Quantum Theory of Field Attenuation*

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(Received 4 August 1967; revised manuscript received 30 October 1967)

A quantum-mechanical theory of photon detection is presented which takes fully into account the attenuation of the field due to the detection process. The time evolution of the joint quantum state of the detector and the field is found, to all orders in perturbation theory. Formulas are derived for the probabilities of absorbing specified numbers of photons within a given time interval, and for the correlations in the positions and times at which absorptions take place. The results are free from the inconsistencies which arise in the conventional theory of photon detection, and remain valid even when the field becomes appreciably attenuated during the experiment. It is found that an initially coherent field state remains coherent during its interaction with a given detector, and that its amplitude becomes attenuated by an amount which is completely independent of the number of counts which the detector records. This independence is shown to be a simple consequence of the Poisson quantum-number distribution of coherent fields. The counting statistics for arbitrary fields are expressed in a way which shows explicitly the relationship between the attenuation of the field and the absorption of quanta by the detector. The analysis is performed by first treating the case in which the field is confined within a homogeneous detecting medium throughout the experiment, and then generalizing to the case in which the field spends a limited amount of time in a spatially localized detecting region. The detector is assumed to consist of harmonic oscillators, which are shown to represent a suitable formal model for the absorption of radiation by large numbers of conventional detecting atoms.

I. INTRODUCTION

THE conventional theory of photon detection[1–5] is essentially a generalization to many atoms of a perturbation theory calculation of the excitation probability for a single atom. As such, it is subject to the following limitations:

(1) The probability that any particular detecting atom becomes excited must remain small throughout the detection process.

(2) If the initial state of the field contains one photon, the probability that this photon is absorbed must be small. Equivalently, for arbitrary initial field states, the mean number of photons absorbed during the time interval under consideration must be a small fraction of the mean number of photons in the initial state of the field.

Both of these conditions are well satisfied in the majority of photon counting experiments, which are therefore accurately described by the conventional theory of photon detection.

It is important to realize, however, that the first condition, which is a natural one in any photon detection theory, does not imply the second. This can be seen simply by letting the number of detecting atoms approach infinity. Experiments in which the field suffers an appreciable attenuation during the detection process are perfectly possible from both a theoretical and a practical standpoint. They cannot, however, be described within the framework of conventional theory, which does not provide a satisfactory account of the effect of the detector on the field.

The limitations of even the strictly quantum-mechanical formulations[2,4,5] of the conventional photodetection theory may be conveniently illustrated by supposing that the beam passes successively through a large number N of identical (transparent) detectors. If exactly one photon is present initially, one finds from the conventional theory that each detector has an equal probability p of absorbing the photon, that the probability that the photon is absorbed at all is Np, and that the probability that the photon is not absorbed is $1-Np$. These results are clearly meaningless if $Np>1$, and in fact they are valid only for $Np\ll 1$. An equally striking illustration of the breakdown of the conventional theory is the case in which the beam initially consists of a coherent[6] wave packet, with mean photon number $\bar{n}$. In that case one finds that the counting statistics for the individual detectors are independent of each other, and that the photoabsorption probability for each detector is given by a Poisson distribution with mean number $p\bar{n}$. The probability $p_m(t)$ that a total of m photons are absorbed is given by a Poisson distribution with mean number $Np\bar{n}$. For N sufficiently large, this mean number becomes much greater than the initial mean number of photons in the field, a result which violates the law of conservation of energy.

The reason for this behavior, as we have indicated, is that the conventional theory does not adequately take into account the attenuation of the field due to the detection process. The transition probabilities for the

* The research reported in this paper was supported, in part, by the U. S. Air Force Cambridge Research Laboratories, Office of Aerospace Research, under Contract F19628-68-C-0062.
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[2] R. J. Glauber, Phys. Rev. Letters **10**, 84 (1963).
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[4] R. J. Glauber, *Quantum Optics and Electronics, Les Houches, 1964* (Gordon and Breach Science Publishers, Inc., New York, 1965).
[5] P. L. Kelley and W. H. Kleiner, Phys. Rev. **136**, A316 (1964).
[6] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

system of detecting atoms are calculated by using, in effect, the interaction Hamiltonian

$$H_1(t) = -\sum_j \mathbf{A}^{(+)}(\mathbf{r}_j,t)\cdot\mathbf{J}_j^{(-)}(t), \tag{1.1}$$

where $\mathbf{A}^{(+)}(\mathbf{r},t)$ is the positive-frequency (annihilating) part of the field, and $\mathbf{J}_j^{(-)}(t)$ is the negative-frequency (raising) part of the current operator for the jth atom. This non-Hermitian coupling is adequate as long as conditions (1) and (2) are both satisfied. When the second condition is not satisfied, on the other hand, it becomes necessary to include the effect of virtual transitions in which the atoms return to their ground states. We include such effects by adopting as our interaction Hamiltonian the Hermitian expression

$$H_1(t) = -\sum_j [\mathbf{A}^{(+)}(\mathbf{r}_j,t)\cdot\mathbf{J}_j^{(-)}(t) + \mathbf{A}^{(-)}(\mathbf{r}_j,t)\cdot\mathbf{J}_j^{(+)}(t)]. \tag{1.2}$$

This coupling may be obtained from the exact expression by dropping antiresonant terms, and it therefore represents a suitable approximation as long as the field cannot change greatly during a small number of periods of oscillation.

By adopting a simple model for the detecting atoms, we are able to obtain solutions for the state of the system of field and detector which are free from the difficulties encountered by the conventional theory. The photoabsorption probabilities $p_m(t)$ are all positive, and the mean number of absorbed quanta is never greater than the mean number of photons in the initial state of the field. The attenuation of the field is exhibited, and is shown to be simply related to the absorption of photons by the detector. In the limit in which condition (2) is satisfied, our results reduce to those of the conventional theory.

Our methods also enable us to discuss in some detail the time-dependent correlations in the distribution of photocounts. This question is ordinarily treated by thinking of the detector as registering counts at definite times, and postulating that the photoabsorption probability $p_m(t)$ may be identified as the probability that a total of m counts will be registered between the initial time and the time t. This procedure is somewhat *ad hoc*, inasmuch as the function $p_m(t)$ is calculated as the probability that a *single* measurement of the total number M of excitations in the detector will yield the result m at time t. We are able to justify it, however, by developing a theory of the measurement of the excitation number M of the detector, and finding the time evolution of the system after a measurement is made. We then postulate that the detector is *monitored*, i.e., that *repeated* measurements of M are made throughout the time interval in question.[7] In this way we are able to speak of definite times at which counts are recorded. We are able to show that the monitoring process during some time interval does not affect the probability that m excitations are found in the detector at the end of the interval, and thus we are able to justify the procedure of identifying the function $p_m(t)$, which has been calculated without reference to monitoring, as the probability that a total of m counts are recorded at different times throughout the interval.

When this identification is made, the functions $p_m(t)$ may be used to obtain part of the information implicit in the full statistical distribution of photocounts. By differentiating them, for example, we may easily evaluate the conditional probability of a count being recorded between the times t and $t+\Delta t$, given that any fixed number of counts have been recorded previously. More precise information about the counting statistics cannot, however, be obtained from the functions $p_m(t)$ alone. It is not possible, for example, to determine from them in what way the probability of recording a count between the times t and $t+\Delta t$ depends upon the *times* at which counts have previously been recorded.

Information of this kind is usually obtained by postulating that the field correlation functions, which appear in integrated form in the expressions for the absorption probabilities, may be identified as the rates for recording counts at different positions and times. We are able, by contrast, to solve for the full statistical distribution of photocounts without making assumptions of this kind.[7] In the case in which the initial state of the field is coherent,[6] we find that the probability of recording counts at any number of space-time points is independent of whether or not counts are recorded at other points, and may be calculated by thinking of the excitations in the detecting atoms as caused by a prescribed classical field, the intensity of which decays at a rate simply related to the mean local counting rate. For more general initial fields we find that the probability of recording counts at any number of space-time points may be expressed in terms of a field correlation function which similarly reflects the attenuation of the field due to the detecting process.

The detecting atoms on which our analysis is based are assumed to have the dynamical behavior of harmonic oscillators, with closely spaced frequencies covering a range large compared to the bandwidth of the radiation field. Although harmonic oscillators are physically very different from ionizable atoms, the formal properties of the solutions for the two kinds of absorbing systems are very similar, if the number density of atoms is very large. This correspondence may be seen by noting that a large number of nearby oscillators with a wide range of frequencies constitutes a single system which may undergo first-order transitions to many excited levels, and which therefore has

[7] P. L. Kelley and W. H. Kleiner [Ref. (4)] have developed a similar theory of multiple-time photon detection. These authors assume that the detector returns to its ground state after each measurement, and that the field evolves independently of the detector between successive measurements; they are therefore unable to treat the attenuation of the field due to the detection process. The simplifying assumptions they make, however, enable them to treat the case in which the field is modified by the presence of sources throughout the experiment.

all of the formal properties of conventional detecting atoms. Alternatively, one may understand the correspondence in question by thinking of a conventional detecting atom as consisting of a large number of two-level systems, all with the same ground-state energy, and with upper levels corresponding to the excited levels of the atom. Harmonic-oscillator modes may then be constructed from many nearby atoms by making use of the formal equivalence between a large number of identical two-level systems and a single harmonic oscillator.[8–10]

The advantage of using harmonic oscillators as detecting atoms is that the time evolution of the state of the system of field and detector can be solved for to all orders in perturbation theory, when the resonant approximation (1.2) is made for the interaction Hamiltonian. By introducing the coherent states for the absorbing oscillators as well as for the field oscillators, we are able to reduce the quantum-mechanical problem to a problem of coupled classical harmonic oscillators.[11–13]

The major part of our discussion is devoted to the analysis of a simple model in which the detecting atoms are uniformly distributed throughout a cavity, and the field consists of a single mode of oscillation. This situation leads to the complete absorption of the field in the limit of large times, and thus provides a convenient basis for the discussion of field attenuation. Many of the questions which arise are in fact quite similar to those which arise in the more realistic case, in which the detecting atoms are confined to a limited region of space through which the beam passes. The latter case is treated by a straightforward generalization of the results for our simple model. The only assumption we are required to make is that the number density of detecting atoms is relatively constant over many wavelengths of the field, an assumption which is not unduly restrictive, inasmuch as it does not prevent us from considering very small detecting regions.

In the next six sections of this paper, we treat the case in which the harmonic oscillators which constitute the detector are uniformly distributed throughout a cavity, and the field is represented by a single excited mode. The harmonic-oscillator model of absorption is presented in Sec. II, and the photoabsorption probabilities $p_m(t)$ are evaluated for arbitrary initial states of the field mode in Sec. III. In Sec. IV we discuss the attenuation of the field, as represented by its reduced density operator. Section V is devoted to a discussion of the way the statistical description of the field is altered, in the general case, by the information obtained by measuring the excitation number M of the detector. It is shown, however, that a coherent field, or indeed any field with a Poisson quantum-number distribution, is unaltered by a measurement of M, and thus is attenuated by an amount which is independent of the number of quanta actually absorbed from it. In Sec. VI it is shown how the state of the system of field and detector evolves in time after a measurement of M has been made. The complete statistical description of the detection process for the one-mode case is presented in Sec. VII. We undertake then in Sec. VIII to generalize our results to include cases in which many modes of the field are excited, and in which the detecting oscillators occupy a limited region of space. Section IX is devoted to a proof of the equivalence of absorption by harmonic oscillators and by two-level atoms, and to a discussion of the conditions under which either type of absorbing system may be used to represent absorption by ionizable atoms.

II. HARMONIC-OSCILLATOR MODEL OF ABSORPTION

Let us consider a detector consisting of a large number of systems which have the dynamical behavior of (one-dimensional) harmonic oscillators. The jth oscillator is assumed to consist of a particle of mass m_j and charge $-e$, which oscillates at the frequency ω_j about the mean position $\mathbf{r}_j$, in the direction specified by the unit vector $\hat{u}_j$. The system is assumed to be electrically neutral when the oscillator is unexcited. If we denote by $q_j(t)$ and $p_j(t)$ the canonical coordinate and momentum, respectively, for the jth oscillator, then the free Hamiltonian for the detector may be written in the form

$$H_{0B}(t)=\hbar\sum_j \omega_j b_j^\dagger(t)b_j(t)\,, \tag{2.1}$$

where the lowering operators $b_j(t)$ are defined as

$$b_j(t)\equiv(m_j\omega_j/2\hbar)^{1/2}q_j(t)+i(2m_j\hbar\omega_j)^{-1/2}p_j(t)\,, \tag{2.2}$$

and satisfy the canonical commutation relations

$$\begin{aligned}[b_j(t),b_{j'}^\dagger(t)]&=\delta_{jj'}\,,\\ [b_j(t),b_{j'}(t)]&=0\,.\end{aligned} \tag{2.3}$$

The case in which the field and detecting oscillators are confined within a cavity is a particularly instructive one for the purposes of our analysis, since it corresponds to continuous absorption of the field. If we use periodic boundary conditions, we may express the vector potential for the field as

$$\mathbf{A}(\mathbf{r},t)=c\left(\frac{\hbar}{2V}\right)^{1/2}\sum_k(\omega_k)^{-1/2}\hat{e}_k \times[a_k(t)e^{i\mathbf{k}\cdot\mathbf{r}}+a_k^\dagger(t)e^{-i\mathbf{k}\cdot\mathbf{r}}]\,, \tag{2.4}$$

[8] F. Schwabl and W. Thirring, Ergeb. Exakt. Naturw. **36**, 219 (1964).
[9] A. E. Glassgold and D. Holliday, Phys. Rev. **139**, A1717 (1965).
[10] This formal similarity is the basis of a discussion of photodetection theory presented by R. J. Glauber at the Summer School on Quantum Optics of the Italian Physical Society, Varenna, Italy, 1967 (to be published). Professor Glauber's treatment is also devoted to correcting the nonunitary character of conventional theory, and his analysis is based on a coupled harmonic-oscillator approach similar to the one used in this paper.
[11] R. J. Glauber, Phys. Letters **21**, 650 (1966).
[12] C. L. Mehta and E. C. G. Sudarshan, Phys. Letters **22**, 574 (1966).
[13] B. R. Mollow, Phys. Rev. **162**, 1256 (1967).

where V is the volume of the cavity. The index k specifies the discrete mode with wave vector $\mathbf{k}$, frequency $\omega_k=|\mathbf{k}|c$, and unit polarization vector $\hat{e}_k$; there are two polarization vectors associated with a given wave vector $\mathbf{k}$. The operators $a_k(t)$ and their adjoints satisfy the canonical commutation relations

$$[a_k(t),a_{k'}{}^\dagger(t)]=\delta_{kk'}\,,\qquad [a_k(t),a_{k'}(t)]=0\,, \tag{2.5}$$

and the free Hamiltonian for the field is

$$H_{0F}(t)=\hbar\sum_k \omega_k a_k{}^\dagger(t)a_k(t)\,. \tag{2.6}$$

Let us assume that the root-mean-square displacements $(3\hbar/2m_j\omega_j)^{1/2}$ of the first excited levels of the detecting oscillators are small compared to the wavelengths ω_j/c of radiation oscillating at the frequencies ω_j:

$$(\hbar\omega_j/m_jc^2)^{1/2}\ll 1\,. \tag{2.7}$$

Then if the oscillators are never excited to energies much greater than those of their first excited states, we shall be justified in making the dipole approximation in evaluating the coupling between the oscillators and the field. The interaction Hamiltonian may then be approximated by the expression

$$H_1(t)=-\frac{e}{c}\sum_i \frac{p_j(t)}{m_j}\hat{u}_j\cdot\mathbf{A}(\mathbf{r}_j,t)\,. \tag{2.8}$$

By using Eq. (2.4) for $\mathbf{A}(\mathbf{r},t)$ and making use of Eq. (2.2) to express $p_j(t)$ in terms of $b_j(t)$ and $b_j{}^\dagger(t)$, we see that the interacting system of field and detector may be described in this approximation as a set of coupled harmonic oscillators.

