Multiple Scattering in the Gaussian Approximation: Systematic Improvement of the Small-Angle Treatment

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In the theory of the elastic multiple scattering of a charged particle in a plane, it is usually assumed that all angles involved are small. Working in the Gaussian approximation, we use a power-series expansion to effect a systematic improvement of the results of the small-angle approximation, these being the zero-order approximation to our expansion. A method of integration in function space is used to determine the joint probability of lateral and angular displacements of the scattered particle, in principle to any order in the expansion parameter; this computation is carried out explicitly to first order in the parameter. This joint probability is employed to obtain first-order corrections to previous results concerning, first, the lateral displacement $y(L/2)$ midway between two selected points a distance L apart on the track of the scattered particle, and second, the mean squared curvature $\langle C^2\rangle_{\mathrm{av}}$ of the track; in the former instance the result is $\langle y^2(L/2)\rangle_{\mathrm{av}}=(L^3/48\lambda)\{1+(23L/96\lambda)\}$, where λ is the scattering parameter, and in the latter, $\langle C^2\rangle_{\mathrm{av}}=(4/3\lambda L)\{1-(25L/96\lambda)\}$. For tracks both in two and in three dimensions, first-order corrections are determined also to results of the small-angle approximation for the actual path length of particles passing through a foil, both when the particles emerge normally to the foil and when they emerge in any direction. The results indicate that the discrepancy reported between theoretical predictions and some experimental measurements of multiple scattering is not attributable to the use of the small-angle approximation in the theoretical description.

1. INTRODUCTION

THE multiple scattering of charged particles in passage through matter is of considerable experimental importance and has been treated theoretically by a number of authors.[1] Because of the use of photographic plates in experimental work, the projection into two dimensions of the actual path of a particle is of particular significance. A useful expression relating to such a path in two dimensions is that for the joint probability density[2]

$$P(y,\alpha;x)=\frac{3^{1/2}\lambda}{\pi x^2}\exp\left\{-\frac{2\lambda}{x}\left(\alpha^2-\frac{3\alpha y}{x}+\frac{3y^2}{x^2}\right)\right\} \quad (1)$$

established by Fermi[3]; here y and α, each measured in a plane, are the lateral and angular displacements of a particle that has traversed a distance x through the scattering medium and λ^{-1} is the mean-squared value of the angular displacement per unit path length.[4]

Three assumptions are involved in obtaining this joint probability density [Eq. (1)]: (i) The elementary scattering process obeys a Gaussian probability law; (ii) the particle suffers no energy loss within the scattering material; and (iii) all angles involved are small. Of these the first two have been examined in great detail by other authors.[1] Here, our concern is with the third assumption and by using a power series in λ^{-1}, or rather in the dimensionless variable x/λ, we present a systematic method of improving the small-angle approximation on which the derivation of (1) is based. In terms of our expansion the small-angle approximation corresponds to the zero-order term alone.

Our method rests on the replacement of the angle α by a new variable s which is chosen so as to yield a Gaussian distribution with x as parameter; the lateral displacement y may be expressed as a functional of $s(x)$,

$$y(x)=\int_0^x F[s(\tau)]d\tau,$$

and the joint probability of y and of s may then be calculated by means of a standard procedure requiring the solution of a certain partial differential equation. In Sec. 3 we carry out this calculation to first order in our expansion parameter. Although in principle computations may be performed in closed form to as high an order as is desired, in practice such computations soon become prohibitively tedious.

The first-order probability density obtained in this way is applied in Sec. 4 to determine both the distribution of the sagitta, the lateral displacement midway between two selected points on the track, and also the mean-squared curvature of the track. In the following section we study a related problem, that of finding the actual path length of particles passing through foils, again by introducing the variable s; in this way we obtain first-order corrections to predictions of the small-angle approximation for tracks both in two and

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[1] See the recent review article on small-angle multiple scattering by W. T. Scott, Rev. Mod. Phys. **35**, 231 (1963), and other references quoted therein.

[2] We use a semicolon to separate given quantities on the right from quantities on the left whose probability distribution is under discussion.

[3] See B. Rossi and K. Greisen, Rev. Mod. Phys. **13**, 240 (1941).

[4] This differs by a factor of 2 from the corresponding scattering parameter introduced by certain authors, e.g., Ref. 3, in terms of angles measured in three dimensions.

in three dimensions, the latter generalizing some results of Yang.[5]

Our results are direct consequences of the Gaussian assumption, but in order to compare them with experimental results one must take into account corrections due to deviations from the Gaussian form. These have been worked out by Molière and others[1,6] and are the dominating corrections when $\lambda \gg x$. When $\lambda \simeq x$, however, the small-angle approximation is completely inadequate and our corrections assume major importance.

2. METHOD

The fundamental assumption we make is that the direction of the track is a Gaussian process whose parameter is *the path length* in the plane of projection. In terms of an elementary distance Δx measured in a fixed direction in this plane, the x axis, this basic relation becomes

$$P(\Delta\alpha;\Delta x)=\left(\frac{\lambda\cos\alpha}{2\pi\Delta x}\right)^{1/2}\exp\left\{-\frac{\lambda(\Delta\alpha)^2\cos\alpha}{2\Delta x}\right\},\quad (2)$$

where α is the angle between the x axis and the direction of the track in the plane of projection, and where λ^{-1} is the mean value of $(\Delta\alpha)^2$ per unit path length.

