Relations between Partially Conserved Axial-Vector Current, Axial-Vector Charge Commutation Relations, and Conspiracy Theory*

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The Adler self-consistency condition is shown to be true without reference to currents, provided that the pion trajectory is a member of a conspiracy with $|M|\geqq 1$. It is then shown from the Adler self-consistency condition that the antisymmetric part of the amplitude for forward scattering of low-energy low-mass pions off any target is equal to a universal constant multiplied by the isotopic spin of the target. Thus all restrictions which current algebra imposes on hadron scattering amplitudes can be derived from conspiracy theory; production amplitudes are not examined here. We next investigate the question whether a system which contains massless pions satisfying the Adler self-consistency condition possesses a conserved axial-vector current. Assuming that the usual Omnes-type equation can be solved, we find that it does. Finally, the commutation relations between total axial-vector charges are shown to follow from the conservation equation under very general assumptions.

1. INTRODUCTION

THE object of this paper is to study some of the assumptions made in the theory of partially conserved axial-vector currents (PCAC) and current commutators, and to investigate whether they may be derived from analyticity-unitarity assumptions or from one another. To begin with, we investigate the restrictions which PCAC and current-commutation relations impose on hadron scattering amplitudes. One such restriction is the Adler self-consistency condition, which states that scattering amplitudes involving soft pions of low mass must be small. A further restriction is necessary for the self-consistency of the Adler-Weisberger relation. Expressed as a low-energy theorem, the Adler-Weisberger relation gives the axial-vector renormalization constant in terms of the antisymmetric part of the pion-nucleon scattering amplitude. One can obtain similar relations between the axial-vector renormalization constant and the antisymmetric part of the amplitude for the scattering of pions off any target at low energy. By eliminating the axial-vector renormalization constant, one can relate the antisymmetric part of the amplitude for scattering of low-energy pions by different targets. One easily finds that the antisymmetric part of the amplitude must be equal to a universal constant multiplied by the isotopic spin of the target. We shall refer to this relation as the Adler-Weisberger self-consistency condition.

All experimentally verifiable results of PCAC make use of the low mass of the pion, and it is uncertain whether the hypothesis of a partially conserved current has any content except in this approximation. One can only obtain exact results in a system where the pion mass is equal to zero; partial conservation then becomes exact conservation. We shall take the pion mass to be equal to zero throughout this paper; the results should be true to a good approximation in nature, where the square of the pion mass is a good deal smaller than the square of the mass of any other particle.

Chew has made the suggestion that the Adler self-consistency condition might be a consequence of conspiracy theory,[1–3] which shows that several trajectories with different quantum numbers pass through the point $t=0$ at equal values of α, or at values of α which differ by integers. From our point of view the important aspect of the theory is that it places restrictions on the Regge residues involving such trajectories. If the trajectory passes through an integral value of α at $t=0$, as the pion trajectory does in our approximation, the Regge residues are products of two vertex constants associated with the pion. The Regge residue in a multiparticle reaction will be the product of two scattering amplitudes involving the pion. Conspiracy theory would therefore be expected to put a restriction on such amplitudes.

We shall show in Sec. 2 that the restrictions imposed by conspiracy theory do imply the Adler self-consistency condition, provided we assume that the conspiracy quantum number $|M|$ of the pion trajectory is 1 or greater. In outline, the argument will be that we are able to obtain a relation between the sense and nonsense residues associated with the pion trajectory. This relation shows that the sense amplitude is zero if $|M|$ is greater than the spin of the pion, i.e., if $|M|$ is greater than zero. Conspiracy theory only applies when *all four components of the pion momentum are zero,* and scattering amplitudes involving a soft massless pion therefore vanish. This is the Adler self-consistency condition. We shall mention briefly the arguments for believing that $|M|$ for the pion is equal to 1. Such arguments have been quoted by several authors

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[1] D. V. Volkov and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **44**, 1068 (1963) [English transl.: Soviet Phys.—JETP **17**, 720 (1963)].
[2] M. Toller, Nuovo Cimento **53**, 671 (1968).
[3] D. Z. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).

independently of the relation between the value of $|M|$ and the Adler self-consistency condition.

Our object in Sec. 3 is to derive the Adler-Weisberger self-consistency condition from the Adler self-consistency condition. The antisymmetric part of the scattering amplitude is linear in the pion momenta if these momenta are small. Furthermore, by the Adler self-consistency condition, it has to vanish when either pion momentum is zero. These two requirements place strong restrictions on the amplitude. If we write the antisymmetric part of the amplitude in the form $T_\mu(p_1-p_2)_\mu$, where the pion momenta p_1 and p_2 are small, we shall show that the restrictions imply that T_μ satisfies a divergence condition. The Adler-Weisberger self-consistency condition can then be proved following the arguments by which one proves that an electromagnetic field interacts with a conserved quantity.

The result of Secs. 2 and 3 is therefore that all restrictions which PCAC or current commutators place on hadron scattering amplitudes can be obtained without mentioning currents if the pion has $|M|=1$. We have not examined production amplitudes involving several soft pions, but it would be surprising if the situation were different for that case.

The next question, which will be discussed in Sec. 4, is whether the existence of massless particles satisfying the Adler self-consistency condition implies the existence of a conserved axial-vector current. Our aim is to construct matrix elements of the axial-vector current by solving Omnes-type equations. We are not concerned here with the deeper questions of solubility of the equations; we shall assume that Omnes equations such as those for vector-current matrix elements are soluble. There are, however, additional problems posed by axial-vector current conservation, since certain axial-vector current matrix elements are found to acquire a pole at $t=0$ as a consequence of the conservation equation. In a theory with massless pions which satisfy the Adler self-consistency equation, the poles can be attributed to one-pion intermediate states, and we shall find that everything is consistent. In other words, the existence of a conserved axial-vector current in such a theory is on the same footing as the existence of a conserved vector current in a theory with the appropriate symmetry.

Finally, we wish to treat the axial-vector current commutation relations, which we shall do in Sec. 5. Most successful applications of the commutation relations depend only on the commutator between total axial-vector charges and not on the more detailed commutation relation between current densities. In particular, the Adler-Weisberger relation depends only on total-charge commutation relations. We shall show that the total-charge commutation relations can be derived from the conservation of the axial-vector current without further assumption. One way to obtain the result would be to use the methods followed by Adler or Weisberger. They assumed knowledge of the commutation relations and related these to the antisymmetric part of the pion-nucleon scattering amplitude, but it would be equally possible to assume the results of Sec. 3 and to deduce the commutation relations therefrom. This method assumes that the time-ordered product of two axial-vector currents can be defined, and that it satisfies locality properties from which the reduction formulas can be obtained.