It is convenient to begin our analysis by assuming that only one mode of the field is excited throughout the absorption process. We shall justify this assumption and discuss the more general case in Sec. VIII. Suppressing the index k for the variables which describe the excited mode, we find by making the resonant approximation (1.2) that the system of field oscillator and detecting oscillators is governed by the Hamiltonian

$$H(t)=H_0(t)+H_1(t)\,, \tag{2.9a}$$

where

$$H_0(t)=\hbar\omega a^\dagger(t)a(t)+\hbar\sum_j \omega_j b_j{}^\dagger(t)b_j(t)\,, \tag{2.9b}$$

$$H_1(t)=-i\hbar[a^\dagger(t)\textstyle\sum_j g_j b_j(t)-a(t)\sum_j g_j{}^* b_j{}^\dagger(t)]\,, \tag{2.9c}$$

and the coupling parameter g_j is defined as

$$g_j\equiv\tfrac{1}{2}e\left[\frac{\omega_j}{m_j\omega V}\right]^{1/2}(\hat{e}\cdot\hat{u}_j)e^{-i\mathbf{k}\cdot\mathbf{r}_j}. \tag{2.10}$$

The equations of motion which follow from the Hamiltonian (2.9) are

$$\frac{d}{dt}a(t)=-i\omega a(t)-\textstyle\sum_j g_j b_j(t)\,, \tag{2.11a}$$

$$\frac{d}{dt}b_j(t)=-i\omega_j b_j(t)+g_j{}^* a(t)\,. \tag{2.11b}$$

Before discussing the solutions to these equations, it it is useful to note the existence of a conserved quantity. It is a consequence of the resonant form we have assumed in the interaction Hamiltonian (2.9c) that the total number of quanta in the system of field and absorbing oscillators is conserved, i.e., we have

$$\frac{d}{dt}[a^\dagger(t)a(t)+M(t)]=0\,, \tag{2.12}$$

where we have defined $M(t)$ as the operator for the total number of quanta in the absorbing oscillators,

$$M(t)\equiv\textstyle\sum_j b_j{}^\dagger(t)b_j(t)\,. \tag{2.13}$$

In the Schrödinger picture, if we introduce the operators

$$a\equiv a(0)\,, \tag{2.14a}$$

$$b_j\equiv b_j(0)\,, \tag{2.14b}$$

and

$$M\equiv M(0)=\textstyle\sum_j b_j{}^\dagger b_j\,, \tag{2.15}$$

then we may easily deduce from the operator conservation law (2.12) that if the state of the system initially contains a specified total number n of quanta

$$(a^\dagger a+M)|\ \rangle=n|\ \rangle\,, \tag{2.16}$$

then the time-dependent state vector

$$|t\rangle=e^{-iHt/\hbar}|\ \rangle \tag{2.17}$$

must preserve the eigenvalue n for the quantum-number sum at all times,

$$(a^\dagger a+M)|t\rangle=n|t\rangle\,. \tag{2.18}$$

This relation is a particularly favorable one for our photon detection theory, since it states, in effect, that the quanta which are absorbed from the field must be found in the detector.

The solutions to the linear equations of motion (2.11) take the form

$$a(t)=\mu(t)a+\textstyle\sum_j \zeta_j(t)b_j\,, \tag{2.19a}$$

$$b_j(t)=\lambda_j(t)a+\textstyle\sum_k \nu_{jk}(t)b_k\,. \tag{2.19b}$$

In the Appendix it is shown that the c-number functions $\mu(t)$, $\zeta_j(t)$, $\lambda_j(t)$, and $\nu_{jk}(t)$ may be evaluated by means of

a straightforward Laplace transform technique.[14] We assume that there are $N(\omega_j)d\omega_j$ absorbing oscillators with frequencies between ω_j and $\omega_j+d\omega_j$, and that the oscillator-direction vectors $\hat{u}_j$ are randomly oriented. The solution for $\mu(t)$ is characterized by a decay rate

$$\kappa=\pi N(\omega)|g_j|^2{}_{(\omega_j=\omega)} \tag{2.20a}$$

$$=\pi\frac{e^2}{12}\frac{N(\omega)}{V}\left(\frac{1}{m_j}\right)\Bigg|_{\omega_j=\omega}, \tag{2.20b}$$

and a frequency shift

$$\delta\omega=P\int d\omega'\frac{N(\omega')|g_j|^2{}_{(\omega_j=\omega')}}{\omega-\omega'}, \tag{2.21}$$

where P means principal value. We must require that both of these quantities be small compared to the frequency of the excited mode

$$\kappa,\delta\omega\ll\omega \tag{2.22}$$

and to the frequency bandwidth of the function $N(\omega)$,

$$N(\omega\pm\kappa,\delta\omega)\approx N(\omega), \tag{2.23}$$

but that there be many oscillators throughout the cavity with frequencies lying within a range equal to κ or $\delta\omega$:

$$\kappa,\delta\omega\gg 1/N(\omega). \tag{2.24}$$

It is necessary, finally, to evaluate our solutions at times large compared to the period of oscillation $1/\omega$, and large compared to the reciprocal of the frequency bandwidth $\Delta\omega$ of the function $N(\omega)$,

$$t\gg\frac{1}{\omega}, \tag{2.25a}$$

$$t\gg\frac{1}{\Delta\omega}, \tag{2.25b}$$

but small compared to the reciprocal of the mean interval between the discrete oscillator frequencies,

$$t\ll N(\omega). \tag{2.26}$$

In the work which follows, we shall assume that all times and time intervals satisfy these conditions, even when we speak of infinitesimal time intervals or take the limit $t\to\infty$.

The conditions (2.22)–(2.26) are the only ones we need impose in order to justify our approximate solution to the coupled equations of motion (2.11). It is shown in Sec. VIII, however, that our assumption that only one mode is excited throughout the absorption process requires us to impose somewhat stronger conditions than those expressed by the inequalities (2.24) and (2.26): The function $N(\omega)$, which is the number of absorbing oscillators per unit frequency range throughout the cavity, must be replaced by the number of oscillators per unit frequency range located within a spatial volume equal to the cube of the wavelength c/ω. It is also necessary to assume that the absorbing oscillators are uniformly distributed throughout the cavity, so that the number of oscillators per unit frequency range per unit volume is

$$N(\omega,\mathbf{r})=N(\omega)/V. \tag{2.27}$$

The condition (2.26) must then be replaced by the stronger condition

$$t\ll N(\omega,\mathbf{r})\frac{c^3}{\omega^3}, \tag{2.28}$$

and the conditions (2.24) by

$$N(\omega,\mathbf{r})\kappa c^3/\omega^3\gg 1, \tag{2.29a}$$

$$N(\omega,\mathbf{r})\delta\omega c^3/\omega^3\gg 1. \tag{2.29b}$$

These conditions are automatically satisfied in the limit

$$N(\omega)\to\infty, \tag{2.30a}$$

$$|g_j|^2\to 0, \tag{2.30b}$$

$$N(\omega)|g_j|^2\to\text{const.} \tag{2.30c}$$

When the conditions (2.22), (2.23), (2.25), and (2.27)–(2.29) are all satisfied, the function $\mu(t)$ may be approximated for times $t>0$ by the exponential function

$$\mu(t)=e^{-zt}, \tag{2.31}$$

where

$$z=\kappa+i(\omega+\delta\omega). \tag{2.32}$$

Let us substitute Eq. (2.31) for $\mu(t)$ into Eqs. (2.19), and then substitute the resulting expressions for $a(t)$ and $b_j(t)$ into the relation

$$a^\dagger(t)a(t)+\textstyle\sum_j b_j^\dagger(t)b_j(t)=a^\dagger a+\sum_j b_j^\dagger b_j, \tag{2.33}$$

which follows from the conservation law (2.12). In this way we find four relations, of which the only one we shall need is

$$\textstyle\sum_j|\lambda_j(t)|^2=1-e^{-2\kappa t}. \tag{2.34}$$

We may obtain additional useful conditions on the c-number functions in Eqs. (2.19) by first writing, in matrix notation,

$$a(t)=e^{-zt}a+\tilde{\zeta}(t)b, \tag{2.35a}$$

$$b(t)=\lambda(t)a+\nu(t)b. \tag{2.35b}$$

Here we are thinking of $b(t)$, $\zeta(t)$, and $\lambda(t)$ as column vectors with components $b_j(t)$, $\zeta_j(t)$, and $\lambda_j(t)$, respectively, and of $\nu(t)$ as a matrix with elements $\nu_{jk}(t)$. The quantity $\tilde{\zeta}(t)$ is the transpose of $\zeta(t)$, and hence is a row

[14] The author is indebted to Professor Glauber (private communication) for these solutions to the coupled oscillator Eqs. (2.11). Similar results are presented in a slightly different context in Ref. (8), and by J. P. Gordon, L. R. Walker, and W. H. Louisell, Phys. Rev. **130**, 806 (1963). The basic method of solution was first used in atomic radiation theory, by V. F. Weisskopf and E. P. Wigner, Z. Physik **63**, 54 (1930); **65**, 18 (1930).

vector with components $\zeta_j(t)$. The invariance of the system of field and detector under translations in time implies that the Eqs. (2.35) must remain valid if we replace a and b by $a(t')$ and $b(t')$, respectively, and $a(t)$ and $b(t)$ by $a(t'+t)$ and $b(t'+t)$. By making use of this invariance property we find four relations, of which the only two we shall need are

$$\tilde{\zeta}(t)\lambda(t')\equiv\sum_j \zeta_j(t)\lambda_j(t')=0 \tag{2.36}$$

and

$$\tilde{\zeta}(t)\nu(t')=\tilde{\zeta}(t+t')-e^{-zt}\tilde{\zeta}(t'), \tag{2.37}$$

which are valid for all positive times t and t'.

The time evolution of the Schrödinger state vector for systems of harmonic oscillators governed by Hamiltonians of the form (2.9)—quadratic expressions involving products of one creation operator and one annihilation operator—may be expressed in a particularly simple way in terms of the *coherent states*[6] of the oscillator modes. A coherent state $|\alpha\rangle_F$ for the field mode is an eigenstate of the annihilation operator

$$a|\alpha\rangle_F=\alpha|\alpha\rangle_F, \tag{2.38}$$

and is given by

$$|\alpha\rangle_F=\exp(a^\dagger\alpha-\alpha^* a)|0\rangle_F, \tag{2.39}$$

where $|0\rangle_F$ is the vacuum state. Similarly, a coherent state $|\beta_j\rangle_j$ for the jth harmonic oscillator satisfies the relation

$$b_j|\beta_j\rangle_j=\beta_j|\beta_j\rangle_j, \tag{2.40}$$

and is given by

$$|\beta_j\rangle_j=\exp(b_j^\dagger\beta_j-\beta_j^* b_j)|0\rangle_j, \tag{2.41}$$

where $|0\rangle_j$ is the ground state for the jth oscillator. We define a simultaneously coherent state for the system of absorbing oscillators as

$$|\beta\rangle_B\equiv\prod_j|\beta_j\rangle_j=\exp(\tilde{b}^\dagger\beta-\tilde{\beta}^*\tilde{b})|0\rangle_B, \tag{2.42}$$

where $|0\rangle_B$ is the ground state for the absorbing system,

$$|0\rangle_B=\prod_j|0\rangle_j. \tag{2.43}$$

The state which evolves during time t from the initial product of a coherent state $|\alpha_0\rangle_F$ for the field mode and a coherent state $|\beta_{0j}\rangle_j$ for each of the absorbing oscillators retains this simultaneously coherent character at all times,[11–13]

$$e^{-iHt/\hbar}|\alpha_0\rangle_F|\beta_0\rangle_B=|\alpha(t)\rangle_F|\beta(t)\rangle_B, \tag{2.44}$$

and the time-dependent complex amplitudes $\alpha(t)$ and $\beta_j(t)$ obey the same linear equations of motion as do the Heisenberg operators $a(t)$ and $b_j(t)$. They are therefore given in terms of α_0 and β_{0j} by the relations

$$\alpha(t)=e^{-zt}\alpha_0+\tilde{\zeta}(t)\beta_0, \tag{2.45a}$$

$$\beta(t)=\lambda(t)\alpha_0+\nu(t)\beta_0. \tag{2.45b}$$

If the initial state of the system of absorbing oscillators in Eq. (2.44) is the ground state $|0\rangle_B$, then the state of the system at time t is

$$|t\rangle=e^{-iHt/\hbar}|\alpha_0\rangle_F|0\rangle_B=|e^{-zt}\alpha_0\rangle_F|\lambda(t)\alpha_0\rangle_B. \tag{2.46}$$

The complex amplitude of the field thus decays exponentially with the decay rate κ, and suffers a frequency shift $\delta\omega$.

It is interesting to observe that the state of the field in Eq. (2.46) continues to evolve during a time interval beginning with some time $t'>0$ in the same way that it would if the state of the absorbing oscillators at time t' were the ground state $|0\rangle_B$ rather than the state $|\lambda(t')\alpha_0\rangle_B$. Indeed, it follows from Eqs. (2.44) and (2.45) that if the state of the system at the beginning of some time interval of length t is $|\alpha\rangle_F|\lambda(t')\alpha_0\rangle_B$, where α and α_0 are any two complex numbers and t' is any positive time, then the state of the system at the end of the interval is

$$\begin{aligned}e^{-iHt/\hbar}&|\alpha\rangle_F|\lambda(t')\alpha_0\rangle_B\\&=|e^{-zt}\alpha+\tilde{\zeta}(t)\lambda(t')\alpha_0\rangle_F|\lambda(t)\alpha+\nu(t)\lambda(t')\alpha_0\rangle_B\\&=|e^{-zt}\alpha\rangle_F|\lambda(t)\alpha+\nu(t)\lambda(t')\alpha_0\rangle_B,\end{aligned} \tag{2.47}$$

where the last step follows from Eq. (2.36). It is clear from Eq. (2.47) that the detector state $|\lambda(t')\alpha_0\rangle_B$, which was excited by the coherent field state $|\alpha_0\rangle_F$, behaves exactly like the ground state $|0\rangle_B$ in its influence on *any* coherent state of the field. This property of the detector states which are excited by coherent field states will play an important role in the analysis of Secs. VI and VII, which provide a theory of multiple-time counting statistics.

III. PHOTON DETECTION PROBABILITIES

Let us assume that the initial state of the system of field mode and detector is the product of an arbitrary (mixed) state for the field and the ground state for the detector. The initial density operator for the system is then

$$\rho=\rho^{(F)}(0)|0\rangle_B\,{}_B\langle 0|, \tag{3.1}$$

where $\rho^{(F)}(0)$ is the initial density operator for the field. The system at time t may then be described by means of the Schrödinger density operator

$$\begin{aligned}\rho(t)&=e^{-iHt/\hbar}\rho e^{iHt/\hbar}\\&=e^{-iHt/\hbar}\rho^{(F)}(0)|0\rangle_B\,{}_B\langle 0|e^{iHt/\hbar}.\end{aligned} \tag{3.2}$$

We may evaluate the probability $p_m(t)$ that a total of m quanta will be found in the detector at time t by first introducing a complete set of states $|m,\gamma\rangle_B$ for the detector, where m is the eigenvalue of the number operator M defined by Eq. (2.15),

$$M|m,\gamma\rangle_B=m|m,\gamma\rangle_B, \tag{3.3}$$

and γ represents the additional variables necessary to specify the state. If we then introduce the projection

operator

$$\hat{P}_m \equiv \sum_\gamma |m,\gamma\rangle_B \, {}_B\langle m,\gamma| \tag{3.4}$$

onto the subspace of detector states containing m quanta, we may express the probability of finding m quanta in the detector at time t as

$$p_m(t) = \mathrm{tr}[\rho(t)\hat{P}_m] \tag{3.5}$$

$$= \sum_{n,\gamma} {}_F\langle n| \, {}_B\langle m,\gamma|\rho(t)|m,\gamma\rangle_B |n\rangle_F , \tag{3.6}$$

where $|n\rangle_F$ is the state of the field mode containing exactly n photons. If the detector is initially in its ground state, we find from Eqs. (3.2) and (3.5) that the probability $p_m(t)$ may be expressed as a linear functional of the initial density operator for the field as

$$p_m(t) = \mathrm{tr}_F[\rho^{(F)}(0)\mathcal{P}_m(t)], \tag{3.7}$$

where $\mathcal{P}_m(t)$ is an operator in the state-space of the field mode, which may be written formally as

$$\mathcal{P}_m(t) = \mathrm{tr}_B\{|0\rangle_B \, {}_B\langle 0| e^{iHt/\hbar}\hat{P}_m e^{-iHt/\hbar}\}. \tag{3.8}$$

In these expressions tr_F and tr_B mean trace with respect to the initial states of the field and the detector, respectively.

