d/f ratio, the relation

$$\alpha+\beta+\theta=0.$$

It is hard to see in what sense this could be approximately true in our result.

In conclusion, it is tempting to speculate that perturbation theory is, after all, well suited to strong interactions, except in those cases where its accuracy is *masked* by the presence of resonances. The latter are such a common occurrence that they may have, so far, successfully hidden the basic validity of the Feynman-Dyson techniques. Further, there may be, after all, a well-defined set of particles which are more elementary than the others, and the spin-$\frac{1}{2}$ members of this set could be the baryon octet.

It would be interesting to attempt a program in which all the strong couplings are ultimately determined by the convergence of weak processes.

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Pion Phase-Shift Information from K_{l4} Decays

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The structure of the K_{l4} decay spectrum is displayed in its full generality as a function of a complete set of five kinematic variables; polarization effects are in part similarly described. The nontrivial dynamical aspects of K_{l4} decay reside in certain form factors, which depend on only three of the variables. It is shown that, with large cuts at the two remaining variables, the measurement of the decay spectrum alone suffices to overdetermine the form factors. On certain standard assumptions, these form factors carry information relating to the phase shifts for pion-pion scattering. It is an important practical matter to extract something of this information under conditions of limited statistics, where the spectra have to be treated in partially integrated form. We find that this can be accomplished, with surprising economy, for the energy-dependent phase-shift difference δ_s-δ_p, on the single additional assumption that the dipion system is produced chiefly in s- and p-wave states. The information emerges from the intensity spectrum for K_{e4} decay, and again from the polarization spectrum for $K_{\mu4}$ decay, the spectra being treated as functions of only three variables. Moreover, the structure in two of these variables is simple and therefore relatively undemanding in a statistical sense.

I. INTRODUCTION

AMONG all semileptonic decay processes for which extensive accumulation of data can reasonably be expected in the near future, the K_{l4} reactions $K\to 2\pi+\nu+(e$ or $\mu)$ are singularly rich in their kinematic structure. A number of standard weak-interaction issues arise for these processes: e.g., the validity of the semileptonic $\Delta I=\frac{1}{2}$ and $\Delta S=\Delta Q$ rules, tests of time-reversal invariance, the implications of current algebra, etc. But more exclusively, one has the possibility here of extracting direct information on pion-pion phase shifts over a range of energies.[1–3] Some preliminary experimental results in this connection are already in hand,[4] but much more remains to be done.

The intensity and polarization spectra for K_{l4} decays are functions of five configuration variables. The present

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[1] This was first noted by E. P. Shabalin, Zh. Eksperim. i Teor. Fiz. **44**, 765 (1963) [English transl.: Soviet Phys.—JETP **17**, 517 (1963)]. For further literature on K_{l4} decays see: S. Oneda, Nucl. Phys. **4**, 21 (1957); K. Chadan and S. Oneda, Phys. Rev. Letters **3**, 292 (1959); V. S. Mathur, Nuovo Cimento **14**, 1322 (1959); L. B. Okun and E. P. Shabalin, Zh. Eksperim. i Teor. Fiz. **39**, 345 (1960) [English transl.: Soviet Phys.—JETP **10**, 1252 (1960)]; E. P. Shabalin, Zh. Eksperim. i Teor. Fiz. **37**, 1775 (1959) [English transl.: Soviet Phys.—JETP **12**, 245 (1961)]; G. Ciochetti, Nuovo Cimento **25**, 385 (1962); N. Van Hieu, Zh. Eksperim. i Teor. Fiz. **44**, 162 (1963) [English transl.: Soviet Phys.—JETP **17**, 113 (1963)]; B. Arbuzov, N. Van Hieu, and R. N. Faustov, Zh. Eksperim. i Teor. Fiz. **44**, 329 (1963) [English transl.: Soviet Phys.—JETP **17**, 225 (1963)]; S. Macheda, Progr. Theoret. Phys. (Kyoto) **30**, 724 (1963); A. Toda, *ibid.* **32**, 855 (1964); L. M. Brown and H. Faier, Phys. Rev. Letters **12**, 514 (1964); C. Kacser, P. Singer, and T. N. Truong, Phys. Rev. **137**, B1605 (1965); J. Iliopoulos, Nuovo Cimento **38**, 907 (1965); **39**, 413 (1965); S. Y. Tsai, Progr. Theoret. Phys. (Kyoto) **35**, 281 (1966); F. A. Berends, A. Donnachie, and G. C. Oades, Nucl. Phys. **B3**, 569 1967; Phys. Letters **26B**, 109, 1967.

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[4] For K_{e4} see R. W. Birge *et al.*, Phys. Rev. **139**, B1600 (1965); for $K_{\mu4}$ see D. Cline and W. F. Fry, Phys. Rev. Letters **15**, 293 (1965); D. E. Greiner, W. Z. Osborne, and W. H. Barkas, *ibid.* **13**, 294 (1964); V. Bisi, R. Cester, A. M. Chiesa, and M. Vigone, Phys. Letters **25B**, 572 (1967).

work is principally based on the following crucial remark. If the five variables are properly chosen, then, on the one and only assumption of effectively local coupling of the lepton pairs to hadronic currents, the structure with respect to two of these variables is explicit and independent of hadronic effects. The essential dynamical effects are contained in certain form factors which can depend at most on the remaining three of the five configuration variables. This separation into "2+3" reduces the complexity of the situation and, as we shall see, can be exploited to decompose the over-all structure into intrinsically more interesting parts.

For these purposes the following set of variables, introduced by Cabibbo and Maksymowicz[2] is suitable: the angles θ_π and θ_l which describe the "decay" of the dipion and of the dilepton systems in their respective rest frames; the angle φ between the normals to the planes defined by the dipion and the dilepton; and the invariant masses $\sqrt{s_\pi}$ and $\sqrt{s_l}$ of the dipion and the dilepton. Precise definitions are given in Sec. II. θ_l and φ are the "trivial" variables, dependence on which can be exhibited explicitly.[5] The form factors depend at most on θ_π, s_π, and s_l only.

