Application of a Unitary Impulse Approximation to Pion-Nucleon Scattering*

ROBERT W. FINKEL
Department of Physics, St. John's University, Jamaica, New York

AND

LEONARD ROSENBERG
Department of Physics, New York University, University Heights, New York

(Received 18 December 1967)

A unitary impulse approximation given previously is described here in a more general (multichannel) form. To test the method, we apply it to a soluble model of K^--d scattering and obtain encouraging results. The method is then applied to π-N elastic scattering. Good agreement with the experimentally determined d_{13} amplitude is obtained for energies near the second resonance. While the impulse approximation violates unitarity bounds, the unitary impulse approximation shows no such violations.

1. INTRODUCTION

SOME time ago a modification of the impulse approximation was introduced which incorporates constraints imposed by unitarity.[1] In the present paper we report two numerical applications of this unitary impulse approximation to three-body scattering problems. The first application is to a nonrelativistic model of elastic K^--d scattering with separable two-body potentials. This allows us to test the method, since the exact solution has been obtained numerically by Hetherington and Schick.[2] The second application is to elastic π-N scattering in the d_{13} and f_{15} states. Numerical results are presented in Sec. 3.

In Sec. 2 we review the unitary impulse approximation, presenting a more general version in which rearrangement collisions and effects of the Pauli principle are treated explicitly. The discussion is placed in the context of an effective potential formalism,[3,4] since we believe that this is the most natural way to arrive at the approximation. The formalism also suggests more elaborate approximations.[3] Generalization of the method for systems involving more than three particles is possible but will not be discussed here.

2. UNITARY IMPULSE APPROXIMATION

We consider a submatrix of the complete S matrix which refers only to energy conserving states of a given total angular momentum. This matrix may be represented in the block form

$$\begin{pmatrix} S & S' \\ S' & S'' \end{pmatrix},$$

where the elements of the matrix S correspond to two-body initial and final states. (Each "body" may be a compound system.) The unitarity of the complete S matrix implies that

$$SS^\dagger+S'S'^\dagger=1. \tag{2.1}$$

For an arbitrary column vector a we have

$$\frac{a^\dagger SS^\dagger a}{a^\dagger a}+\frac{a^\dagger S'S'^\dagger a}{a^\dagger a}=1. \tag{2.2}$$

Since each term on the left-hand side of Eq. (2.2) is non-negative we have the inequalities

$$0\leq a^\dagger(1-SS^\dagger)a\leq a^\dagger a. \tag{2.3}$$

If we write $S=1+2if$, then the inequalities of Eq. (2.3) may be expressed as

$$0\leq a^\dagger(Af-ff^\dagger)a\leq\tfrac{1}{4}a^\dagger a. \tag{2.4}$$

Here, and in the following, the symbol A means "anti-Hermitian part," i.e.,

$$Af=(2i)^{-1}(f-f^\dagger). \tag{2.5}$$

Now suppose f is determined as the solution of the Heitler equation

$$f=K+iKf. \tag{2.6}$$

It follows that

$$Af-ff^\dagger=(1-iK)^{-1}AK(1+iK^\dagger)^{-1}, \tag{2.7}$$

and Eq. (2.4) becomes

$$0\leq a^\dagger[(1-iK)^{-1}AK(1+iK^\dagger)^{-1}]a\leq\tfrac{1}{4}a^\dagger a. \tag{2.8}$$

If AK is a non-negative matrix the left-hand inequality will be satisfied. We now show that the right-hand inequality is satisfied with no further restrictions on K. If we define

$$b=(1+iK^\dagger)^{-1}a, \tag{2.9}$$

it then remains to show that

$$b^\dagger AKb\leq\tfrac{1}{4}b^\dagger(1-iK)(1+iK^\dagger)b \tag{2.10}$$

is satisfied for arbitrary b. But by rearranging terms in

* Supported by the National Science Foundation.

[1] L. Rosenberg, Phys. Rev. **131**, 874 (1963).
[2] J. H. Hetherington and L. H. Schick, Phys. Rev. **137**, B935 (1965).
[3] L. Rosenberg, this issue, Phys. Rev., **168**, 1756 (1968).
[4] E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. **B2**, 181 (1967).

Eq. (2.10), we can put it in the form

$$\tfrac{1}{4}b^\dagger(1+iK)(1+iK)^\dagger b\geq 0, \tag{2.11}$$

which is evidently correct.