As an example of some interest, let us consider first the case in which the field is initially in the pure coherent state $|\alpha_0\rangle_F$, so that the state of the system at time t is given by Eq. (2.46). The detector is then described by the simultaneously coherent state

$$|\lambda(t)\alpha_0\rangle_B \equiv \prod_j |\lambda_j(t)\alpha_0\rangle_j . \tag{3.9}$$

It is not difficult to show that the quantum-number distribution for such a state is the same as that for a single harmonic oscillator described by a coherent state with the same mean number of quanta. The photoabsorption probability $p_m(t)$ is therefore given by the Poisson distribution

$$p_m(t) = \frac{[\bar{m}(t)]^m}{m!} e^{-\bar{m}(t)}, \tag{3.10}$$

in which the mean number of absorbed quanta $\bar{m}(t)$ is, by virtue of Eq. (2.34),

$$\bar{m}(t) = |\alpha_0|^2 \sum_j |\lambda_j(t)|^2 = |\alpha_0|^2(1-e^{-2\kappa t}). \tag{3.11}$$

The probability $p_m(t)$ given by Eqs. (3.10) and (3.11) is the same as that which one would write down in a semiclassical theory, if one postulated that the excitations in the detector were produced by the attenuated classical field

$$\alpha(t) = e^{-\kappa t}\alpha_0 . \tag{3.12}$$

For $t \ll 1/\kappa$, the mean absorbed quantum number $\bar{m}(t)$ is just $2\kappa|\alpha_0|^2 t$, and the expression (3.10) reduces to the one given by conventional photodetection theory. For large times ($\kappa t \gg 1$), however, we have

$$\bar{m}(t) \to |\alpha_0|^2, \tag{3.13}$$

and the probability of finding m absorbed quanta is thus the same as the probability of finding m quanta in the initial state of the field. We shall presently show that result is valid for arbitrary initial states of the field, when the field is confined to a homogeneous detecting medium. We may note that the mean quantum numbers $\bar{m}(t)$ and

$$\bar{n}(t) \equiv \mathrm{tr}[\rho(t)a^\dagger a] \tag{3.14}$$

must, by virtue of the conservation law (2.12), obey the identity

$$\bar{n}(t) + \bar{m}(t) = \bar{n}(0) \tag{3.15}$$

at all times.

If the initial density operator for the field has a P representation,[6,15] i.e., if it can be written[16] as a statistical mixture of pure coherent states,

$$\rho^{(F)}(0) = \int P(\alpha_0,0)\,|\alpha_0\rangle_F \, {}_F\langle\alpha_0| \, d^2\alpha_0 , \tag{3.16}$$

where $d^2\alpha_0 \equiv d(\mathrm{Re}\alpha_0)d(\mathrm{Im}\alpha_0)$, then the photoabsorption probability $p_m(t)$ may be obtained by averaging the result for the coherent state $|\alpha_0\rangle_F$ with respect to the weight function $P(\alpha_0,0)$. We have, then,

$$p_m(t) = \int \left\{ \frac{[\bar{m}(\alpha_0,t)]^m}{m!} e^{-\bar{m}(\alpha_0,t)} \right\} P(\alpha_0,0)\, d^2\alpha_0 , \tag{3.17}$$

where $\bar{m}(\alpha_0,t)$ is given by Eq. (3.11). If the weight function $P(\alpha_0,0)$ which appears in Eq. (3.16) is nonnegative, we may think of the photoabsorption probability $p_m(t)$ as generated by an ensemble of classical fields,[3,4] each of which becomes attenuated during the detection process.

The solution for $p_m(t)$ corresponding to arbitrary initial states of the field can be obtained by noting that Eqs. (3.10) and (3.11) may be expressed in the form

$$p_m(t) = \frac{(1-e^{-2\kappa t})^m}{m!} \cdot {}_F\langle\alpha_0|\, a^{\dagger m} {:}\exp[-a^\dagger a(1-e^{-2\kappa t})]{:}\, a^m |\alpha_0\rangle_F , \tag{3.18}$$

where the colons indicate normal ordering. Let us compare this expression to the form Eq. (3.7) takes for $\rho^{(F)}(0) = |\alpha_0\rangle_F \, {}_F\langle\alpha_0|$. If we then make use of the theorem that any operator for a single harmonic oscillator is

[15] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).

[16] J. R. Klauder [Phys. Rev. Letters **16**, 534 (1966)] has shown that a representation of the form (3.16) exists for arbitrary density operators, provided that one is willing to allow weight functions even more singular than the tempered distributions. The usefulness of the representation in such singular cases is rather limited, however, in that many simple manipulations must be performed with great care, or are prohibited altogether.

uniquely determined[17] by its diagonal matrix elements in the coherent-state representation, we see that the operator $\mathcal{P}_m(t)$ is given by

$$\mathcal{P}_m(t)=\frac{(1-e^{-2\kappa t})^m}{m!}a^{\dagger m}:\exp[-a^\dagger a(1-e^{-2\kappa t})]:a^m \quad (3.19)$$

$$=\frac{(1-e^{-2\kappa t})^m}{m!}a^{\dagger m}e^{-2\kappa t a^\dagger a}a^m, \quad (3.20)$$

where the operator identity[18]

$$:\exp(wa^\dagger a):=\exp[a^\dagger a\ln(1+w)] \quad (3.21)$$

was used to reach the latter expression. Equations (3.7) and (3.19) or (3.20) specify the photoabsorption probabilities for arbitrary initial states of the field.

If the field initially contains exactly n photons, then it follows from Eqs. (3.7) and (3.20) that the probability of finding m quanta absorbed at time t is given by the binomial distribution

$$p_m{}^{(n)}(t)=\binom{n}{m}(1-e^{-2\kappa t})^m(e^{-2\kappa t})^{n-m} \quad \text{for} \quad m\leq n \quad (3.22a)$$

$$=0 \quad \text{for} \quad m>n. \quad (3.22b)$$

We cannot find more than n quanta in the detector at any time, since we have begun at $t=0$ with exactly n quanta in the system of detector and field. We may note that the probability given by Eq. (3.22) is just what one would write down if one pictured the initial state of the field as containing n distinguishable quanta, and postulated that each of these has the probability $1-e^{-2\kappa t}$ of being absorbed between the initial time and time t.

If the initial state of the field is described by an arbitrary density matrix

$$\rho^{(F)}(0)=\sum_{n,n'}|n\rangle_F\ \rho_{nn'}{}^{(F)}(0)\ {}_F\langle n'|, \quad (3.23)$$

then the photoabsorption probability $p_m(t)$ is, according to Eqs. (3.7), (3.20), and (3.22),

$$p_m(t)=\sum_{n=0}^{\infty}\rho_{nn}{}^{(F)}(0)\,{}_F\langle n|\mathcal{P}_m(t)|n\rangle_F$$

$$=\sum_{n=0}^{\infty}\rho_{nn}{}^{(F)}(0)p_m{}^{(n)}(t) \quad (3.24a)$$

$$=(1-e^{-2\kappa t})^m\sum_{n=m}^{\infty}\rho_{nn}{}^{(F)}(0)\binom{n}{m}(e^{-2\kappa t})^{n-m}. \quad (3.24b)$$

[17] This theorem follows simply from the "over-completeness" of the coherent states. See for example R. J. Glauber (Ref. 6); C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. 138, B274 (1965); K. Cahill, Phys. Rev. 138, B1566 (1965).

[18] See for example W. H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill Book Co., N. Y., 1964), Eq. (3.68).

We may note that the probability $p_m(t)$ given by Eq. (3.24) satisfies the relations

$$0\leq p_m(t)\leq 1 \quad (3.25)$$

and

$$\sum_{m=0}^{\infty}p_m(t)=1 \quad (3.26)$$

for arbitrary initial states of the field.

For times $t\ll 1/\kappa$, Eq. (3.24b) reduces to

$$p_m(t)\sim(2\kappa t)^m\sum_{n=m}^{\infty}\rho_{nn}{}^{(F)}(0)\binom{n}{m}(1-2\kappa t)^{n-m}, \quad (3.27)$$

which is the result given by conventional photodetection theory. In the limit of large times ($\kappa t\gg 1$), on the other hand, Eq. (3.24b) reduces to

$$p_m(t)\sim\rho_{mm}{}^{(F)}(0), \quad (3.28)$$

which is the probability of finding m quanta in the initial state of the field.

The photoabsorption probabilities $p_m(t)$ given by Eqs. (3.24) depend only upon the diagonal n-quantum-state matrix elements of $\rho^{(F)}(0)$, i.e., only upon the intensity of the initial field, and not upon the phase. It is interesting to observe that this fact can be deduced, within the context of our model, directly from the conservation law (2.12). Let us expand the density operator $\rho(t)$ for the system in terms of the complete set of states $|n\rangle_F|m,\gamma\rangle_B$, so that a typical element in the expansion will be proportional to

$$|n\rangle_F|m,\gamma\rangle_B\ {}_B\langle m',\gamma'|\ {}_F\langle n'|. \quad (3.29)$$

Terms of this kind may be conveniently classified according to whether

$$(1)\quad n+m=n'+m'$$

or

$$(2)\quad n+m\neq n'+m'.$$

It is clear from the discussion leading to Eq. (2.18) that this classification is invariant in time, i.e., that a term in either of these two classes at a given time can contribute only terms in the same class to the density operator at any other time. Since $m=m'=0$ in the initial state of the detector, it follows that a term in the initial density operator for the field for which $n\neq n'$ can contribute only to those terms in the density operator for the system at time t for which $n+m\neq n'+m'$. It is clear from Eq. (3.6), however, that terms of this kind cannot contribute to the photoabsorption probabilities $p_m(t)$.

The probabilities $p_m(t)$ may be expressed in terms of a generating function[4]

$$Q(\xi,t)=\sum_{m=0}^{\infty}(1-\xi)^m p_m(t) \quad (3.30)$$

by means of the relation

$$p_m(t)=(m!)^{-1}\left(-\frac{\partial}{\partial\xi}\right)^m Q(\xi,t)\,|_{\xi=1}. \tag{3.31}$$

The function $Q(\xi,t)$ may be written, by virtue of Eq. (3.7), as a linear expression in the initial density operator for the field,

$$Q(\xi,t)=\mathrm{tr}_F[\rho^{(F)}(0)\hat{Q}(\xi,t)], \tag{3.32}$$

where the *generating operator* $\hat{Q}(\xi,t)$ is defined as

$$\hat{Q}(\xi,t)=\sum_{m=0}^{\infty}(1-\xi)^m\mathcal{P}_m(t). \tag{3.33}$$

If we substitute Eq. (3.19) for $\mathcal{P}_m(t)$ into this relation and carry out the indicated summation, we find

$$\hat{Q}(\xi,t)=:\exp[-\xi(1-e^{-2\kappa t})a^\dagger a]:, \tag{3.34}$$

and if we make use of the operator identity (3.21), we find

$$\hat{Q}(\xi,t)=\exp\{a^\dagger a\ln[1-\xi(1-e^{-2\kappa t})]\} \tag{3.35a}$$

$$\equiv[1-\xi(1-e^{-2\kappa t})]^{a^\dagger a}. \tag{3.35b}$$

The generating function for an initial n-quantum state of the field is thus

$$Q(\xi,t)=[1-\xi(1-e^{-2\kappa t})]^n, \tag{3.36}$$

and the generating function for an initial coherent state $|\alpha_0\rangle_F$ is

$$Q(\xi,t)=e^{-\xi\bar{m}(t)}, \tag{3.37}$$

where the mean number of absorbed quanta $\bar{m}(t)$ is given by Eq. (3.11).

IV. ATTENUATION OF FIELD

Before proceeding further with our analysis of the photodetection statistics, it is convenient to discuss the attenuation of the field caused by its interaction with the detector. The state of the field at any time t may be described by means of the reduced density operator

$$\rho^{(F)}(t)\equiv\mathrm{tr}_B\rho(t) \tag{4.1}$$

$$=\mathrm{tr}_B\{e^{-iHt/\hbar}\rho^{(F)}(0)\,|0\rangle_B\,{}_B\langle 0|\,e^{iHt/\hbar}\}, \tag{4.2}$$

since the detector is initially in its ground state. This relation expresses the time-dependent density operator for the field as a linear functional of its initial value. It will facilitate further discussion if we express this relationship formally as

$$\rho^{(F)}(t)=\mathfrak{U}[\rho^{(F)}(0);\,t], \tag{4.3}$$

where $\mathfrak{U}$ is a super-operator,[19] i.e., a linear operator functional of the operators in the state-space of the field mode. It may be defined for an arbitrary operator argument F as

$$\mathfrak{U}[F;\,t]\equiv\mathrm{tr}_B\{e^{-iHt/\hbar}F\,|0\rangle_B\,{}_B\langle 0|\,e^{iHt/\hbar}\}. \tag{4.4}$$

We have shown that when the initial state of the field is the coherent state $|\alpha_0\rangle_F$, the state of the field at time t is the coherent state $|e^{-zt}\alpha_0\rangle_F$, so that the reduced density operator for the field is

$$\rho^{(F)}(t)=\mathfrak{U}[|\alpha_0\rangle_F\,{}_F\langle\alpha_0|\,;\,t]$$

$$=|e^{-zt}\alpha_0\rangle_F\,{}_F\langle e^{-zt}\alpha_0|\,. \tag{4.5}$$

If the initial state of the field is described by a P representation, then we find by substituting Eq. (3.16) for $\rho^{(F)}(0)$ into Eq. (4.3) and changing the variables of integration from α_0 to $\alpha=e^{-zt}\alpha_0$ that a P representation exists at all times:

$$\rho^{(F)}(t)=\int P(\alpha,t)\,|\alpha\rangle_F\,{}_F\langle\alpha|\,, \tag{4.6}$$

and that the time-dependent weight function is related to its initial value by means of the complex scale transformation

$$P(\alpha,t)=e^{2\kappa t}P(e^{zt}\alpha,0). \tag{4.7}$$

Thus the P function behaves exactly like the probability density for finding the field oscillator with complex amplitude α, in a classical system in which the complex oscillator amplitude suffers an exponential decay with decay constant κ, and a frequency shift $\delta\omega$. By making use of the formula[6]

$$\mathrm{tr}_F[\rho^{(F)}(t)a^{\dagger n}a^l]=\int\alpha^{*n}\alpha^l P(\alpha,t)d^2\alpha \tag{4.8}$$

for the expectation values of normally ordered products, we find from Eq. (4.7) the identity

$$\mathrm{tr}_F[\rho^{(F)}(t)a^{\dagger n}a^l]=\mathrm{tr}_F[\rho^{(F)}(0)(e^{-z^*t}a^\dagger)^n(e^{-zt}a)^l]. \tag{4.9}$$

Thus the time dependence of the mean values of normally ordered products may be found by replacing the annihilation operator a by the operator $e^{-zt}a$ in the expression for the same quantity at $t=0$.

It is worth noting that the identity (4.9) may be derived from Eq. (4.5) without assuming that the initial density operator for the field possesses a P representation. The expectation value in Eq. (4.9) is given by

$$\mathrm{tr}_F[\rho^{(F)}(t)a^{\dagger n}a^l]=\mathrm{tr}[\rho(t)a^{\dagger n}a^l]$$

$$=\mathrm{tr}\{\rho[a^\dagger(t)]^n[a(t)]^l\}, \tag{4.10}$$

where $a(t)$ is the Heisenberg annihilation operator for the field mode. By substituting Eq. (3.1) for ρ into this relation, we find

$$\mathrm{tr}_F[\rho^{(F)}(t)a^{\dagger n}a^l]=\mathrm{tr}_F[\rho^{(F)}(0)T_{nl}(t)], \tag{4.11}$$

where

$$T_{nl}(t)\equiv\mathrm{tr}_B\{|0\rangle_B\,{}_B\langle 0|[a^\dagger(t)]^n[a(t)]^l\}. \tag{4.12}$$

If we use the result (4.5) to evaluate the left-hand side

[19] The concept of a super-operator has been used in a somewhat different sense in Ref. 5.

of Eq. (4.11) for an initially coherent state of the field, we find

$$ {}_F\langle\alpha_0|\,T_{nl}(t)\,|\alpha_0\rangle_F = (e^{-z^*t}\alpha_0{}^*)^n(e^{-zt}\alpha_0)^l $$
$$ = {}_F\langle\alpha_0|\,(e^{-z^*t}a^\dagger)^n(e^{-zt}a)^l\,|\alpha_0\rangle_F\,, \qquad (4.13) $$

and if we make use of the theorem[17] that an operator is determined by its diagonal matrix elements in the coherent-state representation, we see from Eqs. (4.13) and (4.11) that the identity (4.9) must hold for arbitrary initial states of the field.