The crux of the present method is the substitution

$$s(\alpha)\equiv\int_0^\alpha(\lambda\cos\alpha')^{1/2}d\alpha',\qquad ds/d\alpha=(\lambda\cos\alpha)^{1/2},\quad (3)$$

which converts (2) into

$$P(\Delta s;\Delta x)=(2\pi\Delta x)^{-1/2}\exp\{-(\Delta s)^2/2\Delta x\}.\quad (4)$$

Thus s is a Wiener process whose parameter is x.

The joint probability of this variable s and of the ordinate

$$y(x)=\int_0^x\tan\alpha(\tau)\,d\tau\quad (5)$$

is now to be sought.

Since y and α are not expressible in simple closed form as functions of s we expand (3) and (5) to obtain

$$\alpha=(s/\lambda^{1/2})+\tfrac{1}{12}(s/\lambda^{1/2})^3+\tfrac{11}{480}(s/\lambda^{1/2})^5+\cdots\quad (6)$$

and

$$y=\int_0^x d\tau\left[\frac{s(\tau)}{\lambda^{1/2}}+\frac{5}{12}\left\{\frac{s(\tau)}{\lambda^{1/2}}\right\}^3+\frac{23}{96}\left\{\frac{s(\tau)}{\lambda^{1/2}}\right\}^5+\cdots\right].\quad (7)$$

The second expression is simplified by the introduction of the variable

$$\eta\equiv\lambda^{1/2}y,\quad (8)$$

since then

$$\eta=\int_0^x V[s(\tau)]d\tau,\quad (9)$$

where

$$V(s)\equiv s+\tfrac{5}{12}(s^3/\lambda)+\tfrac{23}{96}(s^5/\lambda^2)+\cdots.\quad (10)$$

To determine $P(\eta,s;x)$, the joint probability density of η and of s, we use a theorem due to Kac,[7] which relates the transform of the probability we seek to a Wiener integral that in turn is determined by the solution of a certain partial differential equation; thus we have

$$\begin{aligned}ds\int_0^\infty&\exp(-u\eta)P(\eta,s;x)\,d\eta\\&=E\left[\exp\left\{-u\int_0^x V[s(\tau)]d\tau\right\};\,s<s(x)<s+ds\right]\\&=Q(s;x)\,ds,\end{aligned}\quad (11)$$

where $Q(s;x)$ satisfies

$$\begin{aligned}\partial Q/\partial x&=\tfrac{1}{2}(\partial^2Q/\partial s^2)-uV(s)Q,\\Q(s;0)&=\delta(s).\end{aligned}\quad (12)$$

Here u is an arbitrary parameter and the Wiener measure is determined by (4); it is assumed that $\alpha=0$ at $x=0$.

By expanding $Q(s;x)$ in the form

$$Q(s;x)=\sum_{\kappa=0}^\infty\lambda^{-\kappa}Q_\kappa(s;x)\quad (13)$$

and equating coefficients of equal powers of λ in (12) we obtain the set of equations

$$LQ_0=0,\quad (14)$$

$$LQ_1=-(5/12)us^3Q_0,\quad (15a)$$

$$LQ_2=-(5/12)us^3Q_1-(23/96)us^5Q_0,\quad (15b)$$

$$\cdots$$

where L is the linear operator

$$L\equiv(\partial/\partial x)-\tfrac{1}{2}(\partial^2/\partial s^2)+us\quad (16)$$

and where the boundary condition is

$$Q_\kappa(s,0)=\delta_{\kappa0}\delta(s).\quad (17)$$

First we solve the zero-order Eq. (14) by taking its Fourier transform. The solution satisfying the boundary condition $Q_0(s;x)=\delta(s)$ is found readily to be

$$Q_0(s;x)=(2\pi x)^{-1/2}\exp\{-[(s+\tfrac{1}{2}ux^2)^2/2x]+\tfrac{1}{6}u^2x^3\}.\quad (18)$$

That this corresponds to the small-angle approximation

[5] C. N. Yang, Phys. Rev. **84**, 599 (1951).

[6] G. Molière, Z. Naturforsch. **2a**, 133 (1947); **3a**, 78 (1948); **10a**, 177 (1955).

[7] M. Kac, *Probability and Related Topics in Physical Sciences* (Interscience Publishers, Inc., New York, 1959), pp. 161–182.

may be seen by inverting the equation

$$\int_0^\infty \exp(-u\eta)P_0(\eta, s; x)d\eta = Q_0(s; x), \qquad (19)$$

which is the zero-order form of (11), to obtain

$$P_0(\eta, s; x) = (3^{1/2}/\pi x^2)\exp\{-(2/x)(s^2 - 3(\eta s/x) + 3(\eta^2/x^2))\}. \qquad (20)$$

Since $\eta \equiv \lambda^{1/2} y$ and since to zero order $s = \lambda^{1/2}\alpha$, this probability density is that of Fermi quoted in Eq. (1).