While we have no reason to doubt these assumptions we shall show that they are not necessary to derive the commutation relations. One can derive them directly from the properties of amplitudes involving soft pions, by examining the intermediate states involved in the commutator. We shall show that the only intermediate state which gives a contribution is that obtained from the initial or final state by the addition of one soft pion and we shall be able to calculate the contribution from this state explicitly. Our derivation will thus place the axial-vector charge commutation relations on a similar footing to the vector charge commutation relations; they can be obtained from the conservation law by considering possible intermediate states. The type of intermediate state which contributes is different in the two cases. For the vector charge, the intermediate states are the same as the initial states, except possibly for an isotopic-spin rotation, while for the axial-vector charge, they differ from the initial states by the presence of a soft pion. The commutation relations are therefore closely bound up with the properties of scattering amplitudes involving soft pions and, in particular, with the Adler-Weisberger self-consistency condition.

Our axial-vector charge commutation relations will of course involve an arbitrary constant, since the normalization of the charge is arbitrary. The normalization is usually *defined* so that this constant is unity. If we make the Gell-Mann universality assumption that such a normalization is appropriate for the weak interactions, we obtain the Adler-Weisberger relation in the usual way.

2. ADLER SELF-CONSISTENCY CONDITION

In this section we wish to show that scattering amplitudes involving a zero-mass pion with conspiracy quantum number $|M|$ unequal to zero must necessarily satisfy the Adler self-consistency condition. Before doing so we make one or two comments on the reasons for believing that $|M|$ for the pion is in fact equal to 1.

We assume that the residue associated with the pion trajectory in nucleon-nucleon scattering does not vanish at $t=0$, since measurements of backward proton-neutron scattering and of photoproduction of pions seem to require a nonzero residue. It then follows that the pion trajectory must be a member of one of the three types of conspiracy described by Freedman and Wang.[3] Their type-I conspiracy has no pion trajectory, their type-II conspiracy has a pion trajectory which conspires with an axial-vector trajectory, and their type-III conspiracy has a pion trajectory which conspires with a scalar trajectory. If the mass of the pion is small, the axial-vector trajectory of the type-II conspiracy would pass through $\alpha=1$ at approximately the mass of the pion and would choose sense at this point. We would therefore have an axial-vector meson of mass approximately equal to that of the pion. Since no such particle is observed in nature, we reject a type-II conspiracy. With a type-III conspiracy, the scalar trajectory would pass through $\alpha=0$ at approximately the mass of the pion, but it would choose nonsense at this point. It would therefore not give rise to any unobserved particle. Accordingly, we assume that the pion trajectory is a member of a type-III conspiracy. Since a type-III conspiracy has $|M|=1$, we conclude that this value of M is associated with the pion trajectory.[4]

We now return to the problem of obtaining the Adler self-consistency condition from the assumption that the pion is a member of a conspiracy with $|M|=1$. Our starting point is the relation between Lorentz poles and Regge poles. A Lorentz pole at position λ gives rise to a series of Regge poles at $\lambda-n$, where $n=1, 2, 3\cdots$.[5] The residue at the Regge pole factorizes in the form

$$\beta_{m_1m_2,m_3m_4}{}^n=\beta_{m_1m_2}{}^n\beta_{m_3m_4}{}^n, \tag{2.1}$$

where m_1, m_2, m_3, and m_4 are the crossed-channel helicities of the particles. The factor $\beta_{m_1m_2}{}^n$ is then given by the equation

$$\beta_{m_1m_2}{}^n=\sum_s C(s_1,s_2,s,m_1,m_2,m)\gamma_s V_{s,m}{}^{M,\lambda,n}, \tag{2.2}$$

where s_1 and s_2 are the spins of the two particles, m is the total crossed-channel helicity $(m=m_1-m_2)$, and γ_s is a constant related to the residue at the Lorentz pole. The factor V is a purely kinematical coefficient which depends on the properties of the groups $O(3,1)$ and $O(2,1)$ and which has been calculated by Sciarrino and Toller.[6] This coefficient is zero unless

$$|M|\leqq|s|. \tag{2.3}$$

We now observe that the right-hand side of (2.2) involves the helicity only in the known kinematical coefficient V. This feature is crucial to our result. It is valid only at $t=0$, where the Lorentz-pole theory applies; at other values of t the dependence of the Regge residues on the helicity is not given by kinematical considerations. If λ is integral, so that the Regge trajectories pass through integral values of α at $t=0$, the ratio between the sense and nonsense helicities will be determined by the kinematics. From this fact it will be possible to derive the Adler self-consistency condition.

We therefore have to investigate the dependence of the coefficient $V_{s,m}{}^{M,\lambda,n}$ on m. The pion trajectory passes through $\alpha=0$ at $t=0$. Since $\alpha=\lambda-n$ and we are interested in the leading trajectory, we conclude that $\lambda=1$, $n=1$. We have agreed to take $|M|=1$. Furthermore, the Clebsch-Gordan coefficient in (2.2) vanishes unless $m\leqq s$. We then find from Ref. 6, Eqs. (6.6) and (4.25), that

$$V_{s,m}{}^{\pm1,1,1}=0, \qquad m=0 \tag{2.4a}$$

$$V_{s,m}{}^{\pm1,1,1}\neq0, \quad |m|>0. \tag{2.4b}$$

At $\alpha=0$, the value $m=0$ is the sense value while all other values of m are nonsense. It follows that V is nonzero if and only if m has a nonsense value. This last result is valid for other values of M, λ, and n, provided that the inequalities $|m|\leqq s$, $|M|\leqq s$ are satisfied, as they are for Lorentz poles, and provided also that $|M|\geqq\lambda-n$.

From (2.2) and (2.4) we now conclude that either $\beta_{m_1m_2}(t)$ is infinite from some $m>0$ (if γ_3 is infinite) or $\beta_{m_1m_2}(t)=0$ for $m=0$ (if γ_s is finite). It is not difficult to see that the first possibility must be rejected. The factor $\beta_{m_1m_2}(t)$ could be infinite at $t=0$ if there were fixed poles in the J or s planes, if there were singularities other than poles, or if two Regge trajectories intersected at $t=0$. Fixed poles in the J plane are excluded by unitarity, fixed poles in the s plane by assumptions of analyticity in s and t, and singularities other than poles by our assumption of Regge asymptotic behavior. (The cuts in the angular momentum plane which are known to be present cannot result in β being infinite, as the discontinuities associated with them vanish when the angular momentum takes on an integral value of the correct signature.) The possibility of two trajectories intersecting cannot be rejected on *a priori* grounds, and we shall assume that this possibility is not realized. Even if it were, it is probable that our conclusions could still be obtained, but the argument would be rather more involved.