The solution (4.5) for the density operator which evolves from an initially coherent state will enable us to obtain the solutions for arbitrary initial states of the field. It follows from the linear character of the superoperator $\mathfrak{U}$ that the n-quantum-state matrix elements

$$ \rho_{nn'}(t) \equiv {}_F\langle n|\,\rho^{(F)}(t)\,|n'\rangle_F \qquad (4.14) $$

of the time-dependent density operator for the field may be expressed in terms of their initial values by means of the relation

$$ \rho_{nn'}{}^{(F)}(t) = \sum_{l,l'} \mathfrak{U}_{nn',ll'}(t)\rho_{ll'}{}^{(F)}(0)\,, \qquad (4.15) $$

where the c-number functions $\mathfrak{U}_{nn',ll'}(t)$ are defined as

$$ \mathfrak{U}_{nn',ll'}(t) \equiv {}_F\langle n|\,\mathfrak{U}[|l\rangle_F\,{}_F\langle l'|\,;\,t]\,|n'\rangle_F\,. \qquad (4.16) $$

The functions $\mathfrak{U}_{nn',ll'}(t)$ may be generated from the solution (4.5) by introducing the states[6,20]

$$ \|\alpha\rangle_F \equiv e^{(1/2)|\alpha|^2}|\alpha\rangle_F \qquad (4.17) $$

$$ = \sum_{l=0}^{\infty} (l!)^{-1/2}\alpha^l\,|l\rangle_F\,, \qquad (4.18) $$

in terms of which the relation (4.5) may be expressed as

$$ \mathfrak{U}[\|\alpha\rangle_F\,{}_F\langle\alpha\|\,;\,t] = \|e^{-zt}\alpha\rangle_F\,{}_F\langle e^{-zt}\alpha\| $$
$$ \times\exp[|\alpha|^2(1-e^{-2\kappa t})]\,. \qquad (4.19) $$

If we use Eq. (4.18) to express both sides of this relation in terms of n-quantum states, then by equating coefficients of $\alpha^l\alpha^{*l'}$ and substituting the resulting expression for $\mathfrak{U}[|l\rangle_F\,{}_F\langle l'|\,;\,t]$ into Eq. (4.16), we find

$$ \mathfrak{U}_{nn',ll'}(t) = \delta_{n-n'-(l-l')}e^{-nzt-n'z^*t}(1-e^{-2\kappa t})^{l-n} $$
$$ \times\left[\binom{l}{n}\binom{l'}{n'}\right]^{1/2} \quad \text{for} \quad n\leq l \qquad (4.20a) $$
$$ = 0 \qquad \text{for} \quad n>l\,. \qquad (4.20b) $$

Thus the matrix elements $\rho_{nn'}{}^{(F)}(t)$ depend only upon those initial matrix elements $\rho_{ll'}{}^{(F)}(0)$ for which $l-l' = n-n'$, and $l\geq n$. These relations may be deduced directly from the conservation law (2.12). If we multiply the expression

$$ |l\rangle_F\,|0\rangle_B\,{}_B\langle 0|\,{}_F\langle l'| \qquad (4.21) $$

on the left by $\exp(-iHt/\hbar)$ and on the right by $\exp(iHt/\hbar)$, then it is clear from Eq. (2.12) that in the expansion of the resulting expression in terms of the complete set (3.29), only those terms can appear for which $n+m=l$ and $n'+m'=l'$. The restrictions on the indices n, n', l, and l' in Eqs. (4.20) follow from these relations and from the observation that the trace of (3.29) with respect to the detector variables is nonvanishing only if $m'=m$.

By substituting Eq. (4.20) for $\mathfrak{U}_{nn',ll'}(t)$ into Eq. (4.15) and differentiating the resulting expression with respect to time, we find that the matrix elements of $\rho^{(F)}(t)$ satisfy the differential equation[21]

$$ \frac{1}{\kappa}\frac{d}{dt}\rho_{nn'}{}^{(F)}(t) = -(n+n')\rho_{nn'}{}^{(F)}(t) $$
$$ +2[(n+1)(n'+1)]^{1/2}\rho_{n+1,n'+1}{}^{(F)}(t)\,, \qquad (4.22) $$

where

$$ \rho_{nn'}{}^{(F)}(t) \equiv e^{i(\omega+\delta\omega)(n-n')t}\rho_{nn'}{}^{(F)}(t)\,. \qquad (4.23) $$

The probabilities of finding specified numbers of quanta in the field thus obey the differential equations[22]

$$ \frac{d}{dt}\rho_{n'n'}{}^{(F)}(t) $$
$$ = 2\kappa[(n'+1)\rho_{n'+1,n'+1}{}^{(F)}(t) - n'\rho_{n'n'}{}^{(F)}(t)]\,. \qquad (4.24) $$

These equations for the quantum-number probabilities $\rho_{n'n'}{}^{(F)}(t)$ are the same as those one would write down in a time-independent classical Markov process,[23] in which an n-quantum state is postulated to have the probability $2\kappa n dt$ of becoming an $(n-1)$-quantum state between the times t and $t+dt$. In the following sections we shall show that this resemblance to a Markov process is not merely a formal one, but actually reflects the way the quantum numbers change when repeated measurements are made on the system.

By evaluating Eqs. (4.15) and (4.20) for $n=n'$, we see that the probabilities $\rho_{n'n'}{}^{(F)}(t)$ are given in terms of their initial values by the relation

$$ \rho_{n'n'}{}^{(F)}(t) $$
$$ = \sum_{l=n'}^{\infty}\binom{l}{n'}(1-e^{-2\kappa t})^{l-n'}(e^{-2\kappa t})^{n'}\rho_{ll}{}^{(F)}(0)\,. \qquad (4.25) $$

[20] V. Bargmann, Commun. Pure Appl. Math. 14, 187 (1961); Proc. Natl. Acad. Sci. U. S. 48, 199 (1962).

[21] M. Scully, W. E. Lamb, Jr., and M. J. Stephen [*Physics of Quantum Electronics*, edited by P. L. Kelley, B. Lax, and P. E. Tannenwald (McGraw-Hill Book Co., New York, 1966)] have constructed a model of laser action in which the effect of damping leads to a term identical to the right-hand side of Eq. (4.22) in the equation of motion for the density matrix. Professor Scully has also shown [M. Scully, thesis, Yale University, 1965 (unpublished)] that Eq. (4.22) leads to the attenuation relation (4.5) for coherent states.

[22] R. J. Glauber [Summer School on Quantum Optics of the Italian Physical Society, Varenna, Italy, 1967 (to be published)] presented a quantum-mechanical theory of the damped harmonic oscillator characterized by equations which reduce to Eqs. (4.24) when the temperature of the heat bath is set equal to zero.

[23] The Markoffian behavior of the density matrix in a broad class of physical contexts has been discussed by M. Lax [Phys. Rev. **145**, 110 (1966)] and by M. Lax and W. H. Louisell [J. Quant. Electron. **QE-3**, 47 (1967)].

If the field initially contains exactly n photons, the probability of finding n' photons remaining in the field at time t is thus

$$\rho_{n'n'}{}^{(F)}(t)=\binom{n}{n'}(1-e^{-2\kappa t})^{n-n'}(e^{-2\kappa t})^{n'} \quad (4.26)$$

$$=p_{n-n'}{}^{(n)}(t), \quad (4.27)$$

where $p_m{}^{(n)}(t)$, as given by Eq. (3.22), is the probability of finding m quanta in the detector at time t for an initial n-quantum state of the field.

The relation (4.27) follows from the conservation law (2.12), which requires that the total number of quanta in the system of field and detector be equal to n at all times, if it is equal to n initially. Thus the analysis of this section leading to Eq. (4.26) provides an alternative derivation of the photoabsorption probability $p_m{}^{(n)}(t)$. If we recall that the relation (3.24a), which expresses the fact that only the *diagonal* n-quantum state matrix elements of $\rho^{(F)}(0)$ can contribute to $p_m(t)$, is a consequence of the conservation law (2.12), we see that the analysis of this section may be used to obtain the photoabsorption probabilities for arbitrary initial field states. The advantage of the present derivation is that it does not depend in a detailed way upon the assumption that the detecting atoms are harmonic oscillators. The results of this section have all been derived from the single relation (4.5) for the time-dependent density operator which evolves from an initially coherent state of the field. The probabilities $p_m(t)$ given in Sec. III are therefore valid for any system of detecting atoms which attenuates a coherent field state in the manner expressed by Eq. (4.5), and which is characterized by an excitation number $M(t)$ satisfying Eq. (2.12).

V. EFFECT OF MEASURING M

It is useful at this point to ask what additional information about the state of the system we obtain by making a measurement of the total number of quanta absorbed by the detector. It is clear that if the system of detector and field is described immediately before the measurement by the density operator $\rho(\tau)$, and if the measurement of M at time τ yields the result m, then the system must be described immediately after the measurement by the density operator

$$\rho^{(m)}(\tau)=\frac{1}{p_m(\tau)}\hat{P}_m\rho(\tau)\hat{P}_m, \quad (5.1)$$

in which $\hat{P}_m$ is the projection operator defined by Eq. (3.4), and the factor $1/p_m(\tau)$ is necessary to ensure that $\mathrm{tr}[\rho^{(m)}(\tau)]=1$. The reduced density operator for the field immediately after the measurement is thus

$$\rho^{(m,F)}(\tau)\equiv \mathrm{tr}_B\rho^{(m)}(\tau) \quad (5.2a)$$

$$=\frac{1}{p_m(\tau)}\mathrm{tr}_B[\hat{P}_m\rho(\tau)], \quad (5.2b)$$

since $\hat{P}_m{}^2=\hat{P}_m$. The probability of finding n quanta in the field at time τ, given that we have found m quanta in the detector, is then

$$p(n,\tau\,|\,m,\tau)={}_F\langle n|\rho^{(m,F)}(\tau)|n\rangle_F \quad (5.3)$$

$$=\frac{1}{p_m(\tau)}\mathrm{tr}[|n\rangle_F\,{}_F\langle n|\hat{P}_m\rho(\tau)]$$

$$=\frac{1}{p_m(\tau)}p(n,m,\tau), \quad (5.4)$$

in which $p(n,m,\tau)$ is the joint probability of finding n quanta in the field and m quanta in the detector at time τ,

$$p(n,m,\tau)=\mathrm{tr}[|n\rangle_F\,{}_F\langle n|\hat{P}_m\rho(\tau)]. \quad (5.5)$$

This probability is easily evaluated when the detector is initially in its ground state. In that case it is clear from the discussion following (3.29) that no contribution can come from the off-diagonal n-quantum-state matrix elements of the initial density operator for the field. It follows from the conservation law (2.12) that the only diagonal matrix element of $\rho^{(F)}(0)$ that can contribute is the one corresponding to $n+m$ quanta. We have, therefore,

$$p(n,m,\tau)=\rho_{n+m,n+m}{}^{(F)}(0)p_m{}^{(n+m)}(\tau) \quad (5.6)$$

$$=\rho_{n+m,n+m}{}^{(F)}(0)\binom{n+m}{m}$$

$$\times(1-e^{-2\kappa\tau})^m(e^{-2\kappa\tau})^n, \quad (5.7)$$

where the latter relation follows from the formula (3.22) for $p_m{}^{(n)}(t)$. Substituting Eq. (5.7) for $p(n,m,\tau)$ into Eq. (5.4), we have

$$p(n,\tau\,|\,m,\tau)=\frac{1}{p_m(\tau)}\rho_{n+m,n+m}{}^{(F)}(0)\binom{n+m}{m}$$

$$\times(1-e^{-2\kappa\tau})^m(e^{-2\kappa\tau})^n. \quad (5.8)$$

In the general case, the conditional probability $p(n,\tau\,|\,m,\tau)$ depends upon the number m of quanta which have been found in the detector. It is interesting to observe, however, that there is an exception to this rule. We have seen that when the initial state of the system is the product of a coherent state for the field and the ground state for the detector, then the state of the system at any time remains the product of a state for the field and a state for the detector; the

statistical properties of the field in this case are therefore unaltered by a measurement of the detector. We may note that all initial density operators for the field whose diagonal matrix elements are given by a Poisson distribution,

$$\rho_{nn}{}^{(F)}(0)=\frac{(\bar{n}(0))^n}{n!}e^{-\bar{n}(0)}, \tag{5.9}$$

give rise, according to Eq. (5.7), to the joint quantum-number distribution

$$p(n,m,\tau)=\left[\frac{(\bar{m}(\tau))^m}{m!}e^{-\bar{m}(\tau)}\right]\left[\frac{(\bar{n}(\tau))^n}{n!}e^{-\bar{n}(\tau)}\right], \tag{5.10}$$

where

$$\bar{n}(\tau)=\bar{n}(0)-\bar{m}(\tau)=\bar{n}(0)e^{-2\kappa\tau}. \tag{5.11}$$

Thus the joint probability distribution for quantum numbers in detector and field continues to factor into Poisson distributions at all times, if the initial distribution of the field is Poisson. The probability distribution for the field after a measurement has found m quanta in the detector is

$$p(n,\tau\,|\,m,\tau)=\frac{(\bar{n}(\tau))^n}{n!}e^{-\bar{n}(\tau)}, \tag{5.12}$$

which is independent of the result of the measurement.

It is interesting to observe that Eq. (5.12) remains valid even for $m=0$, i.e., even when the detector is found to be unexcited at time τ. The mean quantum number of the field given by Eq. (5.11) is smaller than its initial value, and thus the field is attenuated even when a measurement reveals that no quanta have been absorbed by the detector. This is in fact a general feature of the attenuation process we are considering. By evaluating Eqs. (5.8) and (3.24b) for $m=0$ we find, for arbitrary initial fields,

$$p(n,\tau\,|\,m=0,\tau)=\frac{\rho_{nn}{}^{(F)}(0)(e^{-2\kappa\tau})^n}{\sum_{n'=0}^{\infty}\rho_{n'n'}{}^{(F)}(0)(e^{-2\kappa\tau})^{n'}}. \tag{5.13}$$

Thus the quantum-number distribution of the field at time τ, immediately after a measurement has found the detector to be unexcited, is in general different from the initial quantum-number distribution $\rho_{nn}{}^{(F)}(0)$. The new distribution has a mean quantum number

$$\bar{n}_0(\tau)\equiv\sum_{n=0}^{\infty}np(n,\tau\,|\,m=0,\tau), \tag{5.14}$$

which may be shown to satisfy the inequality

$$\bar{n}_0(\tau)\leq\bar{n}(0)\equiv\textstyle\sum_n n\rho_{nn}{}^{(F)}(0). \tag{5.15}$$

It should not surprise us to learn that the statistical description of the field can change even when a measurement has revealed that the state of the detector has not changed. Since the probability of exciting the detector during a given time interval is an increasing function of the initial field intensity, it follows that if no excitations occur, we should be led to reduce our estimate of the number of quanta in the field. The unique exception to this rule occurs when the initial state of the field is the occupation-number eigenstate $|n'\rangle_F$. In that case, if no quanta are found in the detector at time τ, then the field must still be in the state $|n'\rangle_F$, and indeed it follows from Eq. (5.13) that

$$p(n,\tau\,|\,m=0,\tau)=\delta_{nn'} \tag{5.16}$$

if $\rho_{nn}{}^{(F)}(0)=\delta_{nn'}$.

VI. NONRADIANT STATES OF DETECTOR

In order to develop our theory of photon detection further, it is necessary to be able to discuss the time evolution of the system of field and detector after a measurement has been made of the total number of quanta in the detector. The conventional way[24] of dealing with this question is to assume that the further absorption of quanta by the detector proceeds as if the detector had returned to its ground state after the measurement. One may, indeed, define an ideal detector as one for which this assumption is valid. It is necessary, therefore, to justify it in the context of a particular model of the detection process. This is especially important for the model we are investigating, which describes significant attenuation of the field, and strong excitations of the detector.

We shall gain some insight into the nature of the excitations which are induced in the detector by examining in greater detail the solutions which were found in Sec. IV for the time evolution of the reduced density operator for the field. Those solutions were based on the assumption that the detector is in its ground state at $t=0$, and it is therefore interesting to observe that the equations obtained for $\rho^{(F)}(t)$ are invariant under time translation: It follows from the fact that the coefficients in the differential equations (4.22) are independent of time that if $\rho^{(F)}(\tau)$ is the reduced density operator for the field at time $\tau>0$, then the field at any later time $\tau+t$ is described by the density operator

$$\rho^{(F)}(\tau+t)=\mathfrak{U}[\rho^{(F)}(\tau);t], \tag{6.1}$$

where the super-operator $\mathfrak{U}$ is defined by Eq. (4.4). Thus the density operator for the field at time $\tau+t$ is just what it would have been if the density operator for the system at time τ had been

$$\rho^{(F)}(\tau)\,|0\rangle_B\,{}_B\langle 0|=[\mathrm{tr}_B\rho(\tau)]\,|0\rangle_B\,{}_B\langle 0|, \tag{6.2}$$

rather than the operator $\rho(\tau)$ given by Eq. (3.2). Of course the actual state of the system at time τ contains excited states of the detector, and in general specifies

[24] See, for example, Ref. (5).

correlations between the variables which describe the field and those which describe the detector. Evidently the states of the detector which are excited by the field behave just like the ground state in their influence on the further evolution of the reduced density operator for the field.

We have discussed in Sec. II an example of such a state of the detector. According to Eq. (2.47), any state of the form $|\lambda(t')\rangle_B$, where t' is any positive time, behaves just like the ground state in its influence on any coherent state of the field. Since the formulas of Sec. IV for the time evolution of $\rho^{(F)}(t)$ were all derived from the single relation (4.5) for an initially coherent state, it follows that the state $|\lambda(t')\rangle_B$ is like the state $|0\rangle_B$ in its influence on arbitrary initial states of the field: If the initial state of the system is

$$\rho=\rho^{(F)}(0)|\lambda(t')\rangle_B\,{}_B\langle\lambda(t')|\,, \tag{6.3}$$

then the density operator for the field at time t is given by Eq. (4.3).