For energies below the three-particle continuum threshold the two-body submatrix S contains all the open channels and is therefore unitary. For these energies we require $AK=0$, a familiar result.

At this stage we could state our choice of approximate K matrix and verify that it satisfies the unitarity constraint. A more meaningful procedure, we believe, is to place our approximation in the larger context of an effective-potential approach to the three-body problem. In this approach one represents the transition matrix for scattering processes in which a pair of particles is bound in the initial state and a (possibly different) pair is bound in the final state as the solution of the integral equation (in operator form)

$$T_{\alpha\beta}=V_{\alpha\beta}+\sum_\gamma V_{\alpha\gamma}G_\gamma T_{\gamma\beta}. \tag{2.12}$$

Here G_γ is the two-body Green's function for channel γ and $V_{\alpha\beta}$ is the effective-potential matrix. It has been shown[3,4] that in the three-body potential scattering problem the effective potential is rigorously defined as the solution of a set of integral equations of the Faddeev type. They differ from the usual Faddeev equations[5] in that the input involves not the two-particle amplitudes T_γ (the subscript indicates the particular two-particle subsystem) but the modified amplitudes $\hat{T}_\gamma$ obtained from T_γ by removal of the bound-state pole contributions. For example, if there is only one bound state for subsystem γ, at energy $E_{B\gamma}$, then $\hat{T}_\gamma$ may be defined as

$$\hat{T}_\gamma=T_\gamma-\Gamma_\gamma(E-E_{B\gamma})^{-1}\Gamma_\gamma^\dagger, \tag{2.13}$$

where Γ_γ is the exact vertex function. We write Eq. (2.12) in the matrix-operator form

$$T=V+VGT \tag{2.14}$$

and define a reaction matrix R[6] as the solution of

$$R=V+RG_PV, \tag{2.15}$$

where the subscript P denotes the principal-value prescription. Equations (2.14) and (2.15) imply that

$$T=R+RG_IT, \tag{2.16}$$

where $G_I=G-G_P$ has energy δ functions along the main diagonal. We work with the momentum-space representation of Eq. (2.16), using states of well-defined total angular momentum. If we write $k_\gamma^2dk_\gamma=\rho_\gamma dE_\gamma$, where k_γ is the relative momentum variable in channel γ, the energy integrals can be performed. It is convenient to remove the factor ρ_γ by introducing the amplitudes

$$f_{\alpha\beta}=-\pi(\rho_\alpha\rho_\beta)^{1/2}T_{\alpha\beta}, \tag{2.17}$$

$$K_{\alpha\beta}=-\pi(\rho_\alpha\rho_\beta)^{1/2}R_{\alpha\beta}. \tag{2.18}$$

In terms of these amplitudes, evaluated on the energy shell, Eq. (2.16) reduces to Eq. (2.6).

The above discussion shows that unitarity requires AR to be nonpositive. Now Eq. (2.15), along with its adjoint, gives the pair of equations

$$R=(1+RG_P)V, \tag{2.19}$$

$$R^\dagger=V^\dagger(1+RG_P)^\dagger. \tag{2.20}$$

It follows from these two equations, along with the relation $G_P=G_P^\dagger$, that

$$AR=(1+RG_P)AV(1+RG_P)^\dagger. \tag{2.21}$$

Clearly AR will be nonpositive if AV is nonpositive. We observe that, by virtue of its definition as a solution of a set of modified Faddeev equations, the *exact* effective potential satisfies a unitarity relation (modified by omission of two-body intermediate states) which expresses its anti-Hermitian part in a manifestly nonpositive form. This is the expected result since the Faddeev formulation of the three-body problem is known to preserve unitarity.

We now consider a simple *approximation* to the effective potential, obtained by retaining only the leading terms in the defining integral equation. This approximation takes the form[3]

$$V_{\alpha\beta}{}^{(1)}=(\Phi_\alpha|E-H_0|\Phi_\beta)(1-\delta_{\alpha\beta}) +\sum_\gamma(1-\delta_{\alpha\gamma})(1-\delta_{\beta\gamma})(\Phi_\alpha|\hat{T}_\gamma|\Phi_\beta). \tag{2.22}$$