All the higher-order Eqs. (15a), (15b), etc., are of the form

$$LQ_\kappa = f(s; x), \qquad (21)$$

where $f(s; x)$ is a known function. To deal with these inhomogeneous equations the standard Green's function procedure is used, whereby if $G(s, s'; x)$ is the Green's function satisfying the homogeneous equation and boundary condition

$$LG = 0, \qquad G(s, s'; 0) = \delta(s - s'), \qquad (22)$$

the solution of the inhomogeneous Eq. (21) that satisfies the boundary condition $Q_\kappa(s; 0) = 0$ is

$$Q_\kappa(s; x) = \int_0^x dx' \int_{-\infty}^\infty ds' G(s, s'; x - x') f(s'; x'). \qquad (23)$$

Thus the higher-order Eqs. (15a), (15b), etc., may be solved in principle by finding a Green's function that satisfies (22). In fact the Green's function required is

$$G(s, s'; x) = \exp(-us'x)Q_0(s - s'; x) = (2\pi x)^{-1/2}\exp\{-[(s-s')^2/2x] - \tfrac{1}{2}(s+s')ux + (1/24)u^2x^3\}. \qquad (24)$$

Before proceeding to calculate the first-order joint probability density $P^{(1)}(\eta, s; x)$ we first generalize these results. So far only the situation that $\alpha = 0$ at $x = 0$ has been dealt with; now it is desired to extend the results by allowing an arbitrary value to the initial angle. In place of s we therefore work with a variable σ such that

$$\sigma \equiv s - s_0, \qquad (25)$$

where $s_0 = s(\alpha_0)$, α_0 being the value of α at $x = 0$. Instead of (10) this gives

$$V(\sigma; s_0) \equiv (\sigma + s_0) + (5/12)\lambda^{-1}(\sigma + s_0)^3 + (23/96)\lambda^{-2}(\sigma + s_0)^5 + \cdots; \qquad (26)$$

the zero-order equation for $Q_0(\sigma; x, s_0)$ takes the form

$$\partial Q_0/\partial x - \tfrac{1}{2}(\partial^2 Q_0/\partial\sigma^2) + u(\sigma + s_0)Q_0 = 0, \qquad Q_0(\sigma; 0, s_0) = \delta(\sigma), \qquad (27)$$

with the solution

$$Q_0(\sigma; x, s_0) = (2\pi x)^{-1/2} \times \exp\{-(\sigma^2/2x) - \tfrac{1}{2}\sigma ux + (1/24)u^2x^3 - us_0x\}, \qquad (28)$$

and the Green's function is

$$G(\sigma, \sigma'; x, s_0) = (2\pi x)^{-1/2}\exp\{-[(\sigma-\sigma')^2/2x] - \tfrac{1}{2}ux(\sigma + \sigma' + 2s_0) + (1/24)u^2x^3\}. \qquad (29)$$

3. FIRST-ORDER JOINT PROBABILITY DENSITY

From (23) and (15a) it follows that

$$Q_1(\sigma; x, s_0) = -(5/12)u\int_0^x dx' \int_{-\infty}^{+\infty} d\sigma'(\sigma' + s_0)^3 \times G(\sigma, \sigma'; x - x')Q_0(\sigma'; x', s_0). \qquad (30)$$

Integration over σ' and x' in turn yields the result

$$\begin{aligned}Q_1(\sigma; x, s_0) = -(5/12)Q_0(\sigma; x, s_0)ux &\times[(s_0^3 + \tfrac{3}{2}s_0^2\sigma + s_0\sigma^2 + \tfrac{1}{4}\sigma^3 + \tfrac{1}{2}s_0x + \tfrac{1}{4}\sigma x)\\ &- u(\tfrac{1}{4}s_0^2x^2 + \tfrac{1}{4}s_0\sigma x^2 + (3/40)\sigma^2x^2 + (1/20)x^3)\\ &+ u^2((1/40)s_0x^4 + (1/80)\sigma x^4) - (1/1120)u^3x^6],\end{aligned}$$

and the first-order approximation to $Q(\sigma; x, s_0)$ is

$$Q^{(1)}(\sigma; x, s_0) \equiv Q_0 + \lambda^{-1}Q_1. \qquad (31)$$

Inverting (31) in accordance with (11) gives

$$P^{(1)}(\eta, \sigma; x, s_0) = \frac{3^{1/2}}{\pi x^2}\exp\left\{-6\frac{Z^2}{x^3} + 6\frac{Z\sigma}{x^2} - 2\frac{\sigma^3}{x}\right\} \times \left[1 + \frac{x}{\lambda}\sum_{\kappa=0}^{4} a_\kappa \sigma^\kappa\right], \qquad (32)$$

where

$$Z \equiv \eta - s_0x \qquad (33)$$

and

$$\begin{aligned}a_0 &\equiv -(1/56x^6)[5x^6 + 70s_0^2x^5 + 112s_0x^4Z - 280s_0^3x^3Z + 48x^3Z^2 - 840s_0^2x^2Z^2 - 1008s_0xZ^3 - 432Z^4],\\ a_1 &\equiv -(1/28x^5)[7s_0x^4 + 70s_0^3x^3 + 4x^3Z + 210s_0^2x^2Z + 336s_0xZ^2 + 180Z^3],\\ a_2 &\equiv -(1/56x^4)[5x^3 - 196s_0xZ - 144Z^2],\\ a_3 &\equiv -(1/14x^3)[14s_0x + 5Z],\\ a_4 &\equiv -1/7x^2.\end{aligned} \qquad (34)$$

From this there follows easily the first-order marginal probability density

$$P^{(1)}(\eta; x, s_0) = \int_{-\infty}^\infty P^{(1)}(\eta, \sigma; x, s_0)d\sigma = \left(\frac{3}{2\pi x^3}\right)^{1/2}\exp\left\{\frac{-3Z^2}{2x^3}\right\}\left[1 + \frac{x}{\lambda}\sum_{\kappa=0}^{4} b_\kappa Z^\kappa\right], \qquad (35)$$

where

$$b_0 \equiv -(1/224x)[31x+280s_0^2],$$

$$b_1 \equiv -(1/8x^3)[21s_0x-10s_0^3],$$

$$b_2 \equiv -(1/224x^4)[339x-840s_0^2],$$

$$b_3 \equiv (9/2)s_0/x^5,$$

$$b_4 \equiv (27/14)/x^6. \qquad (36)$$

In Fig. 1 are plotted graphs of the zero-order and first-order terms of this marginal probability density in the special instance in which the initial angle α_0 has the value zero. The Molière corrections,[1] not included here, would add a tail to the distribution shown in Fig. 1, especially important where the Gaussian distribution P_0 is smaller than about 0.1.