It follows that if the state $\rho^{(F)}(0)$ in Eq. (6.3) represents the vacuum state of the field, then the field remains in the vacuum state at all times. The state $|\lambda(t')\rangle_B$ therefore does not radiate quanta into the vacuum. Of course this result depends strongly upon the approximations made in Sec. II; we may note in particular that according to Eq. (2.28) we are justified only in treating times small compared to the reciprocal of the mean interval between the discrete frequency levels of the detecting oscillators located within the volume c^3/ω^3. The lifetime of a state such as $|\lambda(t')\rangle_B$ is not infinite, but it is evidently much greater than that of a detector state consisting of incoherent excitations of the oscillators, with the same mean quantum number.[25] The lifetime of the state $|\lambda(t')\rangle_B$ approaches infinity in the limit in which the conditions (2.22), (2.23), (2.25), and (2.27)–(2.29) are perfectly satisfied, and can clearly be much larger than any of the time intervals we need consider.

Let us define a *nonradiant state* $\rho^{(B)}$ of the detector as any state which affects the field, in the approximation we are making, in the same way as the ground state does, i.e., as any state for which the relation

$$\mathrm{tr}_B\{e^{-iHt/\hbar}\rho^{(F)}(0)\rho^{(B)}e^{iHt/\hbar}\}=\mathfrak{U}[\rho^{(F)}(0);\,t] \tag{6.4}$$

holds for arbitrary initial field states $\rho^{(F)}(0)$. If $\rho^{(F)}(0)$ corresponds to the pure coherent state $|\alpha\rangle_F$, we have, for arbitrary α,

$$\mathrm{tr}_B\{e^{-iHt/\hbar}|\alpha\rangle_F\,{}_F\langle\alpha|\rho^{(B)}e^{iHt/\hbar}\} =|e^{-zt}\alpha\rangle_F\,{}_F\langle e^{-zt}\alpha|\,, \tag{6.5}$$

a relation which has been shown to imply the more general relation (6.4). It is not difficult to show that whenever the reduced density operator for the field represents a pure coherent state, then the density operator for the system must be the product of a coherent state for the field and a state for the detector, so that we must have

$$e^{-iHt/\hbar}|\alpha\rangle_F\,{}_F\langle\alpha|\rho^{(B)}e^{iHt/\hbar} =|e^{-zt}\alpha\rangle_F\,{}_F\langle e^{-zt}\alpha|\rho^{(B)}(t;\,\alpha_0)\,, \tag{6.6}$$

where $\rho^{(B)}(t;\,\alpha_0)$ is some state for the detector. The equations (6.4), (6.5), and (6.6) are all equivalent, and any of them may be used to define nonradiant states.

In order to ascertain the general class of nonradiant states, let us consider the state which evolves during time t from the product of an arbitrary coherent state $|\alpha\rangle_F$ for the field and some pure state $|\ \rangle_B$ for the detector,

$$|t\rangle=e^{-iHt/\hbar}|\alpha\rangle_F\ |\ \rangle_B\,, \tag{6.7}$$

and then let us evaluate the quantity $a|t\rangle$. If we make use of Eq. (2.35a) and the relation

$$a(t)=e^{iHt/\hbar}ae^{-iHt/\hbar}\,, \tag{6.8}$$

we find

$$\begin{aligned}a|t\rangle&=e^{-iHt/\hbar}[e^{-zt}a+\xi(t)b]|\alpha\rangle_F\ |\ \rangle_B\\&=e^{-zt}\alpha|t\rangle+e^{-iHt/\hbar}\xi(t)b|\alpha\rangle_F\ |\ \rangle_B.\end{aligned} \tag{6.9}$$

If the state $|\ \rangle_B$ is nonradiant, i.e., if the density operator

$$\rho^{(B)}=|\ \rangle_B\,{}_B\langle\ | \tag{6.10}$$

satisfies Eq. (6.6) for all α, then the state $|t\rangle$ must be equal to the product

$$|t\rangle\equiv e^{-iHt/\hbar}|\alpha\rangle_F|\ \rangle_B=|e^{-zt}\alpha\rangle_F|t;\,\alpha\rangle_B\,, \tag{6.11}$$

where $|t;\,\alpha\rangle_B$ is some state of the detector, and we have, therefore,

$$a|t\rangle=e^{-zt}\alpha|t\rangle. \tag{6.12}$$

If we compare this relation with Eq. (6.9) we see that any pure nonradiant state $|\ \rangle_B$ must satisfy the relation

$$\xi(t)b|\ \rangle_B=0 \tag{6.13}$$

for all $t>0$. It is equally true that any state $|\ \rangle_B$ which satisfies this relation must be nonradiant, since if we substitute it into Eq. (6.9) we obtain Eq. (6.12), and thus Eq. (6.11). The equation (6.13) may therefore be used as a definition of nonradiant pure states of the detector. By the obvious generalization of the steps leading to Eq. (6.13) we find that any nonradiant mixed state $\rho^{(B)}$ must satisfy the relations

$$[\xi(t')b]\rho^{(B)}=\rho^{(B)}[\xi(t')b]^\dagger=0 \tag{6.14}$$

for all $t'>0$, and that any state which satisfies these relations must be nonradiant.

We may note that if the initial density operator ρ for the *system* satisfies the relation

$$[\xi(t')b]\rho=\rho[\xi(t')b]^\dagger=0 \tag{6.15}$$

[25] The existence of anomalously long-lived states of systems consisting of many coherently excited atoms has been noted by R. H. Dicke [Phys. Rev. 93, 99 (1954)], who also shows that there exist anomalously short-lived, or "super-radiant," states of such systems.

for all $t'>0$, then the time-dependent density operator

$$\rho(t)=e^{-iHt/\hbar}\rho e^{iHt/\hbar} \tag{6.16}$$

must possess a reduced density operator $\rho^{(F)}(t)$ for the field which is given in terms of its initial value by Eq. (4.3). To prove this, let us express ρ by means of the expansion

$$\rho=\sum_{n,n'}|n\rangle_F\,{}_F\langle n'|\rho_{nn'}{}^{(B)}, \tag{6.17}$$

where the quantities $\rho_{nn'}{}^{(B)}$ are operators in the state-space of the detector, which, by virtue of the Hermiticity of ρ, satisfy the relation

$$\rho_{nn'}{}^{(B)\dagger}=\rho_{n'n}{}^{(B)}. \tag{6.18}$$

The reduced density operator for the field at the beginning of the interval is, according to Eq. (6.17),

$$\rho^{(F)}(0)=\sum_{n,n'}|n\rangle_F\,{}_F\langle n'|\,\mathrm{tr}_B(\rho_{nn'}{}^{(B)}). \tag{6.19}$$

We wish to show that if the initial density operator (6.17) for the system satisfies the relations (6.15), then the reduced density operator for the field at time t is the same as it would be if the state of the system at the beginning of the interval were just

$$\rho^{(F)}(0)|0\rangle_B\,{}_B\langle 0| =\sum_{n,n'}|n\rangle_F\,{}_F\langle n'|[\mathrm{tr}_B\rho_{nn'}{}^{(B)}]|0\rangle_B\,{}_B\langle 0|. \tag{6.20}$$

We are able in fact to prove a stronger theorem: Each term in the expansion (6.17), corresponding to a particular value of n and n', propagates in time in such a way that its contribution to the reduced density operator for the field at time t is the same as the one which arises from the corresponding term in the expansion (6.20). To prove this, let us note that by virtue of Eqs. (6.15) and (6.17), the operator $\rho_{nn'}{}^{(B)}$, for any values of n and n', must satisfy Eq. (6.14), and hence may be constructed from nonradiant states of the detector. It should be noted that the equivalence between Eqs. (6.14) and (6.4) was derived under the assumption that $\rho^{(B)}$ is a Hermitian operator with unit trace. It is not difficult to show that if $\rho^{(B)}$ does not have unit trace, then an additional factor of $\mathrm{tr}_B\rho^{(B)}$ must be introduced into the right-hand side of Eq. (6.4). The contribution of a particular term in Eq. (6.17) to $\rho^{(F)}(t)$ is then

$$\mathrm{tr}_B\{e^{-iHt/\hbar}|n\rangle_F\,{}_F\langle n'|\rho_{nn'}{}^{(B)}e^{iHt/\hbar}\} =[\mathrm{tr}_B\rho_{nn'}{}^{(B)}]\mathfrak{U}[|n\rangle_F\,{}_F\langle n'|;t], \tag{6.21}$$

which is the same as the contribution from the corresponding term in the expansion (6.20). By summing over n and n' in Eq. (6.21) and making use of Eq. (6.19) we find that $\rho^{(F)}(t)$ is given in terms of its initial value $\rho^{(F)}(0)$ by Eq. (4.3).

There are many nonradiant states of the detector besides the ground state and the states of the form $|\lambda(t')\rangle_B$. Indeed, as we shall now show, states of the detector which are excited by arbitrary initial fields are nonradiant. Let us consider the density operator $\rho(\tau)$ which evolves from the initial state (3.1), and evaluate the quantity $[\xi^\dagger(t')b]\rho(\tau)$. If we make use of Eq. (2.35b) and the relation

$$b(\tau)=e^{iH\tau/\hbar}be^{-iH\tau/\hbar}, \tag{6.22}$$

we find

$$[\xi^\dagger(t')b]P(\tau) =e^{-iH\tau/\hbar}\xi^\dagger(t')[\lambda(\tau)a+\nu(\tau)b] \times\rho^{(F)}(0)|0\rangle_B\,{}_B\langle 0|e^{iH\tau/\hbar} \tag{6.23}$$

$$=e^{-iH\tau/\hbar}\xi^\dagger(t')\lambda(\tau)a\rho^{(F)}(0)|0\rangle_B\,{}_B\langle 0|e^{iH\tau/\hbar}, \tag{6.24}$$

since $b|0\rangle_B=0$. If we then make use of Eq. (2.36), we find

$$[\xi^\dagger(t')b]\rho(\tau)=0=\rho(\tau)[\xi^\dagger(t')b]^\dagger, \tag{6.25}$$

the second part of which is the Hermitian conjugate of the first. Equation (6.25) is not surprising, since we know that the reduced density operator for the field at any time $\tau+t$ is given in terms of its value at time τ by Eq. (6.1). It is worth noting, however, that Eq. (6.25) is valid not only when the initial state of the detector is the ground state, but also whenever the initial state of the detector is nonradiant. Indeed, if the density operator for the system satisfies Eq. (6.15) initially, then it must do so at all times. To prove this we need only substitute ρ for

$$\rho^{(F)}(0)|0\rangle_B\,{}_B\langle 0|$$

in Eq. (6.23), and make use of Eqs. (2.37) and (2.36). We have shown, then, that *the set of nonradiant states is invariant under the action of the Hamiltonian*, within the approximations we are making.

These results enable us to show that the absorption of quanta by the detector during a time interval following a measurement of M proceeds exactly as if the detector were in its ground state immediately after the measurement. Let us begin by noting that if the system at some time τ is described by a pure state $|\tau\rangle$ which contains only nonradiant states of the detector, i.e., if the state $|\tau\rangle$ satisfies the relation

$$[\xi^\dagger(t')b]|\tau\rangle=0 \tag{6.26}$$

for all $t'>0$, then so must the state $\hat{P}_m|\tau\rangle$, where $\hat{P}_m$ is the projection operator defined by Eq. (3.4). To prove this, let us expand the state $|\tau\rangle$ in terms of eigenstates of M,

$$|\tau\rangle=\textstyle\sum_m\hat{P}_m|\tau\rangle, \tag{6.27}$$

so that we have

$$\textstyle\sum_m[\xi^\dagger(t')b]\hat{P}_m|\tau\rangle=0. \tag{6.28}$$

Since the state $\hat{P}_m|\tau\rangle$ contains exactly m quanta, and since the operators b_j are lowering operators, the state

$[\tilde{\xi}(t')b]\hat{P}_m|\tau\rangle$ must contain exactly $m-1$ quanta. Thus the individual terms in the sum (6.28), corresponding to different values of m, are orthogonal to each other, and since the sum vanishes, then so must each term,

$$[\tilde{\xi}(t')b]\hat{P}_m|\tau\rangle=0. \tag{6.29}$$

By similar reasoning we may show that if the state of the system immediately before a measurement is mixed, and if the density operator $\rho(\tau)$ which describes it satisfies Eq. (6.25), then so must the operator $\rho^{(m)}(\tau)$, which is defined by Eq. (5.1) as the density operator which must be used to describe the system immediately after the measurement if m quanta are found. By steps similar to those leading to Eq. (6.29) we find

$$[\tilde{\xi}(t')b]\rho^{(m)}(\tau)=0=\rho^{(m)}(\tau)[\tilde{\xi}(t')b]^\dagger. \tag{6.30}$$

It follows from this relation that the reduced density operator for the field evolves in time after the measurement exactly as if the density operator for the system immediately after the measurement were

$$\rho^{(m,F)}(\tau)|0\rangle_B\,{}_B\langle 0|\,, \tag{6.31}$$

where $\rho^{(m,F)}(\tau)$ is defined by Eq. (5.2a). The reduced density operator for the field at time $\tau+t$ is then

$$\rho^{(F)}(\tau+t)=\mathfrak{U}[\rho^{(m,F)}(\tau);t], \tag{6.32}$$

if m quanta were found in the detector at time τ.

This relation is easily seen to imply that the detector states following a measurement of M behave like the ground state in their absorption of quanta from the field. Let us first consider the case in which the state of the system at time $|\tau\rangle$ is the product

$$|\tau\rangle=|n\rangle_F\,|m\rangle_B\,, \tag{6.33}$$

where $|n\rangle_F$ is an n-quantum state of the field, and $|m\rangle_B$ is a state of the detector containing m quanta, which we assume to be nonradiant,

$$[\tilde{\xi}(t')b]|m\rangle_B=0. \tag{6.34}$$

The reduced density operator for the field at time $\tau+t$ is therefore just what it would be if the state of the detector at time τ were $|0\rangle_B$, and the probability of finding n' quanta in the field at time $\tau+t$ is therefore given by the right-hand side of Eq. (4.26). Since the state of the system contains exactly $m+n$ quanta, the probability of finding n' quanta in the field at any time must be equal to the probability of finding $m+n-n'$ quanta in in the detector. The probability of finding $m+m'$ quanta in the detector at time $\tau+t$ is then, by virtue of the equivalence of the right-hand sides of Eqs. (4.26) and (4.27),

$$p_{m+m'}(\tau+t)=p_{m'}{}^{(n)}(t)\,, \tag{6.35}$$

where the functions $p_{m'}{}^{(n)}(t)$ are given by Eqs. (3.22). Thus the probability of m' quanta being absorbed between times τ and $\tau+t$ is the same when the state of the system at time τ is given by Eq. (6.33) as it would be if the state of the system were $|n\rangle_F\,|0\rangle_B$.

It follows directly that if the state of the system immediately after the measurement takes the form

$$\rho^{(m)}(\tau)=\rho^{(m,F)}(\tau)|m\rangle_B\,{}_B\langle m|\,, \tag{6.36}$$

where $\rho^{(m,F)}(\tau)$ represents a statistical mixture of n-quantum states for the field, then the probability of finding m' quanta absorbed by the detector between times τ and $\tau+t$ is the same as it would be if the density operator for the system immediately after the measurement were given by (6.31). It is not difficult to show that off-diagonal matrix elements of $\rho^{(m,F)}(\tau)$ cannot contribute to the absorption of quanta by the detector between the times τ and $\tau+t$; this can be proved by steps analogous to those used in Sec. III to establish the same result for the case in which the detector is in its ground state at the beginning of the time interval in question. Thus a nonradiant state of the detector containing a fixed number of quanta is like the ground state in its absorption of quanta from an arbitrary mixed state of the field.

We have shown that if the state $\rho^{(m)}(\tau)$ is the product of a state for the field and a state for the detector, then the absorption of quanta by the detector following the measurement proceeds as if the detector were in its ground state. To generalize this result to the case in which the states of the field and detector are correlated after the measurement, we need only express the density operator $\rho^{(m)}(\tau)$ by means of the expansion

$$\rho^{(m)}(\tau)=\sum_{n,n'}|n\rangle_F\,{}_F\langle n'|\rho_{nn'}{}^{(m,B)}(\tau)\,, \tag{6.37}$$

and note that by virtue of Eq. (6.30) each of the operators $\rho_{nn'}{}^{(m,B)}(\tau)$ may be constructed entirely from nonradiant states.