Here Φ_β is the wave function describing the bound subsystem β moving freely in the presence of the third particle, and H_0 is the Hamiltonian for the system with no interaction. The modified two-body scattering operator satisfies the unitarity relation

$$A\hat{T}_\gamma=-\pi\sum_c T_\gamma|\Phi_c)(\Phi_c|T_\gamma^\dagger, \tag{2.23}$$

where $\sum_c$ represents an integral over energy-conserving free-particle intermediate states. States in which the pair γ is bound are omitted from the sum, since the pole contributions have been removed in the definition of $\hat{T}_\gamma$. From Eqs. (2.22) and (2.23) we find that

$$\begin{aligned}(2i)^{-1}&(V_{\alpha\beta}{}^{(1)}-V_{\beta\alpha}{}^{(1)*})\\ &=-\pi\sum_\gamma\sum_c(1-\delta_{\alpha\gamma})(1-\delta_{\beta\gamma})(\Phi_\alpha|T_\gamma|\Phi_c)(\Phi_c|T_\gamma^\dagger|\Phi_\beta)\\ &=-\sum_\gamma\sum_c M_{\alpha c(\gamma)}M_{\beta c(\gamma)}{}^*,\end{aligned} \tag{2.24}$$

where

$$M_{\alpha c(\gamma)}=\pi^{1/2}(1-\delta_{\alpha\gamma})(\Phi_\alpha|T_\gamma|\Phi_c). \tag{2.25}$$

In matrix form we have

$$AV^{(1)}=-MM^\dagger, \tag{2.26}$$

[5] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

[6] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), p. 215.

which is nonpositive as required by unitarity. Furthermore, we verify that $AV^{(1)}$ vanishes, as it should, for states Φ_α and Φ_β with energy below the three-particle continuum threshold. This follows from the fact that $\sum_c$ in Eq. (2.24) does not bring in states with a pair bound, as noted above.

The approximate R matrix is to be obtained from Eq. (2.15) with V replaced by $V^{(1)}$. The solution is then to be used as input in Eq. (2.16) to give the approximate T matrix, $T^{(1)}$. This procedure can be simplified considerably if instead we take $T^{(1)}$ to be the solution of

$$T^{(1)}=V^{(1)}+V^{(1)}G_IT^{(1)}. \tag{2.27}$$

Since $AV^{(1)}$ is nonpositive no unitarity violation is introduced. It is this simplified version of the unitary impulse approximation which is adopted in the applications described in Sec. 3.

In the above discussion the particles were assumed to be distinguishable. Consider now the elastic scattering of a particle which is identical to one or both of the target particles. To satisfy the symmetry requirement of the Pauli principle, we take as our approximate effective potential $V_s^{(1)}$ the appropriate linear combination of direct and exchange matrix elements of the effective potential $V^{(1)}$ which is given in Eq. (2.22). Unitarity requires that $V_s^{(1)}$, considered as a matrix in momentum space, must have a nonpositive anti-Hermitian part. To verify that this condition is satisfied, we need only observe that $V_s^{(1)}$ is an eigenvalue of the channel matrix $V^{(1)}$ and may be expressed in the form

$$V_s^{(1)}=x^\dagger V^{(1)}x, \tag{2.28}$$

where x is the eigenvector. It follows that

$$AV_s^{(1)}=x^\dagger AV^{(1)}x, \tag{2.29}$$

which has been shown to be nonpositive.

A relativistic extension of the above results is easily obtained. For example, one could define a model by setting up a relativistic version of the Faddeev equations.[7] The integral equations can then be rewritten in a form analogous to Eq. (2.12); this is just the Bethe-Salpeter equation with a particular (nonperturbative) expression for the inhomogeneous term (or effective potential). As in the nonrelativistic case, the unitary impulse approximation is defined by using the impulse approximation plus single-particle exchange terms for the effective potential. Of course, one could simply verify directly that the approximate K matrix thus obtained satisfies the unitarity constraint, without invoking relativistic Faddeev equations as theoretical motivation.

3. APPLICATIONS

As a test of the method we compare the unitary impulse approximation (UIA) with the exact solution for a separable potential model of K^--d scattering.[2] The partial-wave amplitudes $f_l^{\rm IA}$, computed according to the impulse approximation (IA), are given by Hetherington and Schick along with the exact solution for laboratory momenta of 105, 194, and 300 MeV/c. These results are reproduced in Table I along with the amplitude

$$f_l^{\rm UIA}=(1-if_l^{\rm IA})^{-1}f_l^{\rm IA}. \tag{3.1}$$

TABLE I. Comparison of exact f_l with the impulse approximation (IA) and unitary impulse approximation (UIA).

