-\varphi(x', y')]^2\rangle_M$$
$$=(\hbar/2\pi\mu\bar{c})\int_0^{k_m}kdk\,(k^2+\lambda_J^{-2})^{-1/2}$$
$$\times\coth[\tfrac{1}{2}\hbar\bar{c}\beta(k^2+\lambda_J^{-2})^{1/2}][1-J_0(kr)], \quad (53)$$

where $J_0(kr)$ is a Bessel function and r is the distance separating the points (x, y) and (x', y'). Apart from the last factor, the integrand is identical with that in Eq. (49), which therefore provides an upper bound

$$\langle[\varphi(x, y)-\varphi(x', y')]^2\rangle_M<3\langle\varphi_M{}^2\rangle_{\text{av}}, \quad (54)$$

because $|1-J_0(x)|<1.5$ for all x. Equation (54) shows that the Meissner state of a junction separating two superconductors is characterized by long-range phase coherence. This result depends crucially on the gap in the spectrum at $k=q=0$. In contrast, the gap in the spectrum of the mixed state is shifted to q_m, and the corresponding phase correlation does not extend over arbitrarily large distances. If the Bloch functions are replaced by plane waves, the correlation function (52) in the mixed state may be evaluated approximately with the spectrum from Eq. (40). For large spatial separations perpendicular to the applied field,

[21] For disturbances varying more rapidly than k_m the superconducting properties of the metals cease to play a role and the phase difference φ cannot be defined.

[22] R. E. DeWames, G. W. Lehman, and T. Wolfram, Phys. Rev. Letters **13**, 749 (1964).

we find

$$\langle[\varphi(x,y)-\varphi(x',y)]^2\rangle_v \approx(\pi\mu\bar{c}\beta F)^{-1}\ln[2\,|\,x-x'\,|/\hbar\bar{c}\beta F], \quad (55)$$

neglecting terms that are independent of $|x-x'|$. Thus, strict phase correlation vanishes as $|x-x'|\to\infty$, but, as in the case of the mean square fluctuations, Eq. (55) is small for all systems of practical dimensions. This expression is very similar to that found by Rice,[14] who investigated phase fluctuations in a two-dimensional superconductor.

V. ZERO-POINT ENERGY

The classical field description of a Josephson junction assumes that quantum-mechanical effects are unimportant. One aspect of this assumption was examined in the previous section. As a different approach, the present section compares the classical and quantum-mechanical contributions to the ground-state energy. We define the mean energy per unit area as

$$\langle\mathcal{H}_i\rangle_{\text{av}}=A^{-1}\langle\mathcal{H}_i\rangle, \quad (56)$$

where $i=0$, 1, or 2, and the angular brackets now denote an average in the ground state specified by a particular solution φ_0.

As a first example, we consider the Meissner state, where φ_0 and $\langle\mathcal{H}_0\rangle_{\text{av}}$ vanish identically. The harmonic contribution to the energy is given by

$$\langle\mathcal{H}_1\rangle_{\text{av}}=(2A)^{-1}\sum_{qk}\hbar\omega_{qk}, \quad (57)$$

and an approximate calculation with Eq. (47) yields

$$\langle\mathcal{H}_{1M}\rangle_{\text{av}}\approx(\hbar\bar{c}/12\pi)[k_m^3+\tfrac{3}{2}\lambda_J^{-2}k_m+\cdots], \quad (58)$$

where the terms omitted are independent of the cutoff $k_m=2\pi/d$. Here the first term represents the zero-point fluctuations in the cavity formed by the two superconductors and is therefore irrelevant to the present considerations. The physically significant quantity is the *change* in the zero-point energy $\langle\delta\mathcal{H}_{1M}\rangle_{\text{av}}$ due to the coupling between the two superconductors. This is given by the second term of Eq. (58),

$$\langle\delta\mathcal{H}_{1M}\rangle_{\text{av}}=\hbar\bar{c}k_m/8\pi\lambda_J^2 =(\hbar j_1/2e)\tfrac{1}{2}\langle\varphi_M^2\rangle_{\text{av}}, \quad (59)$$

where the zero-temperature limit of Eq. (50) has been used to obtain the second line. It is interesting to compare Eq. (59) with the contribution of the anharmonic terms

$$\langle\mathcal{H}_{2M}\rangle_{\text{av}}=(\hbar j_1/2eA)\int d^2r\,\langle1-\tfrac{1}{2}\varphi^2-\cos\varphi\rangle =(\hbar j_1/2e)\{1-\tfrac{1}{2}\langle\varphi_M^2\rangle_{\text{av}}-\exp(-\tfrac{1}{2}\langle\varphi_M^2\rangle_{\text{av}})\}. \quad (60)$$

At zero temperature, Eq. (50) shows that

$$\langle\varphi_M^2\rangle_{\text{av}}=8\pi^3\hbar\bar{c}/\Phi_0^2 =8\pi(\bar{c}/c)\alpha\approx10^{-2},$$

where $\alpha\approx(137)^{-1}$ is the fine-structure constant. Equation (60) may be expanded, and comparison with Eq. (59) yields

$$|\,\langle\mathcal{H}_{2M}\rangle_{\text{av}}/\langle\delta\mathcal{H}_{1M}\rangle_{\text{av}}\,|\approx\tfrac{1}{4}\langle\varphi_M^2\rangle_{\text{av}}\approx\tfrac{1}{4}\times10^{-2}. \quad (61)$$

Thus the anharmonic contributions to the ground-state energy are small.