The other first-order marginal probability density is readily obtainable as

$$P^{(1)}(\sigma; x, s_0) = \int_{-\infty}^{\infty} P^{(1)}(\eta, \sigma; x, s_0)\,d\eta$$
$$= (2\pi x)^{-1/2} \exp\{-\sigma^2/2x\}, \qquad (37)$$

as indeed it must be.

To obtain higher-order approximations by this method is perfectly straightforward. However, even in second order, an unduly large number of terms appear, and so, having displayed the procedure, we rest content with the computation of the first-order approximation.

4. SAGITTA PROBLEM AND THE CURVATURE

The first problem dealt with in this section is that of finding the probability for the lateral displacement y at the midpoint of a chord of length L drawn to the projection of the actual path into a plane; hence may be derived in particular the mean-squared lateral displacement $\langle y^2(L/2)\rangle_{av}$ at the midpoint. The latter quantity has direct experimental significance, since measurements of it are used to yield the value of the scattering parameter λ and this in turn is related to dynamical properties of the scattered particle.

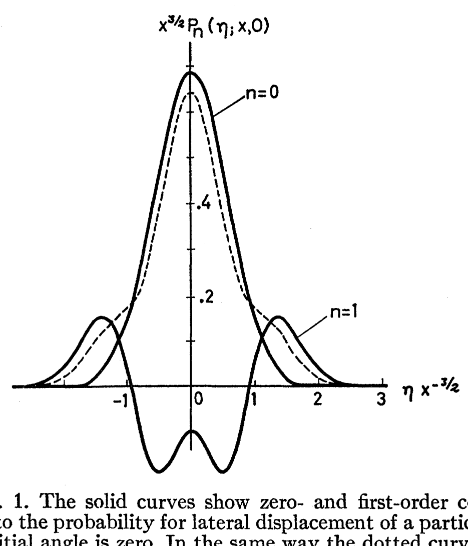

FIG. 1. The solid curves show zero- and first-order contributions to the probability for lateral displacement of a particle when the initial angle is zero. In the same way the dotted curve shows the probability correct to first order, $P^{(1)}=P_0+(x/\lambda)P_1$, at $x=\frac{1}{2}\lambda$.

We consider first a track in two dimensions that starts at the origin with a fixed value of α_0 and that passes through a narrow gate with coordinates $(2x, 0)$. What is sought is the conditional probability density $P(\eta; x, s_0 \mid \eta(2x)=0)$, and, since we are here dealing with a Markov process, an unnormalized measure for this probability density is

$$\int_{-\infty}^{\infty} d\sigma P(\eta, \sigma; x, s_0) P(-\eta; x, \sigma+s_0), \qquad (38)$$

this being obtained by the device of considering an intermediate gate with abscissa x and arbitrary ordinate y. Because the angle of emergence at the final gate is not of interest, the marginal probability density $P(-\eta; x, \sigma+s_0)$ obtainable from (35) is used to describe the passage between the intermediate and final gates.

Integration over σ and subsequent normalization yields the conditional probability density we seek, correct to first order in x/λ, as

$$P^{(1)}(\eta; x, s_0 \mid \eta(2x)=0) = \left(\frac{48}{7\pi x^3}\right)^{1/2}$$
$$\times \exp\left\{-\frac{48}{7}\frac{\eta^2}{x^3}+\frac{36}{7}\frac{\eta s_0}{x^2}-\frac{27}{28}\frac{s_0^2}{x}\right\}\left[1+\frac{x}{\lambda}\sum_{\kappa=0}^{4} f_\kappa \eta^\kappa\right], \qquad (39)$$

where

$$f_0 \equiv \frac{1}{x^2}\left[\frac{19}{5488}x^2+\frac{1865}{76832}s_0^2x-\frac{5031}{67228}s_0^4\right],$$

$$f_1 \equiv \frac{1}{x^3}\left[\frac{87}{2401}s_0x-\frac{5169}{16807}s_0^3\right],$$

$$f_2 \equiv -\frac{1}{x^4}\left[\frac{9759}{4802}x-\frac{37686}{16807}s_0^2\right],$$

$$f_3 \equiv -\frac{97074}{16807}\frac{1}{x^5}s_0,$$

$$f_4 \equiv \frac{152496}{16807}\frac{1}{x^6}.$$

This probability density has been derived for a fixed value s_0 at the origin. To remove this restriction we must average over all initial values of s_0 and to do this we require the distribution of s_0. It is not the *a priori* distribution of s_0 that is needed here but the distribution subject to the restriction $\eta(2x)=0$; thus it is the conditional probability density $P(s_0 \mid \eta(2x)=0)$ that we seek. It is possible to derive this from the *a priori* probability density $P(s_0)$ by the use of Bayes's rule.[8] In zero order s_0 becomes the initial angle α_0 and in this instance we should assign a uniform *a priori* probability to α_0 on the grounds that the physical circumstances do not predetermine the initial angle in any way. In higher order, however, although the same reasoning may indicate that the initial angle remains uniformly distributed, the form (4) of the scattering law suggests that it is s_0 rather than α_0 that is to be regarded as having uniform *a priori* probability. Since we do not regard this as being a convincing argument, nor indeed any argument at all, we prefer to derive the conditional probability density $P(s_0 \mid \eta(2x)=0)$ by a different approach. It may be noted, however, that this alternative approach is consistent with the supposition that s_0 is distributed uniformly, since it yields the same first-order expression for $P(s_0 \mid \eta(2x)=0)$ as does Bayes's rule supplemented by the assumption that $P(s_0)$ is constant.