VII. MULTIPLE-TIME COUNTING STATISTICS

The formulas derived in Sec. III for the probabilities $p_m(t)$ of finding m quanta absorbed by the detector between the initial time and time t are all based on the assumption that the system of field and detector is isolated throughout the time interval in question; the probability $p_m(t)$ refers to a single measurement performed at the end of the interval. In actual photon-counting experiments, on the other hand, the short resolving times of the detectors make it possible to specify quite closely the times at which individual counts are recorded, rather than merely to measure the total number recorded in some interval. It is natural to attempt to describe such an experiment mathematically by supposing that repeated measurements of the total excitation number M of the detector are made, at closely spaced times τ_j. We shall call such a procedure "monitoring the detector."

It is by no means clear *a priori* that the probability of finding m quanta in the detector at time t is the same when the detector has been monitored between the initial time and time t as it is when the detector has not been monitored. We are able, however, to demonstrate this equivalence, within the context of our model of photon detection, by making use of the results of the preceding section.

Let us assume that the detector is in its ground state at time $t=0$, so that the density operator for the system at time τ_1 is given by Eq. (3.2), with $t \rightarrow \tau_1$. The probability $p_m(\tau_1)$ of finding m quanta in the detector at time τ_1 is then given by Eq. (3.5), and if m quanta are found, the density operator which describes the system immediately after the measurement is given by Eq. (5.1). Let us now consider the *ensemble* of systems which result from a measurement of M, corresponding to all possible values of m. The density operator which describes such an ensemble is

$$\rho'(\tau_1)=\sum_m p_m(\tau_1)\rho^{(m)}(\tau_1) \tag{7.1}$$

$$=\sum_m \hat{P}_m\rho(\tau_1)\hat{P}_m. \tag{7.2}$$

The density operator for the system after the measurement is thus, in general, different from the density operator before the measurement, even when the result of the measurement is not taken into account. The reduced density operator for the *field*, on the other hand, cannot be changed by a measurement of the detector when the result of the measurement is not taken into account, and indeed it follows from Eq. (7.2) that the density operator for the field immediately after the measurement is unchanged:

$$\rho'^{(F)}(\tau_1)\equiv \mathrm{tr}_B\rho'(\tau_1)=\rho^{(F)}(\tau_1). \tag{7.3}$$

We have shown in the preceding section that the act of measuring M does not affect the nonradiant character of the states of the detector. It follows that the reduced density operator for the field evolves between the times τ_1 and τ_2 in the same way that it would if no measurement had been made at time τ_1. Since we have shown that the dynamical development of the isolated system of field and detector does not alter the nonradiant character of the states of the detector, we know that the state of the system immediately before the second measurement involves only nonradiant states. By repeating these arguments at each time τ_j we find that the reduced density operator for the field at any time t is the same whether or not measurements of M have been made prior to time t. Of course this result is valid only when we are considering the ensemble of systems which result from a measurement. We have seen in Sec. V that the statistical description of the field can be greatly altered even by a single measurement of M, when the result of the measurement is taken into account.

The fact that the reduced density operator for the field is not altered by the monitoring of the detector enables us to prove the same result for the photoabsorption probabilities $p_m(t)$. If the initial state of the field is the n-quantum state $|n\rangle_F$, then whether or not measurements of M have been made between the initial time and time t, the total number of quanta in the field and detector must be equal to n at all times. The relation (4.27) must therefore hold at all times, and since, as we have shown, $\rho^{(F)}(t)$ is not affected by the monitoring of the detector, then neither is $p_m{}^{(n)}(t)$. It follows immediately that the photoabsorption probabilities $p_m(t)$ are unaffected by the monitoring process in all cases in which the field is initially described by a statistical mixture of n-quantum states. The failure of the off-diagonal n-quantum-state matrix elements of $\rho^{(F)}(0)$ to contribute to $p_m(t)$ when the detector is monitored can be established by much the same reasoning as that which was used in Sec. III to establish this result for the case in which the detector is not monitored; we have only to note that according to Eq. (7.2), the measuring process selects those terms for which $m'=m$ from the expansion of the density operator for the system in terms of the complete set (3.29). It follows then that the monitoring process does not affect the photoabsorption probabilities for arbitrary initial states of the field.

Let us now evaluate the joint probability $p(\{m_j\})$ that m_j quanta are found in the detector at each of the times τ_j. If the system is described initially by the pure state $|\ \rangle$, then the probability of finding m_1 quanta absorbed at time τ_1 is

$$p(m_1)=\langle\ |e^{iH\tau_1/\hbar}\hat{P}_{m_1}e^{-iH\tau_1/\hbar}|\ \rangle, \tag{7.4}$$

and if m_1 quanta are found, the state which describes the system immediately after the measurement is

$$|m_1;\ \tau_1\rangle=[p(m_1)]^{-1/2}\hat{P}_{m_1}e^{-iH\tau_1/\hbar}|\ \rangle, \tag{7.5}$$

where the factor $[p(m_1)]^{-1/2}$ is necessary to insure that the state be properly normalized. The probability of finding m_2 quanta in the detector at time τ_2, given that m_1 quanta have been found at time τ_1, is thus

$$p(m_2|m_1)=\langle m_1;\ \tau_1|e^{iH(\tau_2-\tau_1)/\hbar}\hat{P}_{m_2} \times e^{-iH(\tau_2-\tau_1)/\hbar}|m_1;\ \tau_1\rangle, \tag{7.6}$$

and if m_2 quanta are found, the state which describes the system immediately after the second measurement is

$$|m_1,m_2;\ \tau_2\rangle=[p(m_2|m_1)]^{-1/2}\hat{P}_{m_2} \times e^{-iH(\tau_2-\tau_1)/\hbar}|m_1;\ \tau_1\rangle. \tag{7.7}$$

The joint probability $p(m_1,m_2)$ for finding m_1 quanta at time τ_1 and m_2 quanta at time τ_2 may be found by substituting Eq. (7.5) for $|m_1;\ \tau_1\rangle$ into Eq. (7.6), and then substituting the resulting expression for $p(m_2|m_1)$ into the relation

$$p(m_1,m_2)=p(m_2|m_1)p(m_1). \tag{7.8}$$

We have, then,

$$p(m_1,m_2)=\langle|e^{iH\tau_1/\hbar}\hat{P}_{m_1}e^{iH(\tau_2-\tau_1)/\hbar}\hat{P}_{m_2}\times e^{-iH(\tau_2-\tau_1)/\hbar}\hat{P}_{m_1}e^{-iH\tau_1/\hbar}|\rangle. \quad (7.9)$$

The general formula for the probabilities $p(\{m_j\})$ of finding m_j quanta at each of the times τ_j may be derived by repeating the steps which led to this relation. Let us introduce the operator

$$\hat{R}(\{m_j\})\equiv[\textstyle\prod_j e^{-iH\Delta\tau_j/\hbar}\hat{P}_{m_j}]^\dagger[\prod_j e^{-iH\Delta\tau_j/\hbar}\hat{P}_{m_j}], \quad (7.10)$$

where

$$\Delta\tau_j\equiv\tau_{j+1}-\tau_j, \quad (7.11)$$

and the operator products are understood to be written in chronological order, i.e., with later times to the left of earlier times. Then the probability $p(\{m_j\})$ may be expressed as

$$p(\{m_j\})=\langle|\hat{R}(\{m_j\})|\rangle \quad (7.12a)$$

$$=\mathrm{tr}[\rho\hat{R}(\{m_j\})], \quad (7.12b)$$

in which the latter relation is the obvious generalization to mixed initial states of the system.

By substituting Eq. (3.1) for ρ into Eq. (7.12b) we see that the probability $p(\{m_j\})$ may be expressed as a linear functional of the initial density operator for the field in the form

$$p(\{m_j\})=\mathrm{tr}_F[\rho^{(F)}(0)\mathcal{O}'(\{m_j\})], \quad (7.13)$$

where $\mathcal{O}'(\{m_j\})$ is an operator in the state-space of the field mode, which may be written formally as

$$\mathcal{O}'(\{m_j\})=\mathrm{tr}_B[|0\rangle_B\,{}_B\langle 0|\hat{R}(\{m_j\})]. \quad (7.14)$$

Let us now assume that the time intervals between successive measurements are small enough to satisfy the inequalities

$$\Delta\tau_j\ll\langle n\rangle/\kappa, \quad (7.15)$$

where $\langle n\rangle$ is the initial mean quantum number of the field. For a broad class of initial fields, these inequalities can be well satisfied by time intervals large enough to satisfy the conditions (2.25). When they are satisfied, we may assume that at most one quantum can be absorbed by the detector during the interval $\Delta\tau_j$, i.e., that m_{j+1} is at most m_j+1, and that the probability of this happening is very small. We may then think of the detector as registering counts, i.e., absorbing quanta, during definite small time intervals. Let us define the function $p_m'(t_1\cdots t_m;\,t)$ as the joint probability of recording counts during the time intervals Δt_j immediately following the times t_j, *and of not recording counts during any other time intervals* between the initial time and some time $t\geq t_m$. It is clear from Eq. (7.13) that this probability may be expressed as a linear functional of the initial field state,

$$p_m'(t_1\cdots t_m;\,t)=\mathrm{tr}_F\{\rho^{(F)}(0)\mathcal{O}_m'(t_1\cdots t_m;\,t)\}, \quad (7.16)$$

where $\mathcal{O}_m'(t_1\cdots t_m;\,t)$ may be evaluated by letting the numbers m_j in Eq. (7.14) be the total number of counts recorded by the times τ_j.

It is also useful to introduce the probability $p_m(t_1\cdots t_m)$ of counts being recorded during the time intervals Δt_j immediately following the times t_j, *irrespective of whether or not counts are recorded at any other times*. It is clear that the set of probabilities p_m is linearly related to the set of probabilities p_m', and that therefore the probabilities p_m may also be expressed as a linear functional of the initial density operator for the field,

$$p_m(t_1\cdots t_m)=\mathrm{tr}_F\{\rho^{(F)}(0)\mathcal{O}_m(t_1\cdots t_m)\}. \quad (7.17)$$

The solutions for p_m' and p_m when the initial state of the field is the coherent state $|\alpha_0\rangle_F$ are particularly easy to obtain, with the aid of the results of Sec. VI. We have shown that if the state of the system at the beginning of some interval Δt is the product of a coherent state $|\alpha\rangle_F$ for the field and a nonradiant state of the detector, then the state of the system at the end of the interval is the product of the coherent state $|e^{-z\Delta t}\alpha\rangle_F$ for the field and another nonradiant state of the detector. Since the process of measuring M does not affect the nonradiant character of the states of the detector, it is clear that after any number of measurements, the state of the system at time t remains the product of the coherent state $|\alpha(t)\rangle_F$ for the field and some state for the detector, regardless of the outcome of the measurements. The complex amplitude $\alpha(t)$ of the field state is given by Eq. (3.12), and although the detector state will depend upon the outcome of the measurements, i.e., on the number of counts that have been recorded previous to time t and on when these counts have been recorded, it will nevertheless be a nonradiant state with a specified number of quanta. As such, it will absorb quanta from the field during some time interval Δt following the time t in the same way that the ground state would. The probability of any number of quanta being absorbed between t and $t+\Delta t$ is thus independent of what has happened previously, and may be found by evaluating Eqs. (3.10) and (3.11) with α_0 replaced by $\alpha(t)$, and t by Δt. If we evaluate these expressions for $\Delta t\ll|\alpha_0|^2/\kappa$, we find that the probability of one quantum being absorbed is

$$p_1=|\alpha(t)|^2 2\kappa\Delta t, \quad (7.18a)$$

the probability of no quanta being absorbed is

$$p_0=1-|\alpha(t)|^2 2\kappa\Delta t, \quad (7.18b)$$

and all other possibilities may be neglected.

Since in this case the probability of a count being recorded at any time is independent of what happens at other times, it follows that the probability of counts being recorded within the intervals $\Delta t_1\cdots\Delta t_m$ following the times $t_1\cdots t_m$, irrespective of what happens at

other times, is given by the product

$$p_m(t_1\cdots t_m)=\prod_{j=1}^{m}[|e^{-zt_j}\alpha_0|^2 2\kappa\Delta t_j] \tag{7.19a}$$

$$=\prod_{j=1}^{m}[|\alpha_0|^2 e^{-2\kappa t_j}2\kappa\Delta t_j], \tag{7.19b}$$

since $\mathrm{Re}(z)=\kappa$. This expression is exactly what one would write down in a semiclassical theory, if one postulated that the intensity of the classical field which induces the excitations in the detector decays exponentially. The decay rate for the field intensity—the factor multiplying $-t_j$ in the exponential function in Eq. (7.19b)—is equal to the absorption rate per unit field strength, i.e., the factor multiplying Δt_j.

If the initial density operator for the field possesses a P representation as expressed by Eq. (3.16), then the probability $p_m(t_1\cdots t_m)$ may be expressed in terms of a statistical distribution of attenuated classical fields, as

$$p_m(t_1\cdots t_m)=\int\{\prod_{j=1}^{m}[|\alpha_0|^2 e^{-2\kappa t_j}2\kappa\Delta t_j]\}\times P(\alpha_0,0)d^2\alpha_0. \tag{7.20}$$

A general expression for $p_m(t_1\cdots t_m)$ may be found by noting that Eqs. (7.19) may be expressed as

$$p_m(t_1\cdots t_m)={}_F\langle\alpha_0|(e^{-zt_1}a)^\dagger\cdots(e^{-zt_m}a)^\dagger \times e^{-zt_m}a\cdots e^{-zt_1}a|\alpha_0\rangle_F\prod_{j=1}^{m}2\kappa\Delta t_j \tag{7.21a}$$

$$=\exp[-\sum_{j=1}^{m}2\kappa t_j]\,{}_F\langle\alpha_0|a^{\dagger m}a^m|\alpha_0\rangle_F\times\prod_{j=1}^{m}2\kappa\Delta t_j. \tag{7.21b}$$

By comparing these relations to the form Eq. (7.17) takes for $\rho^{(F)}(0)=|\alpha_0\rangle_F\,{}_F\langle\alpha_0|$, and recalling that any operator $\mathcal{P}_m$ is determined[17] by the function ${}_F\langle\alpha_0|\mathcal{P}_m|\alpha_0\rangle_F$, we see that the general formula for the probability $p_m(t_1\cdots t_m)$, for arbitrary initial fields, may be expressed in the form

$$p_m(t_1\cdots t_m)=G^{(m)}(t_1\cdots t_m;t_m\cdots t_1)\prod_{j=1}^{m}2\kappa\Delta t_j, \tag{7.22}$$

where the function $G^{(m)}(t_1\cdots t_m;t_m'\cdots t_1')$ may be defined for arbitrary $2m$ times $t_1\cdots t_m$, $t_1'\cdots t_m'$ as

$$G^{(m)}(t_1\cdots t_m;t_m'\cdots t_1')\equiv\mathrm{tr}_F\{\rho^{(F)}(0)(e^{-zt_1}a)^\dagger\cdots\times(e^{-zt_m}a)^\dagger e^{-zt_m'}a\cdots e^{-zt_1'}a\}, \tag{7.23}$$

and therefore takes the form, for $t_j=t_j'$,

$$G^{(m)}(t_1\cdots t_m;t_m\cdots t_1)=\exp[-\sum_{j=1}^{m}2\kappa t_j]\times\mathrm{tr}_F\{\rho^{(F)}(0)a^{\dagger m}a^m\}. \tag{7.24}$$

The formula (7.23) for the correlation function $G^{(m)}$ is similar to the one given by Glauber.[4] The important difference is that the attenuation of the field implies that the operator

$$e^{-zt}a=e^{-i(\omega+\delta\omega)t-\kappa t}a \tag{7.25}$$

must be used in place of the interaction-picture operator $e^{-i\omega t}a$.

The probability $p_m(t)$ of recording a total of m counts between the initial time $t=0$ and time t may be evaluated by noting that since the probabilities $p_m(t_1\cdots t_m)$ are given by Eq. (7.22), the generating function defined by Eq. (3.30) is given by[4]

$$Q(\xi,t)=\sum_{m=1}^{\infty}\frac{(-2\kappa\xi)^m}{m!}\int_0^t dt_1\cdots\int_0^t dt_m\, G^{(m)}(t_1\cdots t_m;t_m\cdots t_1). \tag{7.26}$$

If we substitute Eq. (7.24) for $G^{(m)}(t_1\cdots t_m;t_m\cdots t_1)$ into this relation and perform the indicated integrations, we find the result given by Eqs. (3.32) and (3.34). We have thus exhibited by an explicit calculation the result proved earlier in this section, that the probabilities $p_m(t)$ are not affected by repeated measurements of the detector between the initial time and time t.