One might ask whether the coefficient γ_s on the right of (2.2) could be infinite, with the β_m's remaining finite for $|m|\geqq1$ as a result of a cancellation. By proceeding from large to small m and using the inequality $|m|\leqq s$, one can easily reject this possibility. The Clebsch-Gordan coefficients vanish for $|m|\geqq1$ if $s=0$, but this value of s is excluded by (2.3).

We thus conclude that $\beta(t)$ is zero when m takes on its sense value of zero. This means that the coupling constant associated with the vertex $AB\pi$ vanishes, provided $m_A=m_B$. The equality of the masses is certainly a necessary condition for our reasoning to apply, since conspiracy theory, and, in particular, Eq. (2.2), only applies to the equal-mass zero momentum

[4] Another possibility is that the residue associated with the pion trajectory does vanish at $t=0$, but that two other conspiring trajectories pass through $\alpha=0$ or $\alpha=1$ near $t=0$. It has been found possible to fit the data with such trajectories, but they usually imply the existence of particles which have not been seen. I should like to thank Dr. G. Ringland for discussions on this point.

[5] In our definition of λ we are following Toller's notation. The relation between λ and the n of Freedman and Wang is $\lambda=n+1$.

[6] A. Sciarrino and M. Toller, J. Math. Phys. 8, 1252 (1967).

transfer amplitude, where all components of the pion momentum vanish individually.

We obtain the Adler self-consistency condition by extending this result to the case where A and B may be multiparticle systems. The reasoning of the last few paragraphs remains valid in this case, provided always that $m_A=m_B$. If, for instance, we consider the case where A consists of a nucleon and a second pion and B consists of a nucleon alone, the vertex $AB\pi$ becomes the pion-nucleon scattering amplitude. We then obtain the result that the amplitude for the process $N_2\pi_2\rightarrow N_1\pi_1$ vanishes if $m_{N_2\pi_2}=m_{N_1}$ or, in other words, if all four components of the momentum of π_1 are zero. The point where this occurs is at the threshold for the physical region (provided $m_{\pi_2}=m_{\pi_1}=0$). We emphasize again that our reasoning does not imply the vanishing of the amplitude when $m_{N_2\pi_2}\neq m_N$, as conspiracy theory does not apply to the unequal-mass case. Our result is thus that the amplitude vanishes if the pion π_1 is soft, i.e., if all the components of its momentum are zero. This is precisely the Adler self-consistency condition.[7]

The assumption that $|M|\geqq 1$ is crucial to our result. If M were equal to zero, inequalities such as (2.4) would not be true.

In certain cases the results we have quoted are well known and follow from simple kinematics. The Regge residue factor for the process $A\rightarrow A+$pion trajectory has a square-root zero at $t=0$ if $m_\pi=0$. One can show this by making use of the usual kinematic-singularity analysis of helicity amplitudes[8] and, indeed, one can obtain a similar result for the process $A\rightarrow B\pi$, where A and B may be simple or composite systems with equal mass and the same intrinsic parity. The vanishing of the amplitude when A and B have opposite parities does not follow from simple kinematics and requires the assumption that the pion is a member of a conspiracy with $|M|=1$. The Adler self-consistency condition applies to such cases, and, in particular, to the case where A is composed of B together with a soft pion. The Regge residue factor β for the process $A\rightarrow B+$pion trajectory then has to behave like t at $t=0$. A square-root zero would be excluded by the analyticity requirements.

3. ADLER-WEISBERGER SELF-CONSISTENCY CONDITION

We begin this section by showing how one can eliminate the weak-interaction constants from the Adler-Weisberger equations for different processes and thereby obtain relations involving hadron amplitudes alone. We shall then proceed with our main purpose of showing that such relations can be derived from strong-interaction considerations alone, without referring to weak interactions or current commutators.

It is most convenient for our purposes to use the Adler-Weisberger equation in the form of a low-energy theorem. The amplitude for forward scattering of pions by nucleons at rest is written in the usual notation:

$$A_{\alpha\beta}=\delta_{\alpha\beta}A^{(+)}(\nu)+\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\tau_\gamma A^{(-)}(\nu)\,, \qquad (3.1)$$

where $A_{\alpha\beta}$ is a matrix in the charge space of the nucleon, ν the laboratory energy, and τ the isotopic-spin matrix. The function $A^{(-)}$ is usually referred to as the antisymmetric part of A, and the crossing relation implies that it is an odd function of ν. We may therefore write

$$A^{(-)}(\nu)=a\nu+O(\nu^2)\,. \qquad (3.2)$$

The Adler-Weisberger relation then states that

$$g^2(g_V^2/g_A^2)=2m^2a\,, \qquad (3.3)$$

where g^2 is the pion-nucleon coupling constant and g_V^2/g_A^2 the inverse of the axial-vector renormalization constant. Again we are taking the pion mass to be zero.

For the forward scattering of pions by any other particle at rest, one can write a slight generalization of (3.1):

$$A_{\alpha\beta}=\delta_{\alpha\beta}A^{(+,1)}(\nu)+\tfrac{1}{2}\{\rho_\alpha,\rho_\beta\}A^{(+,2)}(\nu)+\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\rho_\gamma A^{(-)}(\nu)\,, \qquad (3.4)$$

where the ρ's are the isotopic-spin matrices appropriate to the target particle. One can again expand $A^{(-)}$ in the form (3.2), and one can again derive (3.3). It therefore follows that the constant a must be the same as for pion-nucleon scattering.

We therefore require the following consistency condition, which we shall call the Adler-Weisberger self-consistency condition: *If the amplitude for the forward scattering of a pion by a target particle is written in the form (3.4), and the antisymmetric part $A^{(-)}$ is expanded in powers of ν around $\nu=0$, the coefficient of ν is equal to a universal constant a, which is independent of the target particle.* We wish to derive this condition without the use of current commutators. We shall show that it is a consequence of the Adler self-consistency condition.

We take q_1 and $-q_2$ to be the four-momenta of the incoming and outgoing pion, p_1 and $-p_2$ to be the four-momenta of the incoming and outgoing target particle. Because of conservation of momentum, there will be three independent four-momenta, which we take to be

$$Q=q_1+q_2=p_1+p_2\,, \qquad (3.5a)$$

$$q=\tfrac{1}{2}(q_1-q_2)\,, \qquad (3.5b)$$

$$p=\tfrac{1}{2}(p_1-p_2)\,. \qquad (3.5c)$$

[7] If we are interested in the behavior of multiparticle amplitudes involving a soft pion, we must apply special treatment to the bremsstrahlung diagrams. Owing to the one-particle pole, the nonsense amplitudes involving such diagrams are infinite and the sense amplitudes are finite. The bremsstrahlung diagrams will only affect P-wave pions and will involve the pion momentum in the form $\mathbf{q}/|\mathbf{q}|$. Our results will still apply to S-wave pions. I should like to thank S. Weinberg and J. Weis for discussions on this point.