The alternative method whereby we determine $P(s_0 \mid \eta(2x)=0)$ is based on the spatial reversibility of the model we are using. Since energy loss is disregarded, there is nothing to distinguish in which direction the particle traverses the track. Consequently, for any track passing between two gates we require that the angle α have the same distribution at each gate.

We consider therefore a track that passes through two gates, one at $(0,0)$ the other at $(x,0)$, and we demand that $P(s_0 \mid \eta(x)=0)$ be the same function of s_0 as $P(s \mid \eta(x)=0)$ is of s. The latter probability density may be obtained from the equation

$$P(s \mid \eta)=\int_{-\infty}^{\infty} ds_0 P(\sigma\equiv s-s_0;\, x,\, s_0 \mid \eta)P(s_0 \mid \eta)$$

by choosing the particular value $\eta(x)=0$. Here, $P(\sigma\equiv s-s_0;\, x,\, s_0 \mid \eta(x)=0)$ is derived from (32)-(34) by putting $Z\equiv\eta-s_0x=-s_0x$; however, what results from substituting this fixed value for η must first be normalized to unity with respect to σ in order to yield $P(\sigma\equiv s-s_0;\, x,\, s_0 \mid \eta(x)=0)$. In this way to first order in x/λ may be found the expression

$$P^{(1)}(s-s_0;\, x,\, s_0 \mid \eta(x)=0)=(2/\pi x)^{1/2} \times \exp\{(-\tfrac{1}{2}s_0^2-2s_0s-2s^2)/x\}\left[1+(x/\lambda)\sum_{\kappa=0}^{4} g_\kappa s^\kappa\right], \tag{40}$$

where

$$g_0\equiv(1/224x^2)(11x^2+11s_0^2x-16s_0^4),$$
$$g_1\equiv(1/14x^2)(s_0x-s_0^3),$$
$$g_2\equiv-(1/56x^2)(5x-8s_0^2),$$
$$g_3\equiv-(1/14x^2)s_0,$$
$$g_4\equiv-1/7x^2.$$

[8] See, for instance, W. Feller, *An Introduction to Probability Theory and its Applications* (John Wiley & Sons, Inc., New York, 1957), 2nd ed., p. 114.

The problem now is to solve the integral equation

$$P(s \mid \eta(x)=0) =\int_{-\infty}^{\infty} ds_0 P(s-s_0;\, x,\, s_0 \mid \eta(x)=0)P(s_0 \mid \eta(x)=0), \tag{41}$$

which is a Fredholm equation of the second kind, homogeneous and of eigenvalue unity. It is convenient to symmetrize the exponential factor in the kernel (40), this being the entire kernel in the zero-order approximation to the integral equation. To this end we first put

$$s\equiv t(\tfrac{1}{3}x)^{1/2}, \qquad s_0\equiv t_0(\tfrac{1}{3}x)^{1/2} \tag{42}$$

and then symmetrize by the substitution

$$P(s \mid \eta(x)=0)\exp(3s^2/4x)\equiv v(t). \tag{43}$$

The integral equation (41) becomes

$$v(t)=\int_{-\infty}^{\infty} dt_0 K(t,t_0)v(t_0), \tag{44}$$

where

$$K(t,t_0)\equiv(2/3\pi)^{1/2} \times \exp\{-(5/12)(t^2+t_0^2)-\tfrac{2}{3}tt_0\}[1+(x/\lambda)F(t,t_0)] \tag{45}$$

and

$$2016F(t,t_0)\equiv 99-60t^2-32t^4+48tt_0-16t^3t_0 +33t_0^2+32t^2t_0^2-16tt_0^3-16t_0^4. \tag{46}$$

To solve this equation we make use of the expansion[9]

$$(2\pi)^{-1/2}(1-e^{-2x})^{-1/2}\exp\left\{\tfrac{1}{4}(y^2+z^2)-\frac{y^2+z^2-2yze^{-x}}{2(1-e^{-2x})}\right\} =\sum_{n=0}^{\infty}\mathfrak{D}_n(y)\mathfrak{D}_n(z)e^{-nx}, \tag{47}$$

in which $\mathfrak{D}_n(x)$ is the *normalized* Weber function of the nth order. By virtue of (47) with $e^{-x}=-\tfrac{1}{2}$ the kernel (45) may be written as

$$K(t,t_0)=\sum_{n=0}^{\infty}(-2)^{-n}\mathfrak{D}_n(t)\mathfrak{D}_n(t_0)[1+(x/\lambda)F(t,t_0)]. \tag{48}$$

By repeated use of the recurrence formula[10]

$$t\mathfrak{D}_n(t)=(n+1)^{1/2}\mathfrak{D}_{n+1}(t)+n^{1/2}\mathfrak{D}_{n-1}(t) \tag{49}$$

the first-order part may also be expressed in terms of the orthonormal set of functions $\mathfrak{D}_n(t)$.