A similar analysis may be carried out for the mixed state, where the classical field φ_0 must now be taken from Eq. (20). Kulik[7] has calculated the classical ground-state energy

$$\langle\mathcal{H}_0\rangle_{\text{av}}=(\hbar j_1/2e)\{4E(\gamma^2K)^{-1}-2\gamma^{-2}+2\}, \quad (62)$$

where the additional constant ensures that $\langle\mathcal{H}_0\rangle_{\text{av}}$ vanishes at $\gamma=1$. The zero-point energy in the mixed state differs from Eq. (58) only because of the altered spectrum. An approximate evaluation with the plasma branch [Eq. (46)] yields

$$\langle\mathcal{H}_{1p}\rangle_{\text{av}}=(\hbar\bar{c}/12\pi)[k_m^3+\tfrac{3}{2}(\gamma\lambda_J)^{-2}k_m(2-\gamma^2-2E/K)], \quad (63)$$

while the vortex oscillations make a smaller contribution of order $\hbar\bar{c}q_m\lambda_J^{-2}\ln(k_m/q_m)$. Finally, an estimate of the anharmonic contributions to the energy of the mixed state gives

$$|\,\langle\mathcal{H}_2\rangle_{\text{av}}/\langle\delta\mathcal{H}_1\rangle_{\text{av}}\,|\approx\tfrac{1}{4}[\langle\varphi_v^2\rangle_{\text{av}}+\langle\varphi_p^2\rangle_{\text{av}}], \quad (64)$$

where $\langle\varphi_v^2\rangle_{\text{av}}$ and $\langle\varphi_p^2\rangle_{\text{av}}$ are taken from Eq. (51) and the second term of Eq. (63), respectively. Like the mean square fluctuations, the anharmonic terms are small except very close to H_{c1}, when $\langle\varphi_v^2\rangle_{\text{av}}$ becomes large.

It is now possible to calculate the energy $\mathcal{E}$ per unit area associated with the mixed state:

$$\mathcal{E}=\langle\mathcal{H}_0+\mathcal{H}_1\rangle_{\text{av}}\,|_{\text{mixed}}-\langle\mathcal{H}_0+\mathcal{H}_1\rangle_{\text{av}}\,|_{\text{Meissner}}. \quad (65)$$

Equations (58) and (63) show that the zero-point energies are identical in leading order, so that the quantum contributions to $\mathcal{E}$ depend only linearly on the cutoff k_m. Explicit evaluation leads to

$$\mathcal{E}=(\Phi_0^2/16\pi^3d\lambda_J)[(4E/\gamma^2K)-2\gamma^{-2}+2] -(\hbar\bar{c}/4d\lambda_J)[(2E/\gamma^2K)-2\gamma^{-2}+2], \quad (66)$$

where the second term represents the zero-point correction to the classical energy [Eq. (62)]. These zero-point fluctuations necessitate a renormalization of the magnetic field. The calculation is most simply performed with the thermodynamic identity[7]

$$\partial\mathcal{E}/\partial\bar{H}=(4\pi)^{-1}(H_0)_rd, \quad (67)$$

and gives

$$(H_0)_r=(2\Phi_0/\pi^2\lambda_Jd)\gamma^{-1}E-(4\bar{c}e/cd\lambda_J) \times[\gamma^{-1}E-K^2(1-\gamma^2)(\gamma E)^{-1}]. \quad (68)$$

The square bracket, which contains the quantum-mechanical corrections, equals unity at $\gamma=1$ and decreases monotonically as $\gamma\to 0$, becoming small of order γ^3. Hence the zero-point effects are most important for $\gamma=1$, which defines the renormalized lower critical field

$$(H_{c1})_r=(2\Phi_0/\pi^2\lambda_J d)-(4\bar{c}e/cd\lambda_J)$$
$$=(H_{c1})_0[1-\tfrac{1}{2}\pi\alpha(\bar{c}/c)]. \quad (69)$$

Here $(H_{c1})_0$ is the "bare" lower critical field [Eq. (24)], and the small quantum corrections ($\approx 10^{-3}$) slightly reduce the theoretical value.