[8] L.-L. Wang, Phys. Rev. **142**, 1187 (1966).

For the forward scattering of massless particles at $\nu=0$, both pions have zero four-momentum. We therefore expand the amplitude in powers of q:

$$T=T^{(0)}(p,Q)+q_\mu T_3^{(1)}(p,Q)+O(q^2). \tag{3.6}$$

We use the symbol T rather than A simply to indicate that we are expressing the amplitude as a function of the components of the momentum. We shall suppress the isotopic-spin indices α and β. By going to the laboratory system it is easily seen that

$$T_0^{(1)}(p_0,0)=\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\rho_\gamma a, \tag{3.7a}$$

where

$$p_0=(m,0,0,0) \tag{3.7b}$$

and a is the constant in (3.2).

The Adler self-consistency condition can now be used to obtain a restriction on the amplitude $T^{(1)}$. We require that T vanish when either q_1 or q_2 is zero. It is of course easy to obtain an amplitude involving two powers of q which satisfies this restriction, but we wish to obtain an amplitude linear in q. Thus, putting $T=0$ when $q_1=0$, i.e., when $q=-\frac{1}{2}Q$, we obtain

$$T^{(0)}(p,Q)-\tfrac{1}{2}Q_\mu T_\mu^{(1)}(p,Q)=O(Q^2). \tag{3.8a}$$

Similarly, putting $T=0$ when $q_2=0$, i.e., when $q=\frac{1}{2}Q$, we obtain

$$T^{(0)}(p,Q)+\tfrac{1}{2}Q_\mu T_\mu^{(1)}(p,Q)=O(Q^2). \tag{3.8b}$$

Subtracting (3.8a) from (3.8b), we obtain the equation

$$Q_\mu T_\mu^{(1)}(p,Q)=O(Q^2). \tag{3.9}$$

Thus $T_\mu^{(1)}$ *is a vector amplitude which satisfies the divergence condition when Q is small.*

We can treat the low-energy forward scattering of massless pions by a multiparticle system in a similar way.[9] We then take as our variables Q and q, defined by (3.5a) and (3.5b), together with a sufficient number of other momenta which we denote collectively by p. The development from (3.7) to (3.9) will remain valid, and again we find that the amplitude $T_\mu^{(1)}$ satisfies the divergence condition when Q is small.

The divergence condition essentially provides us with the result we require, since it is known that a vector interaction which satisfies the divergence condition when $Q=0$ must involve a coupling constant which is proportional to a conserved quantity. This result has been derived from on-shell analyticity properties by Weinberg.[10] He was interested in proving that the electromagnetic interaction is characterized by a conserved charge, but his methods are equally applicable to the present problem. Following his reasoning line by line, we can show that Eq. (3.9) requires a consistency condition which relates the amplitude for scattering of pions by different target particles. The condition is

$$T_{r,0}^{(1)}(p_0,0)=c\sigma_r, \tag{3.10}$$

where p_0 is defined in (3.7b), r is a subscript characterizing the target particle, σ_r is the matrix element between the initial and final states of the target particle, of some conserved quantity, and c is a universal constant.

From (3.7a) we observe that T must transform under isotopic-spin rotations like the matrix $\epsilon_{\alpha\beta\gamma}\rho_\gamma$. The only conserved quantity which does so is the matrix $\epsilon_{\alpha\beta\gamma}\rho_\gamma$ itself. We must therefore identify σ_r with this matrix and, on comparing (3.10) with (3.2), we conclude that a is a universal constant independent of the target particle. This is the result we wished to prove.

4. DEFINITION OF CONSERVED AXIAL-VECTOR CURRENT

We now assume that we have a system containing a massless pseudoscalar particle and we investigate whether it is possible to define a conserved axial-vector current. It is not our aim to investigate the general problem of the solution of the dispersion-theoretic equations which define currents or form factors. We shall assume that such equations normally have solutions and shall concern ourselves with the particular problems raised by axial-vector current conservation. It will be found that such problems do not arise if the system contains a massless pseudoscalar particle and if the scattering amplitudes satisfy the Adler self-consistency condition.

As is well known, a conserved axial-vector current cannot exist in a theory which possesses neither chiral symmetry nor a massless pseudoscalar particle.[11] Indeed, it was the difficulties in this connection which led to the concept of Goldstone bosons. To indicate the nature of the problems involved, we begin by quoting the usual expression for the axial-vector form factor of the $N\bar{N}$ system:

$$j_\mu^5(q^2)=i\gamma_5\gamma_\mu f_1(q^2)+i\gamma_5\sigma_{\mu\nu}q_\nu f_2(q^2)+i\gamma_5 q_\mu f_3(q^2). \tag{4.1}$$

By using the Dirac equation for the nucleons, we easily find the following formula for the divergence of j:

$$q_\mu j_\mu^5(q^2)=2m\gamma_5 f_1(q^2)+i\gamma_5 q^2 f_3(q^2). \tag{4.2}$$

For conservation, we therefore require

$$f_3(q^2)=(2mi/q^2)f_1(q^2), \tag{4.3}$$

so that the formula for a conserved axial-vector current becomes

$$j_\mu^5(q^2)=\gamma_5\{i\gamma_\mu-2mq_\mu/q^2\}f_1(q^2)+i\gamma_5\sigma_{\mu\nu}q_\nu f_2(q^2). \tag{4.4}$$

The first term has a pole at $q^2=0$. We could eliminate this pole by demanding that $f_1(0)=0$, but the matrix element of j_μ^5 would then vanish when taken between

[9] To avoid the bremsstrahlung diagrams we should work with the amplitude from which the P-wave pion states have been projected out.

[10] S. Weinberg, Phys. Rev. **135**, B1049 (1964).

[11] See, for instance, Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).

nucleon states at rest. When discussing a conserved axial-vector current, one usually implies that this matrix element does not vanish, and we shall assume that $f_1(0)\neq 0$.