[9] G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. **36**, 823 (1930). $D_n(x)=\mathfrak{D}_n(x)(n!)^{1/2}(2\pi)^{1/4}$ is the conventional Weber function of nth order.

[10] This relation may be obtained from the formula (Ref. 9) $tD_n(t)=D_{n+1}(t)+nD_{n-1}(t)$.

Since the $\mathfrak{D}_n(t_0)$ form a complete set of functions, we may put

$$v(t_0)=\sum_{m=0}^{\infty} a_m \mathfrak{D}_m(t_0), \tag{50}$$

where

$$a_m \equiv a_m^{(0)}+(x/\lambda)a_m^{(1)}+\cdots. \tag{51}$$

On substituting (48) and (50) into the integral equation (44) we find as an immediate consequence of the orthogonality relations

$$\begin{aligned} a_m^{(0)} &= a_0^{(0)}\delta_{m0}, \\ a_m^{(1)} &= 0, \qquad m=1,3,5,6,7,\cdots, \\ a_2^{(1)} &= -31t_0^2/672, \\ a_4^{(1)} &= -t_0^4/126, \end{aligned} \tag{52}$$

with the coefficients $a_0^{(0)}$ and $a_0^{(1)}$ determined by the normalization requirement. Replacing x by $2x$ we obtain the following solution of the integral equation (41):

$$P^{(1)}(s_0 \mid \eta(2x)=0)=\left(\frac{3}{4\pi x}\right)^{1/2} \times \exp\left(-\frac{3s_0^2}{4x}\right)\left[1+\frac{x}{\lambda}\left\{\frac{47}{336}-\frac{31}{224}\frac{s_0^2}{x}-\frac{1}{28}\frac{s_0^4}{x^2}\right\}\right]. \tag{53}$$

Having thus determined the probability density $P^{(1)}(\eta;\, x,\, s_0 \mid \eta(2x)=0)$ as in (39) for a fixed value of s_0 and the conditional probability density $P^{(1)}(s_0 \mid \eta(2x)=0)$ as in (53) for the distribution of s_0, we now derive the probability density $P^{(1)}(\eta;\, x \mid \eta(2x)=0)$ that we seek by evaluating to first order in x/λ the integral

$$\int_{-\infty}^{\infty} ds_0 P^{(1)}(\eta;\, x,\, s_0 \mid \eta(2x)=0) P^{(1)}(s_0 \mid \eta(2x)=0).$$

On retrieving y, the lateral displacement, by the substitution $\eta \equiv \lambda^{1/2} y$ and on setting $2x=L$, we find the probability density for the lateral displacement at the midpoint of a chord of length L to be

$$P^{(1)}(y(L/2) \mid y(0)=0,\, y(L)=0)=\left(\frac{24\lambda}{\pi L^3}\right)^{1/2} \times \exp\left\{-\frac{24\lambda y^2}{L^3}\right\}\left[1+\frac{L}{2\lambda}\left(\frac{55}{672}-\frac{271}{14}\frac{\lambda y^2}{L^3}+\frac{1728}{7}\frac{\lambda^2 y^4}{L^6}\right)\right], \tag{54}$$

correct to first order in x/λ. The mean-squared lateral displacement at the midpoint of the chord is then, in our first-order approximation,

$$\langle y^2(L/2)\rangle_{\text{av}}=(1/48)(L^3/\lambda)(1+(23/96)(L/\lambda)). \tag{55}$$

The zero-order term here is the same as that known previously.[11]

A further result that follows directly from the probability density (54) is the first-order correction to Bethe's[12] expression for the mean-squared curvature of a track in two dimensions. Here, the curvature C is not the usual local property but is defined as the reciprocal of the radius of that circle which, for any chord of length L drawn to the track, passes through the same three points $(0,0)$, $(L/2, y)$, and $(L, 0)$ as does the track itself. From the properties of a circle it follows that $C=8\,|\,y\,|\,(L^2+4y^2)^{-1}$ and hence that

$$\langle C^2\rangle_{\text{av}}=64L^{-4}\langle y^2\rangle_{\text{av}}-512L^{-6}\langle y^4\rangle_{\text{av}}+\cdots.$$

From (54) there is obtained the mean-squared curvature to first order in x/λ as

$$\langle C^2\rangle_{\text{av}}=\tfrac{4}{3}\lambda L^{-1}(1-(25/96)(L/\lambda)); \tag{56}$$

the zero-order term here is Bethe's result.

5. ACTUAL PATH LENGTH OF PARTICLES IN FOILS

The problem here is to determine the distribution of the actual path length of particles that suffer multiple scattering during their passage through a foil. Here the usual experimental situation is that the particles are incident normally on the foil, and so throughout this section we shall deal only with $\alpha_0=0$.

