New Superconvergent Dispersion Relations for the Forward πN Crossing-Even Amplitude*

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A Liu-Okubo-type dispersion relation is derived for the crossing-even pion-nucleon forward elastic scattering amplitude $T^{(+)}$. Two subtractions are made, one at the physical scattering threshold and the other at a previously determined zero of $T^{(+)}$ on the imaginary axis of the complex ω (pion lab energy) plane. The dispersion relation is well satisfied over the whole allowed range of the Liu-Okubo parameter. Moreover, it is nearly saturated by low-energy scattering for a considerable range of the parameter. It should thus serve as an extremely sensitive test of the low-energy scattering data when such data become more accurately known.

IN a recent letter,[1] Liu and Okubo used a generalization of the method of Gilbert[2] to derive a new πN superconvergent relation. They considered the case of the forward crossing-odd amplitude $T^{(-)}$. In this paper, we use a generalization of their technique to derive several interesting results for the πN forward crossing-even amplitude $T^{(+)}$.

$T^{(+)}(\omega)$ has zeros at $\omega=\pm ia$, where ω is the pion laboratory energy[3,4] and a^2 has the value $0.103\,\mu^2$ (μ is the pion mass).[5,6] $T^{(+)}(\omega)$ has nucleon poles at $\omega=\pm\omega_0=\pm\mu^2/2M$ (M is the nucleon mass) and is assumed to have a high-energy behavior

$$T^{(+)}(\omega)\sim\omega \tag{1}$$

corresponding to constant infinite-energy total cross sections, $\sigma_{\pi^\pm p}(\infty)$. We normalize $T^{(+)}(\omega)$ so that the optical theorem has the form,

$$\begin{aligned}\operatorname{Im}T^{(+)}(\omega)&=(\omega^2-\mu^2)^{1/2}\sigma(\omega)\,,\\ \sigma(\omega)&=\tfrac{1}{2}[\sigma_{\pi^+p}(\omega)+\sigma_{\pi^-p}(\omega)].\end{aligned} \tag{2}$$

The value of $T^{(+)}(\mu)$ is subject to large errors[7]:

$$T^{(+)}(\mu)=-0.010\pm0.040\,\mu^{-1}.$$

Following Liu and Okubo,[1] we consider

$$t(\omega)=\frac{\omega[T^{(+)}(\omega)-T^{(+)}(\mu)]e^{\pi i\beta}}{(\omega^2+a^2)(\omega^2-\mu^2)^\beta}\,, \tag{3}$$

where the function $(\omega^2-\mu^2)^\beta$ is defined as in Ref. 1. For large ω,

$$\operatorname{Im}t(\omega)\sim\omega^{-2\beta}\,, \tag{4}$$

so for $\beta>\frac{1}{2}$, $t(\omega)$ is superconvergent,[8]

$$\int_{-\infty}^{\infty}\operatorname{Im}t(\omega)d\omega=-\pi T^{(+)}(\mu)/(\mu^2+a^2)^\beta. \tag{5}$$

The right side of (5) comes from the poles of $t(\omega)$ at $\pm ia$.

Near $\omega=\mu$, $T^{(+)}$ has the expansion

$$T^{(+)}(\omega)-T^{(+)}(\mu)=C(\omega^2-\mu^2)+i\sigma(\mu)(\omega^2-\mu^2)^{1/2}, \tag{6}$$

where C is a real constant related to the scattering lengths and effective ranges. Hence $t(\omega)$ has no pole at the physical scattering threshold if $\beta<\frac{3}{2}$. The only δ-function contributions to (5) are from the nucleon poles at $\omega=\pm\omega_0$. Thus for $\frac{1}{2}<\beta<\frac{3}{2}$,

$$\frac{2\omega_0 f^2}{(\omega_0^2+a^2)(\mu^2-\omega_0^2)^\beta}=-\frac{2}{\pi}\int_\mu^\infty\frac{\omega d\omega\{\cos\pi\beta\,\operatorname{Im}T^{(+)}(\omega)+\sin\pi\beta\,\operatorname{Re}[T^{(+)}(\omega)-T^{(+)}(\mu)]\}}{(\omega^2+a^2)(\omega^2-\mu^2)^\beta}-\frac{T^{(+)}(\mu)}{(\mu^2+a^2)^\beta}. \tag{7}$$