On taking the matrix element of (4.4) between states at rest, we find

$$\langle j_i^5(0)\rangle=\{\sigma_i-(\boldsymbol{\sigma}\mathbf{q})q_i/\mathbf{q}^2\}f_1(0)\,, \qquad (4.5a)$$

$$\langle j_0^5(0)\rangle=0\,. \qquad (4.5b)$$

The pole in (4.4) therefore causes no infinity in the matrix element, but it does cause a discontinuity at $q=0$. In order for this discontinuity to disappear $f_1(0)$ would have to be equal to zero.

One can generalize this result to apply to the axial-vector form factor of a particle with arbitrary spin by making use of the analyticity properties of helicity amplitudes. It follows from the results of Ref. 8 that the axial-vector form factor of any particle must vanish like q^2 as q^2 approaches zero. If this is not the case the invariant form factors must have a dynamical pole at $q^2=0$, analogous to the pole in (4.4).

If a theory allows the definition of a conserved axial-vector current, it must therefore have massless particles in order to produce the dynamical pole. From (4.4) we observe that the coupling between the axial-vector current and the massless particle must have the form

$$bq_\mu\,, \qquad (4.6)$$

where b is a constant, so that the particle must be pseudoscalar. We shall henceforth refer to it as a pion. If the pion is coupled to a particle such as the nucleon by a matrix of the form $iG\Gamma_5$, the axial-vector form vector will contain a term

$$-bG(q_\mu/q^2)\Gamma_5\,. \qquad (4.7)$$

The Feynman diagram corresponding to such a term is shown in Fig. 1.

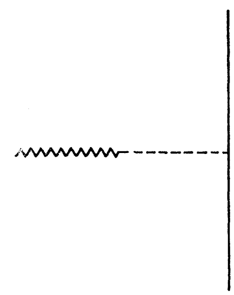

FIG. 1. Pole diagram in a matrix element for a current.

It is not difficult to see that scattering amplitudes involving the pion must satisfy the Adler self-consistency condition. The invariants associated with the axial-vector form factor between two systems of *equal mass but different parity* will have no pole at $q^2=0$. Such invariants have exactly the same properties as those for the vector form factor between two systems of the same parity. On the other hand, Eq. (4.7) indicates that a pole is present unless $G=0$. We therefore conclude that the coupling of the pion between two systems of equal mass but opposite parity vanishes; this is the Adler self-consistency condition.

If our system contains massless particles whose scattering amplitudes satisfy the Adler self-consistency condition, so that poles in the form-factor invariants are allowed, the problem of finding a conserved axial-vector current is very similar to the analogous problem involving the vector current. In general, an axial-vector form factor such as that given in (4.1) will consist of two parts, one being divergenceless and the other the gradient of a pseudoscalar. We shall refer to these parts as the conserved part and the gradient part. By writing dispersion relations in q^2 and using unitarity, one obtains Omnes-type equations for the form factors. Furthermore, the unitarity equations will not mix the conserved part with the gradient part, as the first involves intermediate states with $J=1$, the second intermediate states with $J=0$. One therefore obtains Omnes-type equations for functions such as f_1 and f_2 in (4.4), and they can be treated in the same way as the corresponding equations for the vector form factors.

The value of a form factor at $q=0$ will be fixed once the constant b of (4.6) and the appropriate coupling constant G are known. The residue to the pole in the invariant function will be given by (4.7) and, if this residue is known, one can find the form factor at $q=0$ by an equation such as (4.4). Again the situation is similar to that we encounter with a conserved vector current, when the form factor at $q=0$ is determined by the conservation laws. The situations are not identical since, with a vector current, the form factor at $q=0$ depends only on the quantum numbers of the relevant particle, whereas with an axial current the form factor depends on coupling constants involving the pion.

The normalization of the current is not defined by the Omnes equations, which are linear. The constant b in (4.6) will remain undetermined until the current is normalized. One usually normalizes the axial-vector current through the weak-interaction effective Lagrangian, in which case the function f_1 in (4.4) will be equal to g_A/g_V at $q^2=0$. Comparing the pole terms in (4.4) and (4.7), and putting $G=g$, $\Gamma_5=\gamma_5$ in the latter equation, we then find that

$$b=2mg_A/gg_V\,. \qquad (4.8)$$

The matrix element of the axial-vector current between the one-pion state and the vacuum will be equal to

$$[-i/(2\pi)^{3/2}](1/2p_0)^{1/2}bp_\mu\,, \qquad (4.9)$$

where p is the momentum of the pion. If we assume that this formula is approximately true when the pion mass is small but not zero, and if we use Eq. (4.8) for b, we can obtain the Goldberger-Treiman relation in the usual way.

5. COMPUTATION RELATIONS BETWEEN AXIAL-VECTOR CHARGES

In a theory with a conserved vector current, the commutation relations between total charges follow from the conservation equations, and it is unnecessary to make a separate assumption. We shall show in this section that the same is true of the commutation relations between total axial-vector charges in a theory with a conserved axial-vector current. The intermediate states involved in the axial-vector charge commutator will be different from those involved in the vector charge commutator, since the massless pions play an essential role in the conservation of the axial-vector current, but the final result is similar.

As we explained in Sec. 1, we wish to obtain our result by showing that only a small number of states can contribute to the commutator, and that the matrix elements involving such states can be written down explicitly.

The equal-time commutator between two charge densities is given by the matrix element

$$(2\pi)^2\int dq_{10}dq_{20}\langle N|[j_{0\alpha}{}^5(\mathbf{q}_1,q_{10}),j_{0\beta}{}^5(\mathbf{q}_2,q_{20})]|N\rangle\,,\qquad \mathbf{q}_1=\mathbf{q}_2=0\,, \tag{5.1}$$

where N indicates a nucleon state (or any other state) at rest, and $j_{0\alpha}{}^5$ is the zeroth component of the Fourier transform of the charge density.[12] The subscripts α and β refer to the isotopic spin. With $\mathbf{q}_1=\mathbf{q}_2=0$, axial-vector current conservation would imply that $j_{0\alpha}{}^5$ had a factor $\delta(q_0)$, and the integrations over q_{10} and q_{20} in (5.1) would be trivial. We shall, however, find it necessary to use a limiting procedure in which $\mathbf{q}_1$ and $\mathbf{q}_2$ tend to zero. We therefore leave (5.1) as it stands.

One can now insert a complete set of intermediate states in (5.1). If $\mathbf{q}_1=\mathbf{q}_2=0$, the divergence condition $q_\mu j_\mu(q)=0$ eliminates the intermediate states with energy different from that of the nucleon. On the other hand, the one-nucleon intermediate state itself will not contribute, since we saw in the previous section that the expectation value of $j_0{}^5(0)$ for the one-nucleon state is zero. The only other intermediate state which can possibly contribute is a state consisting of a nucleon and a soft pion. We shall show that the matrix element involving such an intermediate state is singular, and that the state gives a finite contribution to the commutator.