For a foil of thickness l the actual path length is expressed by

$$L=\int_0^l dx\, \sec\alpha(x), \tag{57}$$

where α is the angle between the direction of the track and the incident direction of the particle, this latter direction being taken as the x axis. At this stage the path of interest may be either a track in three dimensions or its projection into a plane as was considered in preceding sections; in the former instance the angle α is not, of course, identical with that appearing in preceding sections. The excess path length is given by

$$\lambda^{-1}\Delta \equiv L-l=\int_0^l dx\{\tfrac{1}{2}\alpha^2+(5/24)\alpha^4+\cdots\}. \tag{58}$$

First we deal with two-dimensional problems. In terms of the variable s, defined in Eq. (3), the excess path length is given by

$$\Delta=\int_0^l U[s(x)]dx, \tag{59}$$

where

$$U(s)\equiv \tfrac{1}{2}s^2+(7/24)\lambda^{-1}s^4+\cdots. \tag{60}$$

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[11] B. Rossi, *High-Energy Particles* (Prentice-Hall, Inc., Englewood Cliffs, N.J., 1952), p. 72, Eq. (13); Rossi's Θ_s is such that $\Theta_s^2=2/\lambda$.

[12] H. A. Bethe, Phys. Rev. **70**, 821 (1946).

This functional integral may be handled in the way described in Sec. 2; by this method we have obtained the transform with respect to Δ of the joint probability density $P^{(1)}(\Delta, s; x, s_0=0)$. Although its form makes it impracticable to invert this transform, we have nevertheless determined moments of Δ with respect to this probability density. We do not give any details, since the quantity of foremost experimental interest, the mean path length, may be determined much more simply by the following procedure.

From (58), (59), and (60) there follows for the mean path length the expression

$$\langle L\rangle_{\text{av}}=\int_0^l dx(1+\tfrac{1}{2}\lambda^{-1}\langle s^2\rangle_{\text{av}}+(7/24)\lambda^{-2}\langle s^4\rangle_{\text{av}}). \quad (61)$$

Using the probability density (37) for s, we obtain the mean path length as

$$\langle L\rangle_{\text{av}}=\int_0^l dx(1+\tfrac{1}{2}\lambda^{-1}x+\tfrac{7}{8}\lambda^{-2}x^2)$$
$$=l+\tfrac{1}{4}(l^2/\lambda)+(7/24)(l^3/\lambda^2). \quad (62)$$

In the problem just considered the particles are allowed to emerge from the foil at any position and in any direction. A second situation of experimental interest is one in which the particles emerge again at any position but only those leaving normally to the foil are counted. The mean path length $\langle L\rangle_{\text{av}}$ is again given by (61) but the probability density for s is now altered. The latter may be obtained in virtue of the Markov nature of the random process $s(x)$ as

$$P(s; x, s_0=0 \mid s(l)=0)=\{2\pi x(l-x)l^{-1}\}^{1/2}$$
$$\times \exp\{-s^2/2x\} \exp\{-s^2/2(l-x)\}, \quad (63)$$

and this yields the mean values

$$\langle s^2\rangle_{\text{av}}=x(l-x)l^{-1}, \qquad \langle s^4\rangle_{\text{av}}=3x^2(l-x)^2l^{-2}.$$

It follows that the mean path length is given by

$$\langle L\rangle_{\text{av}}=l+\tfrac{1}{12}(l^2/\lambda)+(7/240)(l^3/\lambda^2). \quad (64)$$

Both these results (62) and (64) refer to two-dimensional problems. Unlike the sagitta problem and related problems, however, the determination of path lengths is of greatest practical interest for tracks in three dimensions rather than for projections of these tracks. Accordingly we now derive results appropriate to three dimensions, again extending the small-angle approximation by one further order in the expansion parameter. On introducing the angles α_y and α_z as the angles between the x axis, the incident direction of the particle, and the projections of the path into xy and xz planes respectively, we may expand (57) to the order required as

$$L=\int_0^l dx(1+\tan^2\alpha_y+\tan^2\alpha_z)^{1/2}=\int_0^l dx(1+\tfrac{1}{2}\alpha_y^2$$
$$+\tfrac{1}{2}\alpha_z^2+(5/24)\alpha_y^4+(5/24)\alpha_z^4-\tfrac{1}{4}\alpha_y^2\alpha_z^2). \quad (65)$$

This is now to be averaged over all paths in three dimensions and to do this we must decide upon our basic probability assumption, since the form (2) of the elementary scattering law pertains specifically to two dimensions. The alternatives that present themselves immediately are to assume a Gaussian form either for the angle between neighboring elements of the track or for each of the angles α_y and α_z that specify the directions of the projections of the track relative to the x axis. Within the small-angle approximation these alternatives coalesce. Our choice of the second assumption is expedient in that it allows us to employ results used previously. It may be rationalized by noting that the parameter λ that appears in the Gaussian form is determined experimentally from measurements made on tracks in photographic plates, that is, effectively from projections of tracks. In any event, however, the Gaussian assumption is itself only an approximation to the scattering law that is derived from quantum electrodynamics,[1,6] and so too great store should not be set upon insisting on one or other of these choices.

From symmetry considerations we have

$$\langle\alpha_y^2\rangle_{\text{av}}=\langle\alpha_z^2\rangle_{\text{av}}, \qquad \langle\alpha_y^4\rangle_{\text{av}}=\langle\alpha_z^4\rangle_{\text{av}}.$$