It seemed worthwhile to us to investigate theoretically the problem in its full dependence on all five variables, even though a correspondingly complete experimental investigation is surely nowhere in sight. The more practical idea is to find out what are the most efficient ways to extract significant information by partial integrations or "cuts" at the data, with minimal *a priori* assumptions about the form factors. Accordingly, we have worked out the structure of the intensity spectrum in its dependence on all five variables, including the effects of finite charged lepton mass (relevant for $K_{\mu4}$ decay). The results are presented in Sec. II. We have also computed (again relevant for $K_{\mu4}$ decay) the structure of the polarization spectrum in its dependence on all the five variables. For a general choice of polarization direction, the spin-dependent effects are excessively complicated. The general formulas are recorded in the Appendix in a not fully reduced form. However, for the particular component of μ-meson polarization normal to the dilepton plane, a considerable simplification occurs. In Sec. IIB we give a fully reduced form for this polarization component, which, as it turns out, is at the same time of greatest potential diagnostic value.

As a first example of optimal cuts at the data, we mention some tests for the very assumption of lepton-pair locality on which all our work is based:

(1) Integrating the intensity spectrum over all variables but θ_l, there results the simple distribution in θ_l given in Eq. (13) below.

(2) Likewise, integrating over all variables but φ, there results the simple φ distribution given in Eq. (14) below.

Of course, locality is not exact in any event, owing to some of the electromagnetic corrections to the leading hadronic effects. However, apart from nonlocality effects, which are presumably quite small (but not less interesting, therefore), we come in Sec. II to the following general conclusion. Not only is it possible, from the intensity spectrum alone, to arrive in principle at an exact determination of all the form factors, with big (and specified) cuts in θ_l and φ, but, in fact, the form factors are overdetermined by our procedures. This is true for $K_{\mu4}$, while the overdetermination is even stronger for K_{e4}, where we may neglect the charged lepton mass. It should be stressed that this conclusion is independent of the assumptions of (a) T invariance, (b) the $\Delta I=\frac{1}{2}$ semileptonic rule, as long as locality is preserved.

These general results of Sec. II can serve, when high statistics some day will warrant it, to extract full information on the form factors. In Sec. III we turn to the more immediate task of seeing what can be learned, particularly about the pion-pion phase shifts, in limited statistics situations. At this stage, the assumptions (a) and (b) just mentioned do come into play. We make a few brief comments about them.

(*a*) *T invariance.* Insofar as this invariance holds true, the partial-wave amplitudes in a decomposition of the form factors with respect to dipion angular momentum must have the phases of the corresponding pion-pion partial-wave scattering amplitudes. In Sec. III, we point out several ways in which the T-invariance assumption can be tested in K_{l4} decays. Our tests allow for the presence of partial waves up to and including d waves. This would seem quite adequate for the energy regime under consideration. For the purposes of Sec. III we shall adopt the assumption of T invariance.

Nevertheless, there remains the question of principle whether one can extract rigorous phase-shift information in the presence of T violation. This problem has been analyzed in detail by Lee and Wu.[6] It follows from their work that indeed one can get this information from a comparison of the conjugate reactions K_{l4}^+ and K_{l4}^-, provided that CPT invariance obtains. Conversely, such a comparison can serve to test CPT itself, and even more directly and obviously, CP invariance. In the present work, however, we are concerned primarily with the question of getting at the phase-shift information, economically and with a quite reasonable theoretical accuracy, from a single decay channel, in particular $K^+\to\pi^++\pi^-+l^++\nu$.

(*b*) *Semileptonic* $\Delta I=\frac{1}{2}$ *rule.* This rule is surely not exact, if only because of electromagnetic corrections. But there is no indication elsewhere in semileptonic

[5] In Ref. 2, the K_{e4} distribution is integrated from the outset over θ_l. As a result, a certain amount of rigorous information gets lost.

[6] T. D. Lee and C. S. Wu, Ann. Rev. Nucl. Sci. 16, 471 (1966).

reactions of violations of the rule beyond, roughly, the 10% level. This accuracy is probably good enough for present purposes, where one would be delighted to extract even qualitative information on the behavior of the π-π phase shifts. Accordingly, the further analysis in Sec. III is based on a presumed validity of $\Delta I=\frac{1}{2}$, so that the even partial-wave amplitudes correspond to states of definite isospin, $I=0$.

Finally, we study in Sec. III the consequences of the following approximation, which allows us to make *substantial* cuts at the data:

(*c*) *Only s and p states of the dipion system play a role in* K_{l4} *decay.*[7] The full structure of the K_{e4} spectrum in this approximation was first displayed by Lee and Wu.[6] While this approximation does not seem unreasonable, we still give at once a test for it, suitable for limited statistics situations. Namely, in the *s*-*p* approximation the intensity spectrum integrated over all variables but θ_π should yield a distribution in θ_π given by Eq. (16) below.

Proceeding, then, with the *s*-*p* approximation we arrive in Sec. III at the following results:

(1) The phase-shift difference $\delta_s(s_\pi)-\delta_p(s_\pi)$ as a function of s_π can be obtained without further assumption and *without loss of information* by (a) integrating[7] over *all* values of s_l and θ_π, and (b) integrating over large specified domains of θ_l and φ, so that these last two variables can be regarded as discrete in a statistical sense. Thus the *s*-*p* approximation is very suitable for a treatment with modest statistics, but, we repeat, with high statistics this approximation need not be made to get at the form factors.

(2) By taking alternative cuts at θ_l and φ, one can find this phase-shift difference in several *independent* ways.

It is our over-all conclusion that the phase-shift information one is seeking can be extracted within modest statistics, despite the five-dimensional character of the K_{l4} phase space and with a reasonable minimum of *a priori* assumptions—assumptions that can themselves be tested within modest statistics.