In the limit, $\beta=\frac{1}{2}$, we have an extra contribution from the contour integration at infinity which is simply $-\sigma(\infty)$. Therefore we derive

$$\sigma(\infty)=\frac{+T^{(+)}(\mu)}{(\mu^2+a^2)^{1/2}}+\frac{2\omega_0 f^2}{(\omega_0^2+a^2)(\mu^2-\omega_0^2)^{1/2}}+\frac{2}{\pi}\int_\mu^\infty\frac{\omega d\omega\,\operatorname{Re}[T^{(+)}(\omega)-T(\mu)]}{(\omega^2+a^2)(\omega^2-\mu^2)^{1/2}}. \tag{8}$$

The first two terms of (8) are approximately the same as the expression for $\sigma(\infty)$ given by the phase representation[4]. In the other limit, $\beta=\frac{3}{2}$, $t(\omega)$ has poles at $\omega=\pm\mu$ which give a term proportional to $\sigma(\mu)$. Thus we

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[1] Y. C. Liu and S. Okubo, Phys. Rev. Letters **19**, 190 (1967).
[2] W. Gilbert, Phys. Rev. **108**, 1078 (1957).
[3] M. Sugawara and A. Tubis, Phys. Rev. **130**, 2127 (1963).
[4] M. Sugawara and A. Tubis, Phys. Rev. **138**, B242 (1965).
[5] Natural units $\hbar=c=1$ are used throughout this work.
[6] The uncertainty in a^2 is estimated to be about 3% mainly because of the experimental uncertainty in $T^{(+)}(\mu)$.
[7] J. Hamilton [Phys. Letters **20**, 687 (1966)] gives for the s-wave scattering length combination $a_1+2a_3=(-0.002\pm0.008)\mu^{-1}$ so that $T^{(+)}(\mu)=\frac{4}{3}\pi(1+\mu/m)(a_1+2a_3)=(-0.010\pm0.040)\mu^{-1}$; $\sigma(\mu)=-\frac{4}{3}\pi(a_1^2+2a_3^2)=3.4\pm0.2$ mb.
[8] V. de Alfaro *et al.*, Phys. Letters **21**, 576 (1966).

obtain,

$$\frac{-\sigma(\mu)}{\mu^2+a^2}=\frac{-T^{(+)}(\mu)}{(\mu^2+a^2)^{3/2}}-\frac{2\omega_0 f^2}{(\omega_0^2+a^2)(\mu^2-\omega_0^2)^{3/2}}+\frac{2}{\pi}\int_\mu^\infty \frac{\omega d\omega\, \mathrm{Re}[T(\omega)-T(\mu)]}{(\omega^2+a^2)(\omega^2-\mu^2)^{3/2}}. \quad (9)$$

We have performed the numerical evaluation of (7)–(9) by taking the values of Hohler *et al.*[9] and doing a careful numerical integration which emphasizes the low-energy region. Above 30 BeV we took a Regge form for $T^{(+)}(\omega)$ containing the P (Pomeranchon) and P' (f^0 meson) terms,

$$T^{(+)}(\omega)=-\gamma_P\frac{e^{-i\pi\alpha_P/2}}{\sin(\pi\alpha_P/2)}\omega^{\alpha_P}-\gamma_{P'}\frac{e^{-i\pi\alpha_{P'}/2}}{\sin(\pi\alpha_{P'}/2)}\omega^{\alpha_{P'}}, \quad (10)$$

with the parameter values[10]

$$\gamma_P=1.11\mu^{-2},\quad \alpha_P=1;$$
$$\gamma_{P'}=1.82\mu^{-1},\quad \alpha_{P'}=0.39.$$

The high-energy integration is then

$$\int_\Lambda^\infty \frac{\omega d\omega\, \mathrm{Im}[T^{(+)}(\omega)-T^{(+)}(\mu)]e^{\pi i\beta}}{(\omega^2+a^2)(\omega^2-\mu^2)^\beta}=-\gamma_P\frac{\sin[(2\beta-\alpha_P)\pi/2]}{\sin(\pi\alpha_P/2)\Lambda^{2\beta-\alpha_P}(2\beta-\alpha_P)}-\frac{\gamma_{P'}\sin[(2\beta-\alpha_{P'})\pi/2]}{\sin(\pi\alpha_{P'}/2)\Lambda^{2\beta-\alpha_{P'}}(2\beta-\alpha_{P'})}. \quad (11)$$

Because of the factors $\Lambda^{2\beta-\alpha_P}$ and $\Lambda^{2\beta-\alpha_{P'}}$ these contributions rapidly become negligible as β increases from $\frac{1}{2}$.