Furthermore we may replace $\langle\alpha_y^2\alpha_z^2\rangle_{\text{av}}$ by its approximate value $\langle\alpha_y^2\rangle_{\text{av}}\langle\alpha_z^2\rangle_{\text{av}}$, since to evaluate $\langle\alpha_y^2\alpha_z^2\rangle_{\text{av}}$ correct to second order we need the probability distribution of $\alpha_y\alpha_z$ correct only to zero order and to this order, that of the small-angle approximation, the approximation made here is valid. Thus from (65) follows the result

$$\langle L\rangle_{\text{av}}=\int_0^l dx\{1+\langle\alpha_y^2\rangle_{\text{av}}+(5/12)\langle\alpha_y^4\rangle_{\text{av}}-\tfrac{1}{4}\langle\alpha_y^2\rangle_{\text{av}}^2\}, \quad (66)$$

and on using (6), with α_y in place of α, and the distributions (37) and (63) in turn we find

$$\langle L\rangle_{\text{av}}=l+\tfrac{1}{2}(l^2/\lambda)+\tfrac{1}{2}(l^3/\lambda^2) \quad (67)$$

when particles emerging in any direction are counted and

$$\langle L\rangle_{\text{av}}=l+\tfrac{1}{6}(l^2/\lambda)+(1/20)(l^3/\lambda^2) \quad (68)$$

when only particles emerging normally to the foil are counted. Both these formulas agree with Yang's results[5] as regards the terms belonging to the small-angle approximation. It may be noted that the small-angle approximation consistently underestimates the actual path length.

6. COMMENT

Some measurements of multiple scattering[13] have indicated that the scattering is considerably less than that predicted theoretically; our results show that this discrepancy is not attributable to the use of the small-

[13] E. Hisdal, Phil. Mag. 43, 790 (1952); F. F. Heymann and W. F. Williams, Phil. Mag. 1, 212 (1956).

angle approximation, since increased scattering is predicted when this approximation is improved.

As regards the assumption made throughout this paper that the particle suffers no energy loss in the scattering medium, this too may be improved upon by treating the scattering parameter λ as a function of the distance x.[14] The substitutions

$$\xi\equiv\int_0^x d\tau\{\lambda(\tau)\}^{-1},\qquad \nu\equiv\int_0^x d\alpha'\{\cos\alpha'\}^{1/2},$$

when made in the elementary scattering formula (2), yield formula (4) with ξ and ν replacing x and s, respectively; thus the formulas in Secs. 2 and 3 may be adapted directly to the new variables. It should be noted, however, that only the zero-order terms in these formulas are valid, since to suppose that the scattering parameter λ is a function of x rather than of actual path length is to employ the small-angle approximation.

[14] H. Øverås, CERN Report No. 60-18, Synchrocyclotron division (unpublished).

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Nuclear-Magnetic-Resonance Single-Shot Passage in Solids

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We study both theoretically and experimentally the magnetization loss of a nuclear spin system irradiated with a rf field under a linear field sweep through the resonance. Two limiting cases are considered. The first is that of a slightly saturating passage, in which the loss is simply related to the rf field amplitude, a fact which allows an easy calibration of the latter. The second case is that of a quasiadiabatic passage which reverses the magnetization. Three factors contribute to the magnetization loss: (1) The passage is sudden far in the wings of the line, and it becomes rather abruptly adiabatic at a given distance from resonance. An entropy increase accompanies this transition. (2) The passage through the central part is not quite adiabatic because of the finite sweep rate of the field through the line. (3) The finite spin-lattice relaxation time of the spin-spin term causes a loss of magnetization at the passage on the line. The losses are numerically computed for the fluorine spin system in CaF_2 with $\mathbf{H}_0 \parallel [100]$, and they are found to agree with the experimental values.

I. INTRODUCTION

In this article we study the behavior of a nuclear spin system in a solid, irradiated with a rf field during a single-shot passage, which we define as a linear sweep of the applied dc field through the resonance value. This encompasses the well-known fast, or adiabatic, passage, extensively used in nuclear magnetic resonance: When a suitably large rf field is applied, and when the sweep rate is low enough, this passage results in a reversal of the magnetization orientation with only a small loss in magnitude. The theory of the fast passage in solids is based on the spin-temperature concept.[1–3] It is well verified by experiment,[3] which is a check of the validity of this concept. This theory, however, is developed for the case of a strictly adiabatic passage and would be rigorously valid only if the sweep rate was infinitely low. In practice, the slowness of the sweep is limited by the condition that the entire passage must take place in a time much shorter than the spin-lattice relaxation time T_1, whence its name of fast passage. The passage is then never completely adiabatic, and it is the main concern of this work to analyze the lack of adiabaticity of a quasiadiabatic passage, and the resulting loss in magnetization amplitude. A second limiting case is also studied, the case of a slightly saturating single-shot passage: When the rf field is small and the passage through the line is fast, this passage results in a small decrease of the nuclear magnetization with no change of its orientation. The measurement of this magnetization loss provides a simple, fast, and accurate way of calibrating the rf field amplitude.

The theory of these single-shot passages is developed in the frame of the Provotorov theory of saturation.[4,5] Its validity is then restricted to the following cases:

(1) The temperature is high, that is, the nuclear polarization is so low that it is permissible to develop the density matrix to first order as a function of the

[1] A. G. Redfield, Phys. Rev. **98**, 1787 (1955).

[2] A. Abragam, *The Principles of Nuclear Magnetism* (Clarendon Press, Oxford, England, 1961), Chap. XII; C. P. Slichter, *Principles of Magnetic Resonance* (Harper & Brothers, New York, 1963); L. C. Hebel, in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press Inc., New York, 1963), Vol. 15.

[3] C. P. Slichter and W. C. Holton, Phys. Rev. **122**, 1701 (1961).

[4] B. N. Provotorov, Zh. Eksperim. i Teor. Fiz. **41**, 1582 (1961). [English transl.: Soviet Phys.—JETP **14**, 1126 (1962)].

[5] M. Goldman, J. Phys. (Paris) **25**, 843 (1964).