Two singular diagrams for the matrix element of the axial-vector charge between a nucleon state and a nucleon-soft-pion state are shown in Fig. 2. In Fig. 2(a), the pion changes into a current while the nucleon goes on unchanged; in Fig. 2(b) we encounter a pion pole term of the type discussed in the previous section.

[12] We normalize the Fourier transforms in the usual way, i.e., $j(p)=(2\pi)^{-2}\int d^4x\, e^{-ipx}J(x)$. Feynman diagrams will then have a factor $(2\pi)^{-2}$ associated with an external wavy line.

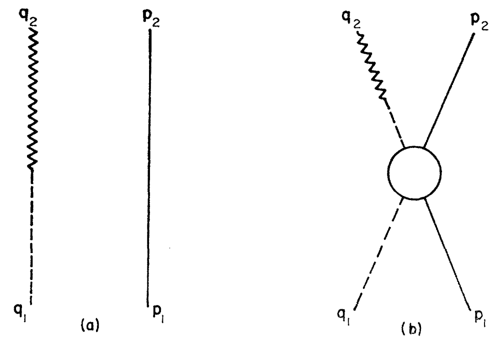

FIG. 2. Important diagrams for a matrix element of an axial-vector charge.

Owing to the singular nature of the quantities involved we shall consider the momenta in (5.1) to be small but nonzero; when we have found the commutator we shall allow the momenta to approach zero. The matrix elements associated with the diagrams are

$$\langle N(\mathbf{p}_2)|j_{\mu,\alpha}{}^5(\mathbf{q}_2q_{20})|N(\mathbf{p}_1),\pi_\beta(\mathbf{q}_1)\rangle_1 = (2\pi)^{1/2}b[q_{2\mu}/(2q_{20})^{1/2}]\delta^3(\mathbf{q}_2-\mathbf{q}_1)\delta(q_{20}-|\mathbf{q}_1|)\times\delta^3(\mathbf{p}_2-\mathbf{p}_1)\delta_{\alpha\beta}\,, \tag{5.2a}$$

$$\langle N(\mathbf{p}_2)|j_{\mu,\alpha}{}^5(\mathbf{q}_2q_{20})|N(\mathbf{p}_1),\pi_\beta(\mathbf{q}_1)\rangle_2 = -\frac{b}{(2\pi)^{5/2}}\frac{1}{(2q_{10})^{1/2}}\frac{q_{2\mu}}{q_{20}{}^2-\mathbf{q}_2{}^2}\delta^3(\mathbf{p}_2+\mathbf{q}_2-\mathbf{p}_1-\mathbf{q}_1)\times\delta(q_{20}-|\mathbf{q}_1|)A_{\alpha\beta}(|\mathbf{q}_1|)\,,\qquad \mathbf{p}_1,\mathbf{p}_2,\mathbf{q}_1,\mathbf{q}_2,q_{20}\text{ small}. \tag{5.2b}$$

The subscripts 1 and 2 on the matrix elements on the left of (5.2) indicate that they correspond to Figs. 2(a) and 2(b), respectively. In writing down (5.2b), we have assumed that $p_{10}=p_{20}=M$, apart from terms of second order which we have dropped.

We draw attention to the factor $1/(q_{20}{}^2-\mathbf{q}^2)$ in (5.2b), which becomes large when q_{20} and $\mathbf{q}$ are small. There will of course be other diagrams for the matrix elements in (5.2), but they will be negligible compared with (5.2b) when q_{20} and $\mathbf{q}$ are small.

We can now evaluate the matrix element

$$\langle N|j_\alpha{}^5 j_\beta{}^5|N\rangle.$$

With pion-nucleon intermediate states, the matrix element becomes $\langle N|j_\alpha{}^5|N\pi\rangle\langle N\pi|j_\beta{}^5|N\rangle$. Each of the two factors will be the sum of two terms corresponding to (5.2a) and (5.2b). The term where both factors correspond to (5.2a) will only be nonzero when $\alpha=\beta$. Since we are interested in finding the commutator of j_α and j_β, we shall restrict ourselves to the case $\alpha\neq\beta$. In the term where both factors correspond to (5.2b), there is an integration over q_{10} and q_{20} [see Eq. (5.1)]; this integration removes the singular factor $1/(q_{20}{}^2-\mathbf{q}^2)$.

The amplitude $A(|\mathbf{q}_1|)$ in (5.2) involves soft pions and is therefore small and, as may easily be verified, the result is that the whole term is small.

We are left with terms where one factor corresponds to (5.2a) and one to (5.2b). The integrations over q_{01} and q_{02} in (5.1), as well as the integration over the intermediate-state pion three-momenta, are trivial because of the δ functions in (5.2a) and (5.2b). We thus obtain the equation

$$(2\pi)^2\int dq_{10}dq_{20}d^3\mathbf{q}'\times\sum_\gamma\langle N(\mathbf{p}_2)|j_{0\alpha}{}^5(\mathbf{q}_2,q_{20})|N(\mathbf{p}_2+\mathbf{q}_2-\mathbf{q}')\pi_\gamma(\mathbf{q}')\rangle_1\times\langle N(\mathbf{p}_2+\mathbf{q}_2-\mathbf{q}')\pi_\gamma(\mathbf{q}')|j_{0\beta}{}^5(\mathbf{q}_1,q_{10})|N(\mathbf{p}_1)\rangle_2$$
$$=\tfrac{1}{2}b^2[|\mathbf{q}_2|/(\mathbf{q}_2{}^2-\mathbf{q}_1{}^2)]A_{\alpha\beta}(|\mathbf{q}_2|)\times\delta^3(\mathbf{p}_1+\mathbf{q}_1-\mathbf{p}_2-\mathbf{q}_2)\,. \quad (5.3\text{a})$$

The corresponding term, with the subscripts 1 and 2 interchanged, is

$$\tfrac{1}{2}b^2[|\mathbf{q}_1|/(\mathbf{q}_1{}^2-\mathbf{q}_2{}^2)]A_{\alpha\beta}(|\mathbf{q}_1|)\times\delta^3(\mathbf{p}_1+\mathbf{q}_1-\mathbf{p}_2-\mathbf{q}_2)\,. \quad (5.3\text{b})$$

In evaluating the amplitude $A_{\alpha\beta}$ in (5.3a) and (5.3b) we shall ignore the symmetric part, as it will not contribute to the commutator of $j_{0\alpha}$ and $j_{0\beta}$. From (3.1) and (3.2) we may therefore write

$$A_{\alpha\beta}(|\mathbf{q}|)=\tfrac{1}{2}a\epsilon_{\alpha\beta\gamma}\tau_\gamma|\mathbf{q}|\,. \quad (5.4)$$