II. GENERAL RESULTS

A. Kinematic Preliminaries

For definiteness we consider the decay $K^+\to\pi^++\pi^-+l^+\nu$, where l stands for electron or muon. The momentum four-vectors of K^+, π^+, π^-, l^+, and ν are denoted, respectively, by K, k_+, k_-, p, q; and the symbols for the particle masses are defined by $K^2=-M^2$, $k_+^2=k_-^2=-\mu^2$, $p^2=-m^2$, $q^2=0$. It will be convenient in what follows to deal with the independent four-vector combinations

$$P=k_++k_-,\quad Q=k_+-k_-,\quad L=p+q,\quad N=p-q. \quad (1)$$

Apart from spin, K_{l4} decay is kinematically parametrized by five variables. For two of the variables, we take the dipion and dilepton squared masses

$$s_\pi=-P^2,\quad s_l=-L^2, \quad (2)$$

and we note the relations

$$\begin{aligned}Q^2&=s_\pi-4\mu^2,\\ P\cdot L&=-\tfrac{1}{2}(M^2-s_\pi-s_l).\end{aligned} \quad (2')$$

For the remaining three variables we choose: θ_π, the angle formed by the π^+ three-momentum vector, in the dipion rest frame, and the line of flight of the dipion as defined in the K-meson rest frame; θ_l, the similar angle formed by the l^+ three-momentum vector, in the dilepton rest frame, and the line of flight of the dilepton as defined in the K-meson rest frame; φ, the angle between the normals to the planes defined in the K-meson rest frame by the pion pair and the lepton pair.[8] In terms of scalar-product invariants, we have

$$\begin{aligned}Q\cdot L&=-(Q^2/s_\pi)^{1/2}X\cos\theta_\pi,\\ P\cdot N&=(m^2/s_l)P\cdot L-(1-m^2/s_l)X\cos\theta_l,\\ Q\cdot N&=(m^2/s_l)Q\cdot L+(Q^2/s_\pi)^{1/2}(1-m^2/s_l)\\ &\quad\times[P\cdot L\cos\theta_\pi\cos\theta_l+(s_\pi s_l)^{1/2}\sin\theta_\pi\sin\theta_l\cos\varphi],\end{aligned} \quad (3)$$

where

$$X\equiv[(P\cdot L)^2-s_\pi s_l]^{1/2}. \quad (3')$$

We also note the relation

$$\begin{aligned}(QPNL)&\equiv\epsilon_{\mu\nu\rho\sigma}Q_\mu P_\nu N_\rho L_\sigma\\ &=i(s_lQ^2)^{1/2}(1-m^2/s_l)X\sin\theta_\pi\sin\theta_l\sin\varphi.\end{aligned} \quad (4)$$

On the usual picture of vector and axial-vector coupling of the leptons to hadronic currents, the transition amplitude for K_{l4} decay is given by

$$(G/\sqrt{2})\sin\theta_c\langle k_+,k_-|A_\lambda+V_\lambda|K\rangle\bar{u}(q)\gamma_\lambda(1+\gamma_5)v(p), \quad (5)$$

where A_λ and V_λ are the strangeness-changing axial-vector and vector currents, θ_c is the Cabibbo angle,[9] and G is the standard weak-interaction coupling constant. The hadronic matrix element has the structure

$$\begin{aligned}\langle k_+,k_-|A_\lambda+V_\lambda|K\rangle=(1/M)[&fP_\lambda+gQ_\lambda+r(K-P)_\lambda\\ &+(h/M^2)\epsilon_{\lambda\mu\nu\sigma}K_\mu P_\nu Q_\sigma],\end{aligned} \quad (6)$$

the first three terms coming from the axial-vector current, the last from the vector current. The (dimensionless) form factors f, g, r, h are functions of the

[7] Reference 2 deals only with the case where d and higher π-π partial waves are neglected, while not all p-wave contributions are systematically included. [More precisely, it is assumed that the form factor f in Eq. (6) below has negligible p-wave contributions.] Furthermore, the π-π phase shifts are discussed under the assumption that the form factors are independent of s_l. The present work shows a way in which one can dispense with this assumption.

[8] More precisely, φ is the angle between the factors $\mathbf{k}_+\times\mathbf{k}_-$ and $\mathbf{p}\times\mathbf{q}$. The sign of φ is fixed by Eq. (4) below.

[9] Our work hardly hinges on the here assumed equality of the vector and axial-vector Cabibbo angles.

invariant variables P^2, $(K-P)^2$, $K\cdot Q$; equivalently, of the variables s_π, s_l, and θ_π.

The probability, or intensity distribution, summed over lepton spins, takes the form

$$d^5w=\frac{\pi^2}{(2\pi)^8}\frac{G^2\sin^2\theta_c}{16M^5}X\left(\frac{Q^2}{s_\pi}\right)^{1/2}\left(1-\frac{m^2}{s_l}\right)^2 \times I(s_\pi,s_l,\theta_\pi,\theta_l,\varphi)ds_\pi ds_l d\cos\theta_\pi d\cos\theta_l d\varphi. \quad (7)$$

The intensity distribution function I will be recorded below. To describe the expectation value $\langle\boldsymbol{\sigma}\rangle$ of the charged lepton spin vector, we decompose in an orthonormal set of base vectors defined by

$$\begin{aligned}\mathbf{e}_{\parallel}&=\mathbf{p}/|\mathbf{p}|,\\ \mathbf{e}_n&=(\mathbf{p}\times\mathbf{q})/|\mathbf{p}\times\mathbf{q}|,\\ \mathbf{e}_\perp&=[(\mathbf{p}\times\mathbf{q})\times\mathbf{p}]/|\mathbf{p}|\cdot|\mathbf{p}\times\mathbf{q}|,\end{aligned} \quad (8)$$

the momentum vectors $\mathbf{p}$ and $\mathbf{q}$ being referred to the rest frame of the K meson. We write the expectation value $\langle\boldsymbol{\sigma}\rangle$, multiplied by the intensity distribution function I, in the form

$$I\langle\boldsymbol{\sigma}\rangle=A_{\parallel}\mathbf{e}_{\parallel}+A_n\mathbf{e}_n+A_\perp\mathbf{e}_\perp. \quad (9)$$

The quantities $A_{\parallel}$, A_n, $A_\perp$ are functions of the variables s_π, s_l, θ_π, θ_l, φ, and an explicit expression for the normal polarization function A_n will be given in Eq. (12) below. Expressions for the other two functions, $A_{\parallel}$ and $A_\perp$, are given in a not fully reduced form in the Appendix.