Thus the zero-point and thermal fluctuations are generally negligible, which arises from a nearly complete cancellation of the zero-point energy in Eqs. (58) and (63). It should be noted that the total zero-point energy far exceeds the classical ground-state energy. Here, however, we are only concerned with the small changes between the mixed state and the Meissner state, or between the state when the superconductors are coupled or uncoupled ($j_1=0$). To obtain a quantitative estimate, we define an effective zero-point magnetic field $H_{\rm eff}$ by the relation

$$(8\pi)^{-1}H_{\rm eff}{}^2 d=\langle\delta\mathcal{H}_1\rangle_{\rm av}\approx(\hbar j_1/2e)\tfrac{1}{2}\langle\varphi^2\rangle_{\rm av}. \quad (70)$$

With the numerical values used previously, $H_{\rm eff}$ is approximately 10^{-2} Oe, which is much smaller than H_{c1} ($\approx$0.4 Oe).

The present work has shown that the quantum-mechanical corrections to Josephson's classical-field equations are usually small. The exception occurs in the region of H_{c1}, where the spectrum of vortex oscillations [Eq. (40)] becomes anisotropic. In this narrow range of applied field, the lattice spacing a may even become comparable with L_x, in which case the sum over q cannot be replaced by an integral. A detailed theoretical and experimental study of fluctuations near the phase transition to the mixed state would be of considerable interest.

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APPENDIX

For completeness, we list here the Bloch functions $b(q,x)$ satisfying Eq. (29) and the exact dispersion relations. These expressions follow from those of Lebwohl and Stephen[9] after a sequence of transformations.[23]

A. Vortex Oscillations

The frequency $\omega_v(q,k)$ and wave number q are determined by the parametric equations

$$[\omega_v(q,k)]^2=\bar{c}^{\,2}[k^2+(\gamma'/\gamma\lambda_J)^2\,\mathrm{sn}^2(\chi,\gamma')], \quad (A1)$$

$$q=(\gamma\lambda_J)^{-1}[E(\chi,\gamma')-\chi(1-E/K)], \quad (A2)$$

where χ varies between 0 and $K'\equiv K(\gamma')$. Here, $\mathrm{sn}(\chi,\gamma')$ is a Jacobian elliptic function and $E(\chi,\gamma')$ is the incomplete elliptic integral of the second kind. Apart from a normalization constant, the corresponding Bloch function is given in terms of ϑ functions as

$$u_q(x)=\frac{\vartheta_3[(\pi x/2\lambda_J\gamma K)+(i\pi\chi/2K)\mid\tau]}{\vartheta_4[(\pi x/2\lambda_J\gamma K)\mid\tau]}, \quad (A3)$$

where $\tau=iK'/K$. For long wavelengths, an expansion of Eqs. (A1) and (A2) yields

$$[\omega_v(q,k)]^2=\bar{c}^{\,2}[k^2+(\gamma'\chi/\gamma\lambda_J)^2], \quad (A4)$$

$$q=\chi E/\gamma\lambda_J K, \quad (A5)$$

which reproduces Eq. (40).

B. Plasma Oscillations

The frequency $\omega_p(q,k)$ and wave number q are determined by the relations

$$[\omega_p(q,k)]^2=\bar{c}^{\,2}[k^2+(\gamma\lambda_J)^{-2}\mathrm{dc}^2(\chi,\gamma')], \quad (A6)$$

$$q-q_m=(\gamma\lambda_J)^{-1}[\chi(1-E/K)+\mathrm{dc}(\chi,\gamma')\,\mathrm{sn}(\chi,\gamma')-E(\chi,\gamma')], \quad (A7)$$

where dc is another Jacobian elliptic function and χ again varies from 0 to K'. Apart from a normalization factor, the Bloch function for the plasma oscillations is equal to

$$u_q(x)=\frac{\vartheta_4[(\pi x/2\lambda_J\gamma K)-(i\pi/2K)(\chi-K')\mid\tau]}{\vartheta_4[(\pi x/2\lambda_J\gamma K)\mid\tau]}. \quad (A8)$$

For $q-q_m\ll q_m$, an expansion of Eqs. (A6) and (A7) yields

$$[\omega_p(q,k)]^2=\bar{c}^{\,2}[k^2+(\gamma\lambda_J)^{-2}+(\chi/\lambda_J)^2], \quad (A9)$$

$$q-q_m=(\chi/\gamma\lambda_J)(1-E/K), \quad (A10)$$

which immediately leads to Eq. (43). The other limit [Eq. (46)] also follows readily by expanding Eqs. (A7) and (A8) near $\chi=K'$.

[23] All of the relevant formulas may be found in E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, England, 1962), 4th ed., Chaps. XXI and XXII and Sec. 23.71. Numerical evaluation is most easily performed with G. W. Spenceley and R. M. Spenceley's work [Smithsonian Inst. Misc. Collections **109**, (1947)] which was used in preparing Fig. 1.