Substituting (5.4) in (5.3) and adding (5.3a) and (5.3b), we are led to the result

$$(2\pi)^2\int dq_{10}dq_{20}\langle N(\mathbf{p}_2)|j_{0\alpha}{}^5(\mathbf{q}_2,q_{20})j_{0\beta}{}^5(\mathbf{q}_1,q_{10})|N(\mathbf{p}_1)\rangle$$
$$=\tfrac{1}{2}b^2\tfrac{1}{2}a\epsilon_{\alpha\beta\gamma}\tau_\gamma\delta^3(\mathbf{p}_1+\mathbf{q}_1-\mathbf{p}_2-\mathbf{q}_2)\,,\quad \mathbf{p}_1,\mathbf{p}_2,\mathbf{q}_1,\mathbf{q}_2\ \text{small}. \quad (5.5)$$

Equation (5.5), together with the corresponding equation in which the order of $j_{0\alpha}{}^5$ and $j_{0\beta}{}^5$ is reversed,

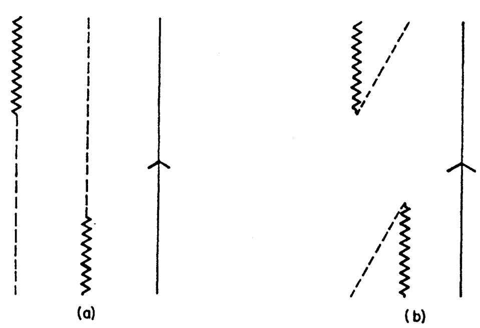

FIG. 3. Diagrams for the matrix element of an axial-vector charge between states containing soft pions.

give us the commutation relation. One can write similar equations for any initial and final states, the matrix τ_γ being replaced by the more general isotopic-spin matrix ρ_γ. Since the initial and final states are arbitrary, we would be tempted to write (5.5) as an operator equation. However, we have not considered the possibility that the initial and final states themselves contain soft pions, which may interact directly with the currents, as in Fig. 3. Figure 3(a) corresponds to an intermediate state with two pions, Fig. 3(b) to an intermediate state with none. It is not difficult to see that the contribution of Fig. 3(a) to the matrix element $\langle j_\alpha{}^5 j_\beta{}^5\rangle$ is equal to the contribution of Fig. 3(b) to the matrix element $\langle j_\beta{}^5 j_\alpha{}^5\rangle$. Such diagrams therefore contribute to the product of two currents but not to their commutator.

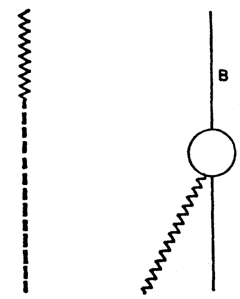

FIG. 4. A diagram which does not contribute to the matrix element of an axial-vector charge.

One may also inquire about diagrams such as Fig. 4, in which only one pion interacts directly with a current. The right half of this diagram is the matrix element of an axial-vector current between states at rest. As before, this matrix element is only nonzero when B consists of A together with a soft pion, i.e., a pion whose momenta are of the same order of magnitude as $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}_1$, and $\mathbf{q}_2$. The phase space associated with the states B which fulfil this condition is of the order of magnitude p^3 and, even if the right half of Fig. 4 contains a pole term such as Fig. 2(b), it is fairly easy to see that the small phase space renders the process unimportant in the limit of vanishing $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}_1$, and $\mathbf{q}_2$. The amplitude associated with Fig. 4 will therefore not contribute to the axial-vector charge commutator.

The presence of soft pions in the initial and final states of (5.5) will thus give rise only to terms which cancel when we take the commutator, and we can write the commutator equation as an operator equation. The matrix τ_γ on the right of (5.5), or its generalization ρ_γ, is just the matrix element of the vector charge between the initial and final states. We may therefore write

$$(2\pi)^2\int dq_{10}dq_{20}[j_{0\alpha}{}^5(\mathbf{q}_2,q_{20})j_{0\beta}{}^5(\mathbf{q}_1,q_{10})]=b^2\tfrac{1}{2}a\epsilon_{\alpha\beta\gamma}j_{0\gamma}(\mathbf{q}_1,0)\delta^3(\mathbf{q}_1-\mathbf{q}_2)\,. \quad (5.6)$$

This is the usual equal-time commutation relation between total axial-vector charges, apart from the factor $\tfrac{1}{2}ab^2$.

One cannot get rid of the numerical factor on the right of (5.6), since the normalization of the current is not defined. One can normalize the current so that the constant $\frac{1}{2}ab^2$ is equal to unity, i.e., so that

$$b^2=2/a. \tag{5.7}$$

Without further assumption one must regard (4.8) and (5.7) as alternative normalizations of the constant b. The Gell-Mann universality assumption is that the axial-vector current, when normalized so that the factor $\frac{1}{2}ab^2$ in (5.6) is unity, enters into the weak-interaction Lagrangian with the same coupling constant as the vector current. With that assumption the normalizations (5.7) and (4.8) are the same and, by eliminating b from the two equations, we obtain the Adler-Weisberger relation

$$g^2g_V^2/g_A^2=2m^2a. \tag{5.8}$$

Nothing in the arguments of Sec. 3 or this section prevents the constant a from being zero. In that case the axial charges would commute, as they do in certain models. There is no theoretical reason for preferring the value $a=0$ to any other value of a, and experimentally a is not equal to zero.

To summarize the contents of this section, the conservation equation shows that most intermediate states give no contribution to the matrix element of the commutator of two axial-vector charges. The important intermediate states are those obtained from the initial state by the addition of one soft pion. In this respect the axial-vector charge is different from the vector charge, where the important intermediate states differ from the initial states only by an isotopic-spin rotation. The difference is due to the fact that conservation of axial-vector charge does not correspond to a symmetry of the system in the usual sense. Nevertheless, the commutation relations have their expected form. They are intimately connected with the Adler-Weisberger self-consistency condition on the antisymmetric part of an amplitude for the forward scattering of a soft pion.

6. CONCLUDING REMARKS

The arguments of Secs. 3 and 4 show that all restrictions which PCAC and current commutators have so far provided on hadron scattering amplitudes can also be obtained from analyticity and unitarity, always on the assumption that the pion trajectory has $|M|=1$. This is not to say that the concepts of PCAC and current commutators play no useful role. Indeed, it was through them that attention was directed to the Adler and Adler-Weisberger self-consistency conditions in the first place. However, the assumption of PCAC and axial-vector charge commutation relations appears to be an alternative to part of the content of the usual strong-interaction assumptions rather than an addition to them.