For purposes of expressing the distribution functions I and A_n in compact form, it is convenient to introduce the following combination of kinematic factors and form factors:

$$\begin{aligned}F_1&=Xf-(P\cdot L)(Q^2/s_\pi)^{1/2}g\cos\theta_\pi,\\ F_2&=(Q^2s_l)^{1/2}g,\\ F_3&=(Q^2s_l)^{1/2}X(h/M^2),\\ F_4&=(P\cdot L)f-s_lr-X(Q^2/s_\pi)^{1/2}g\cos\theta_\pi.\end{aligned} \quad (10)$$

Notice that F_4 is simply related to the matrix element of the divergence of the axial-vector current: $-iMF_4=\langle\pi^+\pi^-|\partial A_\lambda/\partial x_\lambda|K\rangle$.

B. Distribution Functions

The distribution functions I and A_n have a simple explicit structure in those variables, θ_l and φ, which do not enter into the form factors. It is useful, therefore, to group terms according to their behavior with respect to these variables, since this will serve in a crucial way to "decompose" the intensity and polarization spectra.

For the intensity distribution function I, we have the expression

$$I=I_1+I_2\cos2\theta_l+I_3\sin^2\theta_l\cos2\varphi+I_4\sin2\theta_l\cos\varphi +I_5\sin\theta_l\cos\varphi+I_6\cos\theta_l+I_7\sin\theta_l\sin\varphi +I_8\sin2\theta_l\sin\varphi+I_9\sin^2\theta_l\sin2\varphi, \quad (11)$$

where the functions $I_1, \cdots, I_9$ depend on s_π, s_l, θ_π and are given by

$$\begin{aligned}I_1&=\tfrac{1}{4}[(1+m^2/s_l)|F_1|^2+\tfrac{3}{2}(1+m^2/3s_l)\sin^2\theta_\pi\\&\qquad\times(|F_2|^2+|F_3|^2)+(2m^2/s_l)|F_4|^2],\\ I_2&=-\tfrac{1}{4}(1-m^2/s_l)[|F_1|^2-\tfrac{1}{2}\sin^2\theta_\pi(|F_2|^2+|F_3|^2)],\\ I_3&=-\tfrac{1}{4}(1-m^2/s_l)[|F_2|^2-|F_3|^2]\sin^2\theta_\pi,\\ I_4&=\tfrac{1}{2}(1-m^2/s_l)\,\mathrm{Re}(F_1^*F_2)\sin\theta_\pi,\\ I_5&=-\mathrm{Re}[F_1^*F_3+(m^2/s_l)F_4^*F_2]\sin\theta_\pi,\\ I_6&=-\mathrm{Re}[F_2^*F_3\sin^2\theta_\pi-(m^2/s_l)F_1^*F_4],\\ I_7&=-\mathrm{Im}[F_1^*F_2+(m^2/s_l)F_4^*F_3]\sin\theta_\pi,\\ I_8&=\tfrac{1}{2}(1-m^2/s_l)\,\mathrm{Im}(F_1^*F_3)\sin\theta_\pi,\\ I_9&=-\tfrac{1}{2}(1-m^2/s_l)\,\mathrm{Im}(F_2^*F_3)\sin^2\theta_\pi.\end{aligned} \quad (11')$$

For the polarization distribution function A_n, we find

$$A_n=-(m^2/s_l)^{1/2}(A_1\sin\theta_l\sin2\varphi+A_2\sin\varphi +A_3\sin\varphi\cos\theta+A_4\cos\varphi+A_5\cos\theta_l\cos\varphi +A_6\sin\theta_l\cos2\varphi+A_7\sin\theta_l), \quad (12)$$

where

$$\begin{aligned}A_1&=-\tfrac{1}{2}(|F_2|^2-|F_3|^2)\sin^2\theta_\pi,\\ A_2&=\mathrm{Re}(F_4^*F_2-F_1^*F_3)\sin\theta_\pi,\\ A_3&=\mathrm{Re}(F_1^*F_2-F_4^*F_3)\sin\theta_\pi,\\ A_4&=\mathrm{Im}(F_1^*F_2-F_4^*F_3)\sin\theta_\pi,\\ A_5&=\mathrm{Im}(F_4^*F_2-F_1^*F_3)\sin\theta_\pi,\\ A_6&=\mathrm{Im}(F_2^*F_3)\sin^2\theta_\pi,\\ A_7&=-\mathrm{Im}(F_1^*F_4).\end{aligned} \quad (12')$$

We may observe that the quantities I_3 and I_9, when evaluated for lepton mass m set equal to zero, are simply related to quantities appearing in the polarization spectrum

$$\begin{aligned}2I_3(m=0)&=A_1,\\ -2I_9(m=0)&=A_6.\end{aligned} \quad (12'')$$

C. Decomposition via the θ_l and φ Variables

The form-factor combinations $F_1\cdots F_4$ depend in *a priori* unknown ways on the variables s_π, s_l, and θ_π; indeed, it is the elucidation of this dependence that constitutes the goal of K_{l4} experiments. The intensity and polarization spectra also depend on the variables θ_l and φ, but in a simple way. This simplicity stems from the assumption of locality of the coupling of lepton pairs to hadrons, as reflected in Eq. (5) by the factorization of the transition amplitude into a hadronic part and an explicit, simple lepton part. Even for limited statistics situations, this very basic assumption of locality can be subjected to direct experimental test. For example, suppose that one investigates the decay spectrum in its dependence on the single variable θ_l, all other variables being integrated over. Then locality implies the spectrum shape

$$dw/d\cos\theta_l = a + b\cos\theta_l + c\cos 2\theta_l, \qquad (13)$$

a structure of very simple and limited form. Similarly, the one-variable spectrum in φ has the simple shape

$$dw/d\varphi = \alpha + \beta\cos\varphi + \gamma\sin\varphi + \delta\cos 2\varphi + \epsilon\sin 2\varphi. \qquad (14)$$