The joint probability $p_m'(t_1\cdots t_m;t)$ of counts being recorded in the intervals following the times $t_1\cdots t_m$ and of no counts being recorded in any of the other time intervals between the initial time and some time $t\geq t_m$ is easily evaluated for an initially coherent state of the field. If we make use of Eqs. (7.18), we find

$$p_m'(t_1\cdots t_m;t)=\{\prod_{j=1}^{m}[|\alpha(t_j)|^2 2\kappa\Delta t_j]\}\times\{\prod_{k\neq j}[1-|\alpha(t_k)|^2 2\kappa\Delta t_k]\}, \tag{7.27}$$

in which the second product is taken over all time intervals Δt_k during which no counts take place. In the limit $\Delta t_j\to 0$, these intervals comprise the entire time interval, and we may write[26]

$$\prod_{k\neq j}[1-|\alpha(t_k)|^2 2\kappa\Delta t_k]=\exp\{-\sum_k|\alpha(t_k)|^2 2\kappa\Delta t_k\}=\exp\left\{-2\kappa\int_0^t|\alpha(t')|^2dt'\right\}=\exp\{-|\alpha_0|^2(1-e^{-2\kappa t})\}, \tag{7.28}$$

where the last step follows from the definition (3.12) of $\alpha(t)$. If we substitute Eq. (7.28) into Eq. (7.27) and

[26] A similar calculation has been performed by L. Mandel, Phys. Rev. **152**, 438 (1966).

make use of Eq. (3.12), we find

$$p_m'(t_1\cdots t_m;\, t)$$

$$=\exp\{-\sum_{j=1}^{m} 2\kappa t_j-|\alpha_0|^2(1-e^{-2\kappa t})\}$$

$$\times|\alpha_0|^{2m}\prod_{j=1}^{m}2\kappa\Delta t_j \quad (7.29)$$

$$=\exp[-\sum_{j=1}^{m} 2\kappa t_j]_F\langle\alpha_0|\, a^{\dagger m}:\exp\{-a^\dagger a(1-e^{-2\kappa t})\}:$$

$$\times a^m|\alpha_0\rangle_F\prod_{j=1}^{m}2\kappa\Delta t_j, \quad (7.30)$$

and if we compare this relation to Eq. (7.16) and make use of the fact that the operator $\mathcal{O}_m'$ is determined[17] by the function ${}_F\langle\alpha_0|\,\mathcal{O}_m'|\alpha_0\rangle_F$, we find, for arbitrary initial fields,

$$p_m'(t_1\cdots t_m;\, t)$$

$$=\exp[-\sum_{j=1}^{m} 2\kappa t_j]\ \mathrm{tr}_F\{\rho^{(F)}(0)a^{\dagger m}$$

$$\times:\exp[-a^\dagger a(1-e^{-2\kappa t})]:\, a^m\}\prod_{j=1}^{m}2\kappa\Delta t_j \quad (7.31)$$

$$=\exp[-\sum_{j=1}^{m} 2\kappa t_j]\ \mathrm{tr}_F\{\rho^{(F)}(0)a^{\dagger m}e^{-2\kappa t a^\dagger a}a^m\}$$

$$\times\prod_{j=1}^{m}2\kappa\Delta t_j, \quad (7.32)$$

where the identity (3.21) was used to reach the latter expression.

If the field is initially in the n-quantum state $|n\rangle_F$, Eq. (7.32) takes the form

$$p_m'^{(n)}(t_1\cdots t_m;\, t)$$

$$=\exp[-\sum_{j=1}^{m} 2\kappa t_j-2\kappa t(n-m)]$$

$$\times\frac{n!}{(n-m)!}\prod_{j=1}^{m}2\kappa\Delta t_j \quad \text{for} \quad m\leq n \quad (7.33a)$$

$$=0 \quad \text{for} \quad m>n. \quad (7.33b)$$

It is interesting to observe that Eq. (7.33a) may be written in the form

$$p_m'^{(n)}(t_1\cdots t_m;\, t)$$

$$=\prod_{j=1}^{m}[e^{-(n-j+1)2\kappa(t_j-t_{j-1})}2(n-j+1)\kappa\Delta t_j]$$

$$\times e^{-(n-m)2\kappa(t-t_m)}, \quad (7.34)$$

where $t_0\equiv 0$. This is a relation which can be deduced directly with the aid of the results of Sec. VI. Since the states of the detector at any time are nonradiant, the probability of a quantum being absorbed between the tines t and $t+\Delta t$, if the field has exactly n quanta at time t, is the same as it would be if the detector were in its ground state at time t, and is therefore equal to $2n\kappa\Delta t$. The quantum-number distribution for the field is thus described by a time-independent Markov process, a result which was found in Eq. (4.24), for the case in which the detector is not monitored.

The general expression for p_m' is, by virtue of the diagonal nature of the operator expression in Eq. (7.32),

$$p_m'(t_1\cdots t_m;\, t)=\sum_n \rho_{nn}^{(F)}(0)p_m'^{(n)}(t_1\cdots t_m;\, t). \quad (7.35)$$

It is interesting to observe that the Eq. (7.34) is what we would write down if we pictured the initial state of the system as containing n distinguishable quanta, and postulated that the probability of any quantum being absorbed between the time t and $t+\Delta t$, if it has not been absorbed before time t, is $2\kappa\Delta t$. The complete description of photocounts given by Eq. (7.35) would then correspond to a statistical distribution of the initial number of such distinguishable quanta. When a non-negative P representation exists, this way of picturing the absorption process gives the same results as the semiclassical picture described earlier, in which the probability of recording a count is proportional to a classical field intensity, and the initial state of the quantum-mechanical field is thought of as an ensemble of classical fields, each of which becomes attenuated during the detection process.

VIII. SPACE-TIME CORRELATIONS IN PHOTOCOUNTS

In actual photon counting experiments, the spatial distribution of detecting atoms is not uniform, but is nonvanishing in a relatively limited region of space. In describing such experiments, however, we shall be forced to assume that the number density of detecting atoms varies slowly over many wavelengths of the field. We shall then divide the entire region of space in which the experiment takes place into many smaller regions, throughout each one of which we shall assume the number density of detecting atoms to be relatively constant. The assumption of spatial homogeneity in each region is a necessary one if we are to make use of the results of the preceding sections, which were based on the assumption that the field consists of a single mode of oscillation throughout the detection process. If the spatial distribution of atoms in a given region were inhomogeneous, then an initial field state consisting of a single excited mode would become attenuated by different amounts at different points within the region, and hence would eventually consist of many excited modes. The different modes of the field are all coupled to the same detecting atoms, and hence exert an influence on each other even though they are not coupled directly by the Hamiltonian.

It is necessary for this reason to begin our discussion by continuing to assume that the detecting atoms are homogeneously distributed throughout a region of volume V, and establishing the conditions under which the different modes of field may be treated independently of one another. We shall then treat more realistic situations by letting the region in question be a small part of a larger region, over which significant variations in the density of atoms may take place.

If we substitute the general expression (2.4) for the field within a region of volume V into Eq. (2.8) and make use of Eq. (2.2), we find that the interaction Hamiltonian for the system is

$$H_1(t)=-i\hbar\sum_{k,j} a_k^\dagger(t)g_{kj}b_j(t)+\text{H.c.}, \tag{8.1}$$

in which the coupling parameter g_{kj} is given by Eq. (2.10), with ω and $\hat{e}$ replaced by ω_k and $\hat{e}_k$, respectively.

It will simplify our analysis if we assume that the frequencies of the detecting oscillators are allowed to take on only certain (closely spaced) discrete values, and that there are a large number N_ω of oscillators throughout the region in question with any given allowed frequency ω. Since the oscillators are assumed to be homogeneously distributed, the number of oscillators per unit volume with the same frequency ω is

$$N_\omega(\mathbf{r})=N_\omega/V. \tag{8.2}$$

It is convenient to introduce the function

$$g(\omega_k,\omega)\equiv\tfrac{1}{2}e\left[\frac{\omega}{3\omega_k}\frac{N_\omega}{V}\left(\frac{1}{m_j}\right)\Big|_{\omega_j=\omega}\right]^{1/2}, \tag{8.3}$$

in terms of which the coupling parameter g_{kj} may be expressed as

$$g_{kj}=g(\omega_k,\omega_j)\left(\frac{3}{N_{\omega_j}}\right)^{1/2}(\hat{e}_k\cdot\hat{u}_j)\,e^{-i\mathbf{k}\cdot\mathbf{r}_j}. \tag{8.4}$$

The interaction Hamiltonian (8.1) may then be expressed in the form

$$H_1(t)=-i\hbar\sum_{k,\omega} a_k^\dagger(t)g(\omega_k,\omega)b_{k,\omega}(t)+\text{H.c.}, \tag{8.5}$$

where the operator $b_{k,\omega}(t)$ is defined as a phased sum of annihilation operators with frequency ω:

$$b_{k,\omega}(t)\equiv\frac{1}{g(\omega_k,\omega)}\sum_{\omega_j=\omega} g_{kj}b_j(t) \tag{8.6}$$

$$=\left(\frac{3}{N_\omega}\right)^{1/2}\sum_{\omega_j=\omega}(\hat{e}_k\cdot\hat{u}e_j)^{-i\mathbf{k}\cdot\mathbf{r}_j}b_j(t). \tag{8.7}$$

We have introduced the factor $1/g(\omega_k,\omega)$ into the definition (8.6) of the operator $b_{k,\omega}(t)$ so that it satisfies the usual commutation relation for harmonic oscillators: If we recall that the polarization vectors $\hat{u}_j$ of the detecting oscillators are randomly oriented, we find from Eq. (8.7) the commutation relation

$$[b_{k,\omega}(t),b_{k,\omega}^\dagger(t)]=\frac{3}{N_\omega}\sum_{\omega_j=\omega}(\hat{e}_k\cdot\hat{u}_j)^2$$

$$=1. \tag{8.8}$$

It is not difficult to show that in the limit in which the number density of detecting oscillators approaches infinity but remains constant throughout the cavity, the operators $b_{k,\omega}(t)$ and $b_{k',\omega}(t)$, for $k\neq k'$, describe independent harmonic-oscillator modes. The commutator of $b_{k,\omega}(t)$ and $b_{k',\omega'}^\dagger(t)$ is, according to Eq. (8.8),

$$[b_{k,\omega}(t),b_{k',\omega'}^\dagger(t)]$$

$$=\delta_{\omega\omega'}\frac{3}{N_\omega}\sum_{\omega_j=\omega}(\hat{e}_k\cdot\hat{u}_j)(\hat{e}_{k'}\cdot\hat{u}_j)e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}_j}. \tag{8.9}$$

If there are many oscillators with a given frequency ω within a volume equal to the cube of the wavelength $1/k\sim c/\omega$ of the absorbed radiation,

$$N_\omega(\mathbf{r})c^3/\omega^3\gg 1, \tag{8.10}$$

then summations over j in Eq. (8.9) may be replaced by

$$\sum_j\rightarrow\sum_{\hat{u}_j}\int d^3r_j N_\omega(\mathbf{r}_j). \tag{8.11}$$

If we carry out the summation over the randomly oriented vectors $\hat{u}_j$, we find then that Eq. (8.9) reduces to the relation

$$[b_{k,\omega}(t),b_{k',\omega'}^\dagger(t)]=\delta_{\omega\omega'}\frac{1}{N_\omega}(\hat{e}_k\cdot\hat{e}_{k'})$$

$$\times\int d^3r\, N_\omega(\mathbf{r})e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}. \tag{8.12}$$

It is clear that if the spatial distribution of absorbing oscillators is homogeneous, i.e., if $N_\omega(\mathbf{r})$ is independent of $\mathbf{r}$, then this expression vanishes for any two electromagnetic modes indices $k\neq k'$, so that we have, according to Eq. (8.9),

$$[b_{k,\omega}(t),b_{k',\omega'}^\dagger(t)]=\delta_{\omega\omega'}\delta_{kk'}. \tag{8.13}$$

Thus, in the limit we are considering, any two distinct electromagnetic field modes are coupled by the Hamiltonian (8.5) to independent modes of oscillation of the detector. The equations of motion for the variables which describe the different field modes are therefore decoupled from one another, and the problem of the interaction of field and detector may be solved by treating each field mode separately.

We may note that the condition (8.10), which is a necessary one for the decoupling of the modes, leads to stronger conditions on the number density of detecting

oscillators than are required by the approximations made to solve the Eqs. (2.11). Since the derivations in the Appendix require that the frequencies of the absorbing oscillators take on many values between the frequencies ω and $\omega+\kappa$, it follows that if there are $n(\omega)d\omega$ possible oscillator frequencies between ω and $\omega+d\omega$, we must have

$$n(\omega)\kappa\gg 1. \tag{8.14}$$

Since there are $N_\omega(\mathbf{r})$ oscillators per unit volume with any given frequency ω, the number of oscillators per unit volume per unit frequency range is

$$N(\omega,\mathbf{r})=\bar{N}_\omega(\mathbf{r})n(\omega), \tag{8.15}$$

where $\bar{N}_\omega(\mathbf{r})$ is the average of the function $N_\omega(\mathbf{r})$ over many nearby discrete levels. By multiplying the left-hand sides of the inequalities (8.10) and (8.14) and making use of Eq. (8.15), we find the result which was asserted without proof in Eq. (2.29a). In a similar way we see that the restriction

$$t\ll n(\omega), \tag{8.16}$$

which we must impose on the time t leads, by virtue of the inequality (8.10), to the condition (2.28).

When these conditions are satisfied, the results of the preceding sections are easily generalized to the case in which many modes of the field are excited. Each field mode is then characterized by a decay rate κ_k and a frequency shift $\delta\omega_k$, which are given by Eqs. (2.20) and (2.21), with ω replaced by ω_k. The decay rate is then, by virtue of Eq. (2.27),

$$\kappa_k=\tfrac{1}{12}\pi e^2N(\omega_k,\mathbf{r})\left(\frac{1}{m_j}\right)\Big|_{\omega_j=\omega_k}, \tag{8.17}$$

which of course is independent of $\mathbf{r}$. If the initial state of the detector is the ground state and the initial state of the field is the product of coherent states for each mode,

$$|\alpha_0\rangle_F=\exp[\textstyle\sum_k(a_k^\dagger\alpha_{0k}-\alpha_{0k}^*a_k)]|0\rangle_F, \tag{8.18}$$

then the state of the system at any time is the product of a state for the detector and a coherent state $|\alpha(t)\rangle_F$ for the field. The complex amplitudes of the field state are given by

$$\alpha_k(t)=e^{-z_kt}\alpha_{0k}, \tag{8.19}$$

where

$$z_k\equiv\kappa_k+i(\omega_k+\delta\omega_k). \tag{8.20}$$

The probability of finding m quanta absorbed by the detector at time t is still given by the Poisson distribution (3.11), but the mean absorbed quantum number $\bar{m}(t)$ is now equal to the sum of separate contributions from each field mode,

$$\bar{m}(t)=\textstyle\sum_k|\alpha_{0k}|^2(1-e^{-2\kappa_kt}). \tag{8.21}$$

For arbitrary initial fields, the generating function (3.30) for the photoabsorption probabilities may be expressed as in Eq. (3.32), where the generating operator $\hat{Q}(\xi,t)$ is equal to the product of the generating operators for each mode,

$$\hat{Q}(\xi,t)=\exp\{\textstyle\sum_k a_k^\dagger a_k\ln[1-\xi(1-e^{-2\kappa_kt})]\}. \tag{8.22}$$

The monitoring theory developed in Sec. VII to describe the statistics of the temporal distribution of photocounts may be straightforwardly generalized to the many-mode case. We find that for an initially coherent state of the field, the probability of counts being recorded between the times t_1 and $t_1+\Delta t_1$, $\cdots t_m$ and $t_m+\Delta t_m$, irrespective of what happens at other times, is

$$p_m(t_1\cdots t_m)=\prod_{j=1}^m\{\sum_k|\alpha_k(t_j)|^2\kappa_k\}\Delta t_j, \tag{8.23}$$

where $\alpha_k(t)$ is given by Eq. (8.19). For arbitrary initial fields, the probability $p_m(t_1\cdots t_m)$ may be expressed by first introducing the noncanonical operators

$$a_k'(t)\equiv e^{-z_kt}a_k(0), \tag{8.24}$$

which reflect the attenuation of the field. If we then introduce the correlation function

$$\begin{aligned}G^{(m)}{}_{k_1\cdots k_m;k_1'\cdots k_m'}(t_1\cdots t_m;\,t_m'\cdots t_1')\\ \equiv\mathrm{tr}_F\{\rho^{(F)}(0)a_{k_1}'^\dagger(t_1)\cdots a_{k_m}'^\dagger(t_m)a_{k_m'}'(t_m')\cdots\\ \times a_{k_1'}'(t_1')\},\end{aligned} \tag{8.25}$$

the probability $p_m(t_1\cdots t_m)$ may be expressed as

$$\begin{aligned}p_m(t_1\cdots t_m)=\sum_{k_1\cdots k_m}(2\kappa_{k_1})\cdots(2\kappa_{k_m})\\ \times G^{(m)}{}_{k_1\cdots k_m;k_m\cdots k_1}(t_1\cdots t_m;\,t_m\cdots t_1)\prod_{j=1}^m\Delta t_j.\end{aligned} \tag{8.26}$$