The results from Eq. (7) are shown in Table I, and it can be seen that the right- and left-hand sides differ by at most 1%. The convergence is very rapid for $\beta>1$, but for $\beta<1$ the high-energy contribution is quite important. For β close to its limiting value of $\frac{3}{2}$, the integral is almost saturated by the low-energy region $\omega<5$ BeV.

The evaluation of the terms in Eq. (8) yields

$$\frac{T^{(+)}(\mu)}{(\mu^2+a^2)^{1/2}}=0 \text{ mb (Ref. 9)}, \quad (12)$$

$$\frac{2\omega_0 f^2}{(\omega_0^2+a^2)(\mu^2-\omega_0^2)^{1/2}}=28.2 \text{ mb}, \quad (13)$$

TABLE I. The comparison of the left- and right-hand sides of Eq. (7) for various values of the cutoff Λ of the integral.

β	Left-hand side ($\text{mb}\times\mu^{1-2\beta}$)	Right-hand side ($\text{mb}\times\mu^{1-2\beta}$) $\Lambda=5$ BeV	15 BeV	30 BeV	∞
0.501	28.19	−1.27	2.00	3.37	27.65
0.6	28.12	14.22	18.06	19.76	27.84
0.7	28.22	21.81	24.40	25.40	28.00
0.8	28.24	25.38	26.80	27.29	28.10
0.9	28.25	27.03	27.73	27.94	28.18
1.0	28.27	27.83	28.14	28.23	28.30
1.1	28.29	28.29	28.41	28.45	28.47
1.2	28.30	28.61	28.65	28.67	28.67
1.3	28.32	28.87	28.88	28.89	28.89
1.4	28.33	28.77	28.78	28.78	28.78
1.499	28.35	28.67	28.67	28.67	28.67

$$\frac{2}{\pi}\int_\mu^{30\text{ BeV}} \frac{\omega d\omega\, \mathrm{Re}[T(\omega)-T(\mu)]}{(\omega^2+a^2)(\omega^2-\mu^2)^{1/2}}=-3.1 \text{ mb}, \quad (14)$$

$$\frac{2}{\pi}\int_{30\text{ BeV}}^\infty \frac{\omega d\omega\, \mathrm{Re}[T(\omega)-T(\mu)]}{(\omega^2+a^2)(\omega^2-\mu^2)^{1/2}}=-2.3 \text{ mb}, \quad (15)$$

or

$$\sigma(\infty)=22.8 \text{ mb}, \quad (16)$$

compared with the experimental value of 22.1 ± 0.9 mb.[11] Note that in this case the Pomeranchon term in (10) does not contribute because it is pure imaginary.

When we evaluate Eq. (9) we find

$$\sigma(\mu)=2.2\pm0.76 \text{ mb} \quad (17)$$

compared with the "experimental" values of 3.8 ± 0.2 mb[7] and 3.4 ± 0.2 mb.[9] In (17), only the uncertainty in $T^{(+)}(\mu)$ in the first term of (9) is accounted for.

It can easily be shown that all the sum rules become identities if $\mathrm{Re}T^{(+)}(\omega)$ is calculated from the ordinary dispersion relations so we should not be surprised by the good results. Therefore our results are most useful when accurate experimental values of $\mathrm{Re}T^{(+)}(\omega)$ become available. Then they will provide a good test of low-energy values of $\mathrm{Re}\,T^{(+)}(\omega)$ because of the rapid convergence of our integrals in the case $1<\beta<\frac{3}{2}$.

The formalism in this paper is very convenient for deriving finite-energy sum rules[12] from which Regge parameters may be estimated. This application will be discussed in a separate paper.

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[9] G. Hohler, G. Ebel, and J. Gieseke, Z. Physik **180**, 430 (1964). These authors use $a_1=(0.192\pm0.004)\mu^{-1}$, $a_3=(-0.096\pm0.002)\mu^{-1}$ so that $T^{(+)}(\mu)=(0.00\pm0.04)\mu^{-1}$; $\sigma(\mu)=3.8\pm0.2$ mb.

[10] These values are taken from [Y.-C. Liu and S. Okubo, Phys. Rev. **168**, 1712 (1968)] a report which was received while this work was in progress. The treatment of the $T^{(+)}(\omega)$ amplitude in their paper differs somewhat from ours.

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