With more statistics, one could go on to test the joint distribution in θ_l and φ implied by Eq. (11). Such tests are very important, bearing as they do on the lepton pair locality assumption which is at the very root of our picture of the weak interactions.[10]

For the rest of the discussion, however, let us adopt the locality hypothesis. Observe in Eqs. (11) and (12) that the form factors enter in different combinations in the various terms classified according to behavior with respect to the θ_l, φ variables. This affords the possibility of separating the form factors, one from another, in an economical way. The variables θ_l and φ can be regarded as statistically discrete, in the following sense. Suppose, for example, that one wishes to isolate the coefficient I_6 of $\cos\theta_l$ in Eq. (11), for given values of s_π, s_l, θ_π. It is enough for this purpose to group all events into merely two categories, according to $\cos\theta_l > 0$ and $\cos\theta_l < 0$, independent of φ. The difference in the numbers of events in these two groups then arises solely from the term in question in Eq. (11). The other terms in Eqs. (11) and (12) can similarly be isolated in turn, by lumping the data in at most four domains in the variables θ_l and φ, the number of events in each domain being combined with appropriate algebraic signs. The interested reader can easily work out his own table of rules.

It is formally amusing to observe that the intensity function I contains nine distinctive terms as classified with respect to θ_l and φ. On the other hand, for given values of s_π, s_l, and θ_π, we are dealing with seven real quantities to be determined: namely, the magnitude of $F_1\cdots F_4$ and their three relative phases. It is therefore possible to extract complete information on the form factors solely from the intensity spectrum, without use of polarization data; indeed, the problem is overdetermined. For the case of K_{e4} (with $m=0$), F_4 does not come into play, so that the overdetermination becomes even stronger there.

III. PION-PION PHASE-SHIFT INFORMATION

A. Partial-Wave Expansion

The K_{l4} form factors, of course, carry information on the weak interactions, but they also reflect in intricate ways on the strong interactions. Insofar as time-reversal invariance holds to sufficient accuracy, however, there is one outstanding respect in which the form factors carry *direct* strong-interaction information. In a partial-wave expansion of the form factors with respect to angular momentum of the dipion system, a partial-wave amplitude of definite angular momentum and isospin must have the phase of the corresponding pion-pion scattering amplitude. In the odd partial waves in $K^+ \to \pi^+ + \pi^- + l^+ + \nu$ decay, one deals unambiguously with isospin $I=1$ for the dipion system. For the even partial waves, however, one encounters an *a priori* unknown mixture of $I=0$ and $I=2$ states. The two can in principle be disentangled on the basis of a joint study of $K^+ \to \pi^+ + \pi^- + l^+ + \nu$ and $K^+ \to 2\pi^0 + l^+ + \nu$ decays. For practical purposes, however, we shall invoke the celebrated $\Delta I = \frac{1}{2}$ rule of semileptonic weak reactions, and, accordingly, we suppose that the even partial waves correspond to the definite isospin value $I=0$.

The first couple of terms in a partial-wave expansion of the various form factors can now be indicated:

$$\begin{aligned} f &= \tilde{f}_s e^{i\delta_s} + \tilde{f}_p e^{i\delta_p}\cos\theta_\pi + \cdots, \\ r &= \tilde{r}_s e^{i\delta_s} + \tilde{r}_p e^{i\delta_p}\cos\theta_\pi + \cdots, \\ g &= \tilde{g}_p e^{i\delta_p} + \tilde{g}_d e^{i\delta_d}\cos\theta_\pi + \cdots, \\ h &= \tilde{h}_p e^{i\delta_p} + \tilde{h}_d e^{i\delta_d}\cos\theta_\pi + \cdots. \end{aligned} \qquad (15)$$

The quantities $\tilde{f}_s$, $\tilde{f}_p$, etc., which bear the tilde marks, are *real* functions of the variables s_π, s_l; and the phases $\delta(s_\pi)$ are pion-pion phase shifts evaluated at the invariant dipion mass $\sqrt{s_\pi}$. The subscripts s, p, d, $\cdots$ refer to the angular momentum quantum number. Isospin labels are suppressed, on the understanding that even partial waves are taken to correspond to $I=0$, the odd partial waves in any case corresponding to $I=1$.

It is clear that the partial-wave expansion can *in principle* be carried out in full once the form factors have been determined, as discussed in the previous section, in their full dependence on the variables s_π, s_l, θ_π. But it is fanciful to expect such an experimental achievement in the near future. We are therefore concerned here with the art of extracting partial, but useful information under limited statistics conditions where one has to treat the spectra in partly integrated form.

The phase shifts depend on the single variable s_π, which must of course be retained as a free variable, although in practice one would be willing to average over more or less sizable bins. It would be nice, however, to be able to integrate over the variable s_l without total loss of phase-shift information, and without *a priori* assumptions on the variation of the form factors with s_l. This turns out to be possible, under the following hypothesis. Let us suppose that the pion pair is produced exclusively in s and p waves; or rather, that the amplitudes for the higher partial waves are sufficiently small so as not to seriously distort the deductions to be extracted for partial waves that are retained. Clearly, the assumption is a reasonable one for small values of s_π, whereas at the upper end of the spectrum it could prove to be less reasonable. It is an important point that the assumption can be subjected to experimental

[10] Equations (13) and (14) apply both to $K_{\mu 4}$ and K_{e4} decays.

test, even for limited statistics conditions. Namely, on the s- and p-wave hypothesis, the form factors g and h are taken to be independent of θ_π, whereas f and r are at most linear in $\cos\theta_\pi$. In turn, this implies that the quantities F_2 and F_3 of Eq. (10) are independent of θ_π, whereas F_1 and F_4 are at most linear in $\cos\theta_\pi$. It is easily seen from Eq. (11) that the intensity spectrum in the single variable θ_π, all other variables being integrated over, must on the above hypothesis have the simple form

$$dw/d\cos\theta_\pi = A + B\cos\theta_\pi + C\cos 2\theta_\pi. \tag{16}$$

The correctness of this expression, of course, constitutes only a necessary condition, since higher-order trigonometric terms may happen to vanish on integration over the other variables. With more data one could impose more stringent tests, say for the joint distribution in the variables θ_π, θ_l, φ, the implied structure in these variables being easily read off from Eq. (11). Such tests, it seems to us, are very worthwhile.