The Eqs. (8.19) and (8.20) may be regarded as the solutions for the time-dependent Fourier components of a classical field propagating through a medium with a frequency-dependent complex dielectric constant. The positive-frequency part of this field may be expressed by means of the Fourier integral representation

$$\mathfrak{A}(\mathbf{r},t)=(\sqrt{\hbar})c(2\pi)^{-3/2}\int\frac{d^3k}{(2ck)^{1/2}}\boldsymbol{\alpha}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{8.27}$$

where $k\equiv|\mathbf{k}|$, and $\boldsymbol{\alpha}(\mathbf{k},t)$ is perpendicular to $\mathbf{k}$. Let us now assume that the bandwidth of the field is narrow enough so that we may ignore the effects of dispersion and of the frequency dependence of the attenuation parameter κ_k. Then the Fourier amplitudes $\boldsymbol{\alpha}(\mathbf{k},t)$ obey the differential equations

$$\frac{\partial}{\partial t}\boldsymbol{\alpha}(\mathbf{k},t)=[-ikc-\kappa]\boldsymbol{\alpha}(\mathbf{k},t), \tag{8.28}$$

where κ is the value of κ_k at the mean frequency of the field. The time evolution of the function $\mathfrak{A}(\mathbf{r},t)$ is then

governed by the equation

$$\frac{\partial}{\partial t}\mathfrak{A}(\mathbf{r},t)=[-i\Omega-\kappa]\mathfrak{A}(\mathbf{r},t), \tag{8.29}$$

where Ω is a (Hermitian) integral operator representing the free propagation of the field: The expression $\Omega\mathfrak{A}(\mathbf{r},t)$ is an abbreviation for the integral

$$\int d^3r'\Omega(\mathbf{r}-\mathbf{r}')\mathfrak{A}(\mathbf{r}',t), \tag{8.30}$$

where

$$\Omega(\mathbf{r}-\mathbf{r}')=\int\frac{d^3k}{(2\pi)^3}(kc)e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}. \tag{8.31}$$

Our analysis so far has depended upon the assumption that the detecting medium is spatially homogeneous. It is clear, however, that our results will remain valid in an approximate sense if we allow the properties of the medium to vary slowly over many wavelengths of the field. The function $\mathfrak{A}(\mathbf{r},t)$ will then obey the differential equation

$$\frac{\partial}{\partial t}\mathfrak{A}(\mathbf{r},t)=[-i\Omega-\kappa(\mathbf{r})]\mathfrak{A}(\mathbf{r},t), \tag{8.32}$$

where the absorption parameter κ, as given by the right-hand side of Eq. (8.17), is now a function of position.

Let us divide the entire region of space under consideration into many subregions, throughout each one of which the absorption function $\kappa(\mathbf{r})$ is relatively constant, and let us then suppose that measurements of the total number of excitations in the detecting atoms in each region are made, at closely spaced intervals of time. In this way we may speak of counts being registered at specified times, and in specified regions of space.

If the field is coherent, the probability $p_m(t_1,V_1\cdots t_m,V_m)$ of recording counts between the times t_j and $t_j+\Delta t_j$, and within the regions V_j, is simply the product of the probabilities for each region considered individually. If we make use of Eq. (8.23) to evaluate these individual probabilities, and recall that κ_k is assumed to be independent of frequency, we find

$$p_m(t_1,V_1\cdots t_m,V_m)=\prod_{j=1}^{m}\left\{\int_{V_j}|\mathfrak{A}(\mathbf{r}_j,t_j)|^2 s(\mathbf{r}_j)d^3r_j\Delta t_j\right\}, \tag{8.33}$$

where

$$s(\mathbf{r})\equiv\frac{4\omega}{\hbar c^2}\kappa(\mathbf{r}), \tag{8.34}$$

and the complex amplitude function $\mathfrak{A}(\mathbf{r},t)$ obeys Eq. (8.29).

For arbitrary initial fields, we may evaluate the time-dependent photoabsorption probabilities by introducing an operator field $\mathbf{A}'(\mathbf{r},t)$ which satisfies the equation

$$\frac{\partial}{\partial t}\mathbf{A}'(\mathbf{r},t)=[-i\Omega-\kappa(\mathbf{r})]\mathbf{A}'(\mathbf{r},t), \tag{8.35}$$

and which is assumed to reduce to the positive-frequency part of the vector potential in the absence of absorption. If we then introduce the correlation function

$$G^{(m)}{}_{\lambda_1\cdots\lambda_m;\,\lambda_{m'}\cdots\lambda_{1'}}(x_1\cdots x_m;\,x_{m'}\cdots x_{1'})\equiv\mathrm{tr}_F\{\rho^{(F)}(0)A_{\lambda_1}{}'^{\dagger}(x_1)\cdots A_{\lambda_m}{}'^{\dagger}(x_m)A_{\lambda_{m'}}{}'(x_{m'})\cdots\times A_{\lambda_{1'}}{}'(x_{1'})\}, \tag{8.36}$$

where x_j means $(\mathbf{r}_j,t_j)$ and the index λ specifies the Cartesian components of the field, then the probability of recording counts in the detecting regions V_j, between the times t_j and $t_j+\Delta t_j$, may be expressed in the form

$$p_m(t_1,V_1\cdots t_m,V_m)=\int_{V_1}d^3r_1 s(\mathbf{r}_1)\Delta t_1\cdots\int_{V_m}d^3r_m s(\mathbf{r}_m)\Delta t_m\sum_{\lambda_1\cdots\lambda_m}\times G^{(m)}{}_{\lambda_1\cdots\lambda_m;\,\lambda_m\cdots\lambda_1}(x_1\cdots x_m;\,x_m\cdots x_1). \tag{8.37}$$

IX. ABSORPTION BY MORE GENERAL SYSTEMS

In this section we shall extend our formulation of photodetection theory so as to include absorption by two-level systems, and by ionizable atoms. We shall restrict our discussion to the case in which the detecting atoms are uniformly distributed throughout a cavity; the case in which the number density of atoms is a function of position can then be described by the methods of the preceding section.

Let us suppose that the cavity is filled with two-level atoms, each of which is characterized by a ground state $|0\rangle_j$ and a single excited state $|1\rangle_j$ with energy $\hbar\omega_j$. The free Hamiltonian for the jth atom is then

$$H_{0j}=\hbar\omega_j M_j, \tag{9.1}$$

where the occupation number operator M_j is defined by

$$M_j|m\rangle_j=m|m\rangle_j, \tag{9.2}$$

for $m=0$, 1. If we introduce the lowering operator B_j by means of the definition

$${}_j\langle m|B_j|m'\rangle_j\equiv\delta_{m0}\delta_{m'1}, \tag{9.3}$$

then we may express M_j as

$$M_j=B_j^{\dagger}B_j. \tag{9.4}$$

The commutator of B_j and its adjoint is

$$[B_j,B_j^{\dagger}]=1-2M_j, \tag{9.5}$$

and if the atom is isolated, the Heisenberg operator $B_j(t)$ is given in terms of its initial value by the relation

$$B_j(t)=e^{-i\omega_j t}B_j(0). \tag{9.6}$$

If the atoms we are considering are small compared to the wavelength of electromagnetic radiation at the same frequency, then the current associated with the jth atom may be considered to be localized at its center of mass $\mathbf{r}_j$, and the current operator for the system of two-level atoms takes the form

$$\mathbf{J}(\mathbf{r},t)=\sum_j \delta(\mathbf{r}-\mathbf{r}_j)\mathbf{J}_j(t)\,, \tag{9.7}$$

where $\mathbf{J}_j(t)$ is the current operator for the jth atom. The positive-frequency part of this operator may be expressed in the form

$$\mathbf{J}_j^{(+)}(t)=i(2)^{-1/2}\epsilon_j\hat{u}_j B_j(t)\,, \tag{9.8}$$

where $\hat{u}_j$ is a unit vector and ϵ_j is a real parameter which may depend upon the frequency ω_j. If we make the resonant approximation (1.2) in the interaction Hamiltonian, we find

$$H_1(t)=-i\hbar\sum_{k,j}\left[a_k^\dagger(t)g_{kj}B_j(t)-a_k(t)g_{kj}{}^*B_j^\dagger(t)\right], \tag{9.9}$$

where

$$g_{kj}=\tfrac{1}{2}c\epsilon_j\left[\hbar\omega_k V\right]^{-1/2}(\hat{e}_k\cdot\hat{u}_j)e^{-i\mathbf{k}\cdot\mathbf{r}_j}. \tag{9.10}$$

The interaction Hamiltonian (9.9) bears a formal resemblance to the one given by Eq. (8.1). The important difference is that the quantities b_j are lowering operators for harmonic oscillators, while the lowering operators B_j refer to two-level atoms. If the total number of atoms is so large, however, that the probability of any single atom becoming excited during the entire detection process is very small, then the harmonic-oscillator states with energies greater than that of the first excited state may be ignored, and the absorption of radiation by harmonic oscillators, on the one hand, and two-level atoms, on the other, is very nearly the same.[8,9] We may demonstrate this equivalence by defining, analogously to Eq. (8.7),

$$b_{k,\omega}(t)\equiv\left(\frac{3}{N_\omega}\right)^{1/2}\sum_{\omega_j=\omega}(\hat{e}_k\cdot\hat{u}_j)e^{-i\mathbf{k}\cdot\mathbf{r}_j}B_j(t)\,, \tag{9.11}$$

where N_ω is the total number of atoms with frequency $\omega_j=\omega$ throughout the cavity. The interaction Hamiltonian (9.9) may then be expressed as in Eq. (8.5), where here we must put

$$g(\omega_k,\omega)=\tfrac{1}{2}c\left[\frac{N_\omega}{3\hbar\omega_k V}\right]^{1/2}\cdot\epsilon_j\big|_{\omega_j=\omega}. \tag{9.12}$$

The commutator of $b_{k,\omega}(t)$ and $b_{k',\omega'}{}^\dagger(t)$ may be evaluated straightforwardly with the aid of the definition (9.11) and the commutation relation (9.5). If we assume that the number density $N_\omega(\mathbf{r})$ of atoms at frequency ω satisfies Eq. (8.10), then we may sum the contributions from the first term on the right-hand side of Eq. (9.5) in the same way as we did in Sec. VIII, leading to the relation (8.13). By including the contributions from the second term on the right-hand side of Eq. (9.5), we find then

$$\left[b_{k,\omega}(t),b_{k',\omega'}{}^\dagger(t)\right]=\delta_{\omega\omega'}\delta_{kk'}+\delta_{\omega\omega'}\frac{3}{N_\omega}\times\sum_{\omega_j=\omega}(\hat{e}_k\cdot\hat{u}_j)(\hat{e}_{k'}\cdot u_j)e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}_j}(-2M_j). \tag{9.13}$$

It is clear that as N_ω approaches infinity, while the total number of quanta absorbed by the atoms remains fixed, the second term on the right-hand side of this relation approaches zero. The commutator evaluated in it thus approaches the value given by Eq. (8.13), and the operator $b_{k,\omega}(t)$ defined by Eq. (9.11) may be thought of as the annihilation operator for a harmonic-oscillator mode. The Hamiltonians (8.1) and (9.9) thus become formally equivalent in the limit we are considering. By comparing the expressions (8.3) and (9.12) for $g(\omega_k,\omega)$, we see then that the system of two-level atoms coupled to the radiation field by the Hamiltonian (9.9) may be represented formally by a set of harmonic oscillators if we adjust the masses of the oscillators to satisfy the relation

$$e\left[\frac{\hbar\omega}{m_j c^2}\right]^{1/2}\Bigg|_{\omega_j=\omega}=\epsilon_j\big|_{\omega_j=\omega}. \tag{9.14}$$

If we make use of this relation in Eq. (8.17), we find that the absorption rate for the kth mode of the field by the two-level atoms is

$$\kappa_k=\tfrac{1}{12}\pi\frac{\epsilon_k{}^2c^2}{\hbar\omega_k}N(\omega_k,\mathbf{r})\,, \tag{9.15a}$$

where

$$\epsilon_k{}^2\equiv\epsilon_j{}^2\big|_{\omega_j=\omega_k}. \tag{9.15b}$$

These results enable us to discuss the absorption of radiation by atoms containing many energy levels above the ground state. If the probability that a single atom becomes excited during the absorption process is very small, then processes involving transitions between two excited levels of an atom may be ignored. Each atom may then be thought of as consisting of many two-level systems, each with the same ground-state energy, and with excited energies equal to those of the excited levels of the atom. The current operator for the atom may then be approximated by the relation

$$\sum_j|0\rangle\langle 0|\mathbf{J}|j\rangle\langle j|+\text{H.c.}=\sum_j i(2)^{-1/2}\epsilon_j\hat{u}_j B_j+\text{H.c.}\,, \tag{9.16}$$

where

$$B_j\equiv|0\rangle\langle j| \tag{9.17}$$

and

$$\epsilon_j\hat{u}_j\equiv-i(2)^{1/2}\langle 0|\mathbf{J}|j\rangle. \tag{9.18}$$

With these definitions, the Hamiltonian for the system of atoms and field may be expressed as in Eqs. (9.9) and (9.10). The absorption rate κ_k for photons in the kth mode of the field may then be evaluated by making

use of Eq. (9.18) in Eqs. (9.15). In Eq. (9.15a), the number $N(\omega,r)$ of two-level atoms per unit frequency per unit volume must be replaced by the number $N(r)$ of actual many-level atoms per unit volume times the number $n(\omega)$ of excited levels per unit frequency for such an atom. We have, then

$$\kappa_k=\tfrac{1}{6}\pi\frac{c^2}{\hbar\omega_k}\{|\langle 0|J|j\rangle|^2\}_{\omega_j=\omega_k}n(\omega_k)N(\mathbf{r}), \quad (9.19)$$

a result which could have been obtained by calculating an absorption rate for each atom by perturbation theory, and then summing over all the atoms in the detector. The perturbation-theory calculation, however, would in effect be based on approximating the coupling between the field and the atoms by the first term on the right-hand side of Eq. (9.9), and hence would be valid only for times small enough so that the probability of a single photon being absorbed remains small.

ACKNOWLEDGMENT

The author would like to express his gratitude to Professor H. N. Pendleton for the very extensive discussions which led to this paper.

APPENDIX

By substituting Eqs. (2.19) for $a(t)$ and $b_j(t)$ into Eqs. (2.11), we find that the c-number functions $\mu(t)$ $\lambda_j(t)$ satisfy the differential equations

$$\left(\frac{d}{dt}+i\omega\right)\mu(t)=-\textstyle\sum_j g_j\lambda_j(t), \quad (A1a)$$

$$\left(\frac{d}{dt}+i\omega_j\right)\lambda_j(t)=g_j^*\mu(t). \quad (A1b)$$

Since $a(0)\equiv a$ and $b_j(0)\equiv b_j$, the initial values of these functions are $\mu(0)=1$, $\lambda_j(0)=0$. By introducing the Laplace transform functions $\bar{\mu}(s)$ and $\bar{\lambda}_j(s)$, we reduce Eqs. (A.1) to linear algebraic equations. The solution for $\mu(s)$ is found to be

$$\mu(s)=\frac{1}{s+i\omega+\mathfrak{F}(s)}, \quad (A2)$$

where

$$\mathfrak{F}(s)=\sum_j\frac{|g_j|^2}{s+i\omega_j}. \quad (A3)$$

In the limit in which the spectrum of frequencies ω_j approaches a continuum, we may replace summations over j by the integral

$$\sum_j\rightarrow\int d\omega' N(\omega'), \quad (A4a)$$

if we also replace the quantity $|g_j|^2$ by its average value for a given frequency,

$$|g_j|^2\rightarrow\{|g_j|^2\}_{\mathrm{av}}|_{\omega_j=\omega'}\equiv g^2(\omega'). \quad (A4b)$$

If we make these substitutions in Eq. (A3) for $\mathrm{Re}(s)\rightarrow 0$, and then evaluate the result at $\mathrm{Im}(s)=-\omega$, we find the approximate result

$$\mathfrak{F}(s)=\kappa+i\delta\omega, \quad (A5)$$

where

$$\kappa=\pi N(\omega)g^2(\omega) \quad (A6a)$$

and

$$\delta\omega=P\int d\omega'\frac{N(\omega')g^2(\omega')}{\omega-\omega'}. \quad (A6b)$$

By substituting Eqs. (A5) and (A6) into Eq. (A2) and inverting the Laplace transform, we find the result given by Eqs. (2.31) and (2.32).