In this connection it is interesting to note that Gilman and Harari,[13] in their attempt to obtain correlations between resonance parameters from superconvergence relations and current commutators, really only use the current commutators to obtain low-energy theorems of the type discussed in this paper. Their results may therefore be considered as consequences of the usual analyticity and unitarity assumptions.

If one wishes to obtain results related to the conserved axial-vector current itself, as opposed to results related to hadron scattering amplitudes, one has, of course, to make certain assumptions beyond on-shell analyticity. The usual assumption is that a matrix element of a current has the appropriate analyticity and unitarity properties. We are all familiar with such assumptions applied to the vector current and, if one applies them to the axial-vector current, one can put PCAC on the same footing as ordinary current conservation. Furthermore, one can derive the commutation relations for total charge from the conservation of the axial-vector current.

As in the case of the vector charge, the method of obtaining the axial-vector charge commutation relations is to examine the intermediate states involved. Only a small number of intermediate states give a nonzero contribution. The actual intermediate states which come into play with the axial-vector charge commutator are different from those which come into play with the vector charge commutator, owing to the intimate connection of the axial-vector current conservation with soft pions. The details have been discussed in Sec. 4.

The vector and axial-vector charges satisfy the commutation relations of the $SU(2)\times SU(2)$ algebra. Nevertheless, $SU(2)\times SU(2)$ is not a symmetry of the system in the usual sense; the states of the system are multiplets of $SU(2)$ only. Our system is the analog of the canonical field-theoretic model in which the Lagrangian possesses a higher symmetry than the system itself. Such a system must have Goldstone bosons, and it has been known for some time that a massless pion plays the role of a Goldstone boson in a system with a conserved axial current. The existence of such bosons can be studied without reference to a Lagrangian.[11] If a current satisfies a conservation law but the system does not possess the symmetry appropriate to that conservation law, Goldstone bosons must exist. The current commutation relations can then correspond to a larger algebra than the symmetry algebra of the system. In nature it appears that the existence of conserved charges satisfying the $SU(2)\times SU(2)$ algebra is true to a fairly high degree of approximation, whereas $SU(2)\times SU(2)$ symmetry, if it has any meaning at all, is badly broken.

It is important to stress the different footing on which total-charge commutation relations and current-density commutation relations stand. The former are a con-

[13] F. Gilman and H. Harari, Phys. Rev. Letters **18**, 1150 (1967); **19**, 723 (1967); Phys. Rev. **165**, 1803 (1968).

sequence of conservation equations, which in turn are a consequence of symmetry principles (for the vector charges) or of the existence of a massless particle with $|M|\neq 0$ (for the axial charge). The current-density commutation relations represent a further assumption which cannot be obtained from such reasoning. Furthermore, in all applications of current-density commutation relations to experiments which are practicable in the forseeable future, one has to make assumptions regarding the validity of taking the limit $p\rightarrow\infty$ or, alternatively, assumptions regarding unsubtracted dispersion relations for weak amplitudes. Questions regarding the validity of such assumptions, or of the relationship of their validity to the bootstrap assumptions, have not been answered as yet. As we have shown in this paper, most of the sum rules for which we have experimental evidence are consequences of much more general assumptions and have no bearing on such questions. The one exception is the Cabibbo-Radicatti sum rule. Further evidence in its favor, or evidence regarding other current-density sum rules so far untested, will provide answers to questions which go beyond the general arguments of this paper.

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Singularities of the Regge-Pole Model at Mass Zero

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We consider the behavior of a Regge-pole expansion when the mass of a physical particle tends to zero. If this variation of the mass is obtained by letting some parameter λ vary, it is found that the analytic form of the residues must change discontinuously when the mass passes through zero. We call such an effect a transition, by analogy with statistical mechanics. Some consequences are given concerning the reliability of perturbation expansion, zero-mass pions and their Toller classification, and the Pomeranchuk singularity.

1. INTRODUCTION

For the sake of orientation, let us focus on a special case of the kind of problems which we want to consider. Let us assume that we have a reasonable model of strong interactions which predicts a Regge trajectory $\alpha(t)$ and its residue $\beta(t)$. Let us furthermore assume that there is an arbitrary parameter in the model which allows α and β to vary. If $\alpha(t)$ vanishes for some value $t=\mu^2$, and if the value zero of the angular momentum is physical for the reaction we consider, there exists a particle of mass μ. Since the trajectory can vary, we shall take μ^2 as a parameter. We can then make explicit the dependence of α and β as functions of μ^2 by writing them as $\alpha(t,\mu^2)$ and $\beta(t,\mu^2)$.

For positive values of μ^2, $\beta(\mu^2,\mu^2)$ is the coupling constant of the particle (we explicitly assume that $J=0$ makes sense). Letting μ^2 become negative, we generate a ghost. If our model is consistent, it should impose the condition that the residue $\beta(\mu^2,\mu^2)$ vanishes for negative values of μ^2.

Our problem is as follows: We have

$$\begin{aligned}\beta(\mu^2,\mu^2)&\neq 0, \quad \text{for} \quad \mu^2>0\\ \beta(\mu^2,\mu^2)&=0, \quad \text{for} \quad \mu^0<0.\end{aligned}\tag{1.1}$$

There is no known example of a tractable model where the physical quantities are not analytic functions of the parameters. At worst they are different analytic functions for different domains of the parameters. But this "worst" must be realized here for an analytic function $\beta(\mu^2,\mu^2)$ which is identically zero, because a finite set of real values of μ^2 cannot in any case be continued to a function different from zero.

We are then faced with the following problem. A consistent model of strong interactions which allows the mass of a particle to go to zero must give different answers for different sets of values of the parameter. Such a difficulty is quite new in elementary-particle physics, although it is well known in statistical mechanics in connection with phase transitions.[1]

In the present paper, we shall investigate this kind of problem. As we have just seen, it can appear when the mass of a particle goes to zero. In fact, some attention has been given recently to the problem of letting

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[1] See, for instance, K. Huang, *Statistical Mechanics* (John Wiley & Sons, Inc., New York, 1963); M. Kač, G. E. Uhlenbeck, and P. C. Hemmer, J. Math. Phys. 4, 216 (1963); G. E. Uhlenbeck, P. C. Hemmer, and M. Kač, *ibid.* 4, 229 (1963); P. C. Hemmer, M. Kač, and G. E. Uhlenbeck, *ibid.* 5, 60 (1964); P. C. Hemmer, *ibid.* 5, 75 (1964).