For the remainder of the discussion, we accept the low partial-wave hypothesis and go on to consider how the phase-shift difference $\delta_s - \delta_p$ can be determined in an economical way, both from the intensity spectrum of K_{e4} decay and the polarization spectrum of $K_{\mu 4}$ decay.

As is obvious from Eqs. (10) and (15), the functions $F_1 \cdots F_4$ can be expanded under the present assumption in the form[11]

$$\begin{aligned} F_1 &= \tilde{F}_{1,s}e^{i\delta_s} + \tilde{F}_{1,p}e^{i\delta_p}\cos\theta_\pi, \\ F_2 &= \tilde{F}_{2,p}e^{i\delta_p}, \\ F_3 &= \tilde{F}_{3,p}e^{i\delta_p}, \\ F_4 &= \tilde{F}_{4,s}e^{i\delta_s} + \tilde{F}_{4,p}e^{i\delta_p}\cos\theta_\pi, \end{aligned} \tag{17}$$

where the quantities bearing the tilde mark are real functions of s_π and s_l.

B. K_{e4} Intensity Spectrum

For K_{e4} decay, it is an excellent approximation to set the electron mass equal to zero. In this approximation, as is known, the electron polarization is purely longitudinal and there is nothing of dynamical interest to be learned from polarization effects here. Let us, however, consider the intensity spectrum. Suppose that one integrates over the variables s_l and θ_π, thereby regarding the spectrum as a function of only the three variables s_π, θ_l, and φ. According to Eq. (7), this reduced spectrum is given by

$$d^3w = \frac{\pi^2}{(2\pi)^8}\frac{G^2\sin^2\theta_c}{16M^5}\left(\frac{Q^2}{s_\pi}\right)^{1/2} W\langle I\rangle ds_\pi d\cos\theta_l d\varphi, \tag{18}$$

[11] The general partial-wave expansions are obviously given by $F_1 = \sum \tilde{F}_{1,l}e^{i\delta_l}P_l$, $F_2 = \sum \tilde{F}_{2,l}e^{i\delta_l}P_l'$, $F_3 = \sum \tilde{F}_{3,l}e^{i\delta_l}P_l'$, $F_4 = \sum \tilde{F}_{4,l}e^{i\delta_l}P_l$, where $P_l'(z) = dP_l/dz$.

where

$$W = \int\int X ds_l d\cos\theta_\pi = 2\int X ds_l,$$

and where $\langle I\rangle$, which still depends on s_π, θ_l, and φ, is the weighted average defined by

$$W\langle I\rangle = \int\int XI ds_l d\cos\theta_\pi.$$

The function $\langle I\rangle$ has the same decomposition in the variables θ_l and φ as in Eq. (11), with the functions I_i replaced by the corresponding weighted averages $\langle I_i\rangle$, these depending only on the variable s_π.

As we have repeatedly emphasized, the quantities $\langle I_1\rangle \cdots \langle I_9\rangle$ can be separately determined with relative economy, on the basis of the spectral variations in θ_l and φ. Suppose then that this has been accomplished, so that the quantities $\langle I_1\rangle \cdots \langle I_9\rangle$ are separately known as functions of s_π. Recalling that the lepton mass m has been set equal to zero, and invoking the expansions of Eq. (17), we see that

$$\begin{aligned} \langle I_1\rangle &= \tfrac{1}{4}\langle \tilde{F}_{1,s}^2 + \tfrac{1}{3}\tilde{F}_{1,p}^2 + \tilde{F}_{2,p}^2 + \tilde{F}_{3,p}^2\rangle, \\ \langle I_2\rangle &= -\tfrac{1}{4}\langle \tilde{F}_{1,s}^2 + \tfrac{1}{3}\tilde{F}_{1,p}^2 - \tfrac{1}{3}\tilde{F}_{2,p}^2 - \tfrac{1}{3}\tilde{F}_{3,p}^2\rangle, \\ \langle I_3\rangle &= -\tfrac{1}{6}\langle \tilde{F}_{2,p}^2 - \tilde{F}_{3,p}^2\rangle, \\ \langle I_4\rangle &= \tfrac{1}{8}\pi\langle \tilde{F}_{1,}\,\tilde{F}_{2,p}\rangle\cos(\delta_s - \delta_p), \\ \langle I_5\rangle &= -\tfrac{1}{4}\pi\langle \tilde{F}_{1,s}\tilde{F}_{3,p}\rangle\cos(\delta_s - \delta_p), \\ \langle I_6\rangle &= -\tfrac{2}{3}\langle \tilde{F}_{2,p}\tilde{F}_{3,p}\rangle, \\ \langle I_7\rangle &= \tfrac{1}{4}\pi\langle \tilde{F}_{1,s}\tilde{F}_{2,p}\rangle\sin(\delta_s - \delta_p), \\ \langle I_8\rangle &= -\tfrac{1}{8}\pi\langle \tilde{F}_{1,s}\tilde{F}_{3,p}\rangle\sin(\delta_s - \delta_p), \\ \langle I_9\rangle &= 0, \end{aligned} \tag{19}$$

where, of course,

$$\langle \tilde{F}\cdot\tilde{F}\rangle \equiv \int X\tilde{F}\tilde{F}ds_l \Big/ \int X ds_l.$$

The phase-shift difference can evidently be determined in two distinct ways; namely,

$$\tan(\delta - \delta_p) = \tfrac{1}{2}\langle I_7\rangle/\langle I_4\rangle$$

and

$$\tan(\delta - \delta_p) = 2\langle I_8\rangle/\langle I_5\rangle. \tag{20}$$

Equations (20) implement the earlier statements that the phase-shift information in the s-p approximation can be obtained by integrating over s_l and θ_π. Moreover, Eqs. (19) contain several tests of our basic assumptions.

(1) In the absence of d waves, tested for separately by Eq. (16), any observed inconsistency between the two phase-shift determinations given by Eqs. (20) would be an indication for an $I=2$ admixture in the s wave and/or for a breakdown of time-reversal invariance.

(2) The prediction that $\langle I_9 \rangle = 0$ is especially interesting because it continues to hold even if one allows for d-wave states of the dipion system. To the extent that still higher waves are really negligible, a nonvanishing value for $\langle I_9 \rangle$ could arise only from a breakdown of the $\Delta I = \frac{1}{2}$ rule, i.e., from admixture of $I=2$ states in the d wave; and/or, from a breakdown of time-reversal invariance. Notice, however, that F_3 is proportional to h/M^2. Thus if h were comparable to f, g, and r, then F_3 would be small compared to F_1, F_2, and F_4. On this account alone, the quantities $\langle I_5 \rangle$, $\langle I_6 \rangle$, $\langle I_8 \rangle$, and $\langle I_9 \rangle$ might be small. But by the same token, the very presence of any of the terms $\langle I_5 \rangle$, $\langle I_6 \rangle$, $\langle I_8 \rangle$ [and $\langle I_9 \rangle$] can be used, in limited statistics situations, to establish whether the vector current form factor h contributes at all appreciably to K_{l4} decay. For qualitative purposes, one could integrate also over the variable s_π, treating the intensity spectrum as a function solely of the variables θ_l and φ—the "easy" variables.

C. $K_{\mu 4}$ Polarization Spectrum

Recall that the polarization function $A_n(s_\pi, s_l, \theta_\pi, \theta_l, \varphi)$ is defined by the statement that the expectation value of the component of muon spin normal to the dilepton plane is given by $\langle \boldsymbol{\sigma} \cdot \mathbf{e}_n \rangle = A_n / I$. Taking the difference between the decay spectra for spin-up and spin-down muons (quantizing along the normal), we obtain what we shall call the polarization spectrum. It is given by Eq. (7), with the intensity function I replaced by the polarization function A_n. As in the preceding discussion of K_{e4} decay, let us again imagine that one integrates over the variables s_l and θ_π, the polarization spectrum being regarded as a function of s_π, θ_l, and φ. As before, we write this in the form

$$d^3 w_{\text{pol.}} = \frac{\pi^2}{(2\pi)^8} \frac{G^2 \sin^2\theta_c}{16 M^5} \left(\frac{Q^2}{s_\pi}\right)^{1/2} W \langle A_n \rangle ds_\pi d\cos\theta_l d\varphi, \quad (21)$$

where

$$W \langle A_n \rangle = \int\int X (1 - m^2/s_l)^2 A_n ds_l d\cos\theta_\pi .$$

The function $\langle A_n \rangle$ depends on s_π, θ_l, and φ, and it decomposes with respect to θ_l and φ as in Eq. (12), the quantities A_i being replaced by the weighted averages $\langle A_i \rangle$, which depend only on s_π.

Again invoking the expansions of Eq. (17), we find

$$\begin{aligned}
\langle A_1 \rangle &= -\tfrac{1}{3} \langle \tilde{F}_{2,p}{}^2 - \tilde{F}_{3,p}{}^2 \rangle, \\
\langle A_2 \rangle &= \tfrac{1}{4}\pi \langle \tilde{F}_{4,s} \tilde{F}_{2,p} - \tilde{F}_{1,s} \tilde{F}_{3,p} \rangle \cos(\delta_s - \delta_p), \\
\langle A_3 \rangle &= \tfrac{1}{4}\pi \langle \tilde{F}_{1,s} \tilde{F}_{2,p} - \tilde{F}_{4,s} \tilde{F}_{3,p} \rangle \cos(\delta_s - \delta_p), \\
\langle A_4 \rangle &= -\tfrac{1}{4}\pi \langle \tilde{F}_{1,s} \tilde{F}_{2,p} - \tilde{F}_{4,s} \tilde{F}_{3,p} \rangle \sin(\delta_s - \delta_p), \quad (22) \\
\langle A_5 \rangle &= -\tfrac{1}{4}\pi \langle \tilde{F}_{4,s} \tilde{F}_{2,p} - \tilde{F}_{1,s} \tilde{F}_{3,p} \rangle \sin(\delta_s - \delta_p), \\
\langle A_6 \rangle &= \langle A_7 \rangle = 0.
\end{aligned}$$

We again find two distinct determinations of the phase-shift difference

$$\tan(\delta_s - \delta_p) = -\langle A_4 \rangle / \langle A_3 \rangle$$

and

$$\tan(\delta - \delta_p) = -\langle A_5 \rangle / \langle A_2 \rangle . \quad (23)$$

As already observed in connection with Eq. (12″), the quantities A_6 and $I_9(m=0)$ are simply related. So the remarks applied in the preceding subsection to the vanishing of $\langle I_9 \rangle$ also apply to the vanishing of $\langle A_6 \rangle$. On the other hand, the prediction that $\langle A_7 \rangle = 0$ would no longer follow if one were to include d-wave dipion states in the analysis.

D. Concluding Remarks

Apart from the relatively standard, and independently testable, assumptions of the adequacy of the semileptonic $\Delta I = \frac{1}{2}$ rule, and of time-reversal invariance, our single additional assumption has been that s- and p-wave dipion states dominate over the higher partial waves in K_{l4} decay. We have then seen how a study of the intensity spectrum $d^3w/ds_\pi d\cos\theta_l d\varphi$ for K_{e4} decay and of the polarization spectrum $d^3 w_{\text{pol.}}/ds_\pi d\cos\theta_l d\varphi$ for $K_{\mu 4}$ decay can be used to establish the phase-shift difference $\delta_s(s_\pi) - \delta_p(s_\pi)$, along with other information of interest. We have emphasized that the structure in the variables θ_l and φ has a simple explicit form which permits one to treat these variables in "block" form, mitigating thereby the demands on statistics that one would otherwise envisage for the determination of a spectrum in three variables.

We have *not* had to discuss how the form factors vary with s_l, or how the *magnitudes* of the partial-wave amplitudes vary with s_π; in this sense the phase-shift information comes through as if we were dealing with free π-π scattering. (This is no longer true if we go beyond our s, p approximation.) No doubt the magnitudes of the partial-wave amplitudes have a dependence on s_π which reflects on the properties of π-π scattering. But the connection is model-dependent, whereas the phases are directly identifiable with the corresponding π-π phase shifts in a model-independent way. Various models are of course available in the literature for estimating the effects of final-state scattering on the magnitudes of decay amplitudes; but it would seem safest to get at the phase-shift information in a direct way.

APPENDIX

Corresponding to each one of the three polarization functions $A_{||}$, A_n, $A_\perp$ described in the text, we introduce a covariant spin vector S_μ satisfying $S^2 = 1$, $S \cdot p = 0$. Namely,

$$\begin{aligned}
S_\mu{}^{(||)} &= -(1/m\beta^{1/2})[(K \cdot p) p_\mu + m^2 K_\mu], \\
S_\mu{}^{(n)} &= -(i/\rho^{(n)}) \epsilon_{\mu\nu\lambda\sigma} N_\nu L_\lambda P_\sigma, \\
S_\mu{}^{(\perp)} &= (1/\rho^{(\perp)})[\alpha p_\mu + \beta q_\mu + \gamma K_\mu],
\end{aligned}$$

where

$$\alpha = -[(K\cdot p)(K\cdot q)+M^2(p\cdot q)],$$
$$\beta = [(K\cdot p)^2 - m^2M^2],$$
$$\gamma = -[(K\cdot p)(p\cdot q)+m^2(K\cdot q)],$$

and

$$\rho^{(n)} = (1-m^2/s_l)s_l^{1/2}X\sin\theta_l,\quad \rho^{(1)} = \tfrac{1}{2}\beta^{1/2}\rho^{(n)}.$$

Let us also recall the notation introduced in connection with Eq. (4), namely,

$$(abcd) \equiv \epsilon_{\mu\nu\lambda\sigma}a_\mu b_\nu c_\lambda d_\sigma.$$

For any one of three polarization functions, we then find

$$(1-m^2/s_l)A = -m[a_1(P\cdot S)+a_2(q\cdot S) + a_3(Q\cdot S)+ia_4(PQLS)+ia_5(PQNS) + ia_6(PLNS)+ia_7(QLNS)],$$

where the appropriate spin vector S is to be used. The quantities $a_1\cdots a_7$, expressed in terms of scalar (and pseudoscalar) products, but not further reduced, are given as follows:

$$a_1 = |f|^2(P\cdot L - P\cdot N) + \mathrm{Re}\left(f^*g + f^*h\frac{P\cdot L}{M^2} + g^*h\frac{Q\cdot L}{M^2} - r^*h\frac{m^2}{M^2}\right)(Q\cdot L - Q\cdot N)$$
$$+\mathrm{Re}\left(g^*h\frac{Q^2}{M^2} + r^*h\frac{Q\cdot N}{M^2} - f^*r\right)(s_l - m^2) + i\,\mathrm{Im}\left(\frac{f^*h}{M^2}\right)(QPNL),$$

$$a_2 = \left(|f|^2 + |h|^2\frac{(Q\cdot L)^2}{M^4}\right)s_\pi - \left(|g|^2 + |h|^2\frac{X^2}{M^4}\right)Q^2 + m^2|r|^2 + i\,\mathrm{Im}\left(\frac{r^*h}{M^2}\right)(QPNL)$$
$$+\mathrm{Re}\left(f^*h\frac{s_\pi}{M^2}\right)(Q\cdot L - Q\cdot N) + \mathrm{Re}\left(\frac{r^*h}{M^2}\right)[(P\cdot L)(Q\cdot N) - (Q\cdot L)(P\cdot N)]$$
$$+\mathrm{Re}\left(g^*h\frac{Q^2}{M^2}\right)(P\cdot L - P\cdot N) - \mathrm{Re}f^*r(P\cdot L + P\cdot N) - \mathrm{Re}g^*r(Q\cdot L + Q\cdot N),$$

$$a_3 = |g|^2(Q\cdot L - Q\cdot N) + \mathrm{Re}\left(f^*g - f^*h\frac{P\cdot L}{M^2} - g^*h\frac{Q\cdot L}{M^2} + \frac{m^2}{M^2}r^*h\right)(P\cdot L - P\cdot N)$$
$$+\mathrm{Re}\left(f^*h\frac{s_\pi}{M^2} - r^*h\frac{P\cdot N}{M^2} - g^*r\right)(s_l - m^2) + i\,\mathrm{Im}\left(\frac{g^*h}{M^2}\right)(QPNL),$$

$$a_4 = \mathrm{Im}\left(f^*g - \frac{f^*h}{M^2}(P\cdot L - P\cdot N) - \frac{g^*h}{M^2}(Q\cdot L - Q\cdot N) + \frac{r^*h}{M^2}(s_l - m^2)\right) - i\frac{|h|^2}{M^4}(QPNL),$$

$$a_5 = -\mathrm{Im}(f^*g),\quad a_6 = -\mathrm{Im}(f^*r),\quad a_7 = -\mathrm{Im}(g^*r).$$

The full reduction to the variables which we have chosen for K_{l4} decay is tedious, except for A_n, where various simplifications obtain. More important, the reduced expressions for $A_{||}$ and $A_\perp$ contain factors which couple the variables θ_l, s_l, s_π in a nonseparable way, so that the structure does not decompose simply in θ_l upon integration over s_l. But it is just such decomposition that we wished to exploit in this paper, for use in limited statistics situations. Hence our special attention to A_n.