Momentum (MeV/c)	l	$f_l^{\rm IA}$	$f_l^{\rm UIA}$	$f_l^{\rm exact}$
300	0	$0.340+0.463i$	$0.151+0.351i$	$0.165+0.318i$
	1	$0.087+0.120i$	$0.069+0.113i$	$0.096+0.127i$
	2	$0.023+0.033i$	$0.022+0.032i$	$0.023+0.033i$
194	0	$0.454+0.608i$	$0.163+0.424i$	$0.120+0.487i$
	1	$0.068+0.094i$	$0.056+0.089i$	$0.077+0.095i$
	2	$0.011+0.016i$	$0.011+0.016i$	$0.011+0.016i$
105	0	$0.479+0.504i$	$0.192+0.396i$	$0.012+0.436i$
	1	$0.029+0.032i$	$0.028+0.031i$	$0.032+0.029i$
	2	$0.002+0.002i$	$0.002+0.002i$	$0.002+0.002i$

The numerical results show that when the amplitude is large the IA is very misleading and a substantial improvement is obtained in the UIA. For the smaller amplitudes the IA is quite accurate and the UIA does not necessarily improve it.

We now describe an application of the UIA to pion-nucleon scattering, which we treat as a three-body scattering problem with the nucleon in initial and final states considered as a pion-nucleon composite. According to the discussion in Sec. 2, unitarity bounds will be satisfied if we use, as input to the Bethe-Salpeter equation, the sum of amplitudes represented diagrammatically in Fig. 1. In Fig. 1(b) the circle representing the input two-body π-N amplitude has a vertical line through it which is meant to indicate that the nucleon pole contribution has been subtracted out. The diagrams of Figs. 1(b) and 1(c) could in principle be evaluated using only on-mass-shell two-body π-N and π-π input amplitudes. These two-body amplitudes appear, in the t channel, as final-state interaction corrections to the Born approximation. Standard dispersion-theoretic techniques exist for this type of computation.[8] In the simplified calculation reported here, however, we have confined our attention to peripheral collisions and have accordingly retained only the contribution whose singularities in the complex t plane lie closest to the physical region. That is, diagrams 1(a) and 1(b) have been dropped and diagram 1(c) was evaluated in the strip approximation.[9] A ρ resonance approximation was made

[7] V. A. Alessandrini and R. L. Omnes, Phys. Rev. **139**, B167 (1965); D. Z. Freedman, C. Lovelace, and J. M. Namyslowski, Nuovo Cimento **63**, 258 (1966); L. Rosenberg, Phys. Rev. **147**, 1016 (1966).

[8] The solution of the relevant dispersion relation, for a given partial wave, is derived in Ref. 6, p. 908. A dispersion relation for the full amplitude, along with a discussion of its iterative solution, is presented in L. Rosenberg, Nuovo Cimento **28**, 1107 (1963).

[9] An earlier calculation along these lines was performed by J. S. Ball and W. R. Frazer, Phys. Rev. Letters **7**, 204 (1961). The violation of unitarity which appeared in the Ball-Frazer calculation suggested the present application of the unitary impulse approximation to this problem.

for the π-π amplitude and the π-N-N coupling constant was used as vertex function.

The d_{13} amplitude calculated in the impulse (or strip) approximation is plotted for various pion laboratory energies in Fig. 2 along with the results of the UIA, Eq. (3.1). A similar comparison for the f_{15} amplitude appears in Fig. 3. The unitarity bounds $|f|\leq 1$ and $\mathrm{Im}f-|f|^2\leq\frac{1}{4}$ are violated for a wide range of energies by the IA and this is corrected by the UIA. In Fig. 4 the UIA for the d_{13} amplitude is compared with fits to the experimental data.[10] Good agreement is obtained in the resonance region, though not at lower energies where virtual ρ-meson production is not the dominant effect. We hope to report in the future on an extension of the calculation which takes into account diagrams 1(a) and 1(b).

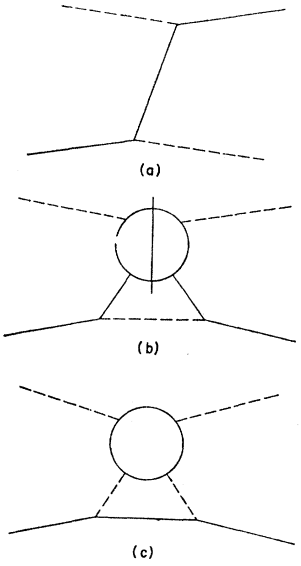

FIG. 1. Diagrams for approximate effective potential for pion-nucleon scattering. Dashed and solid lines represent pions and nucleons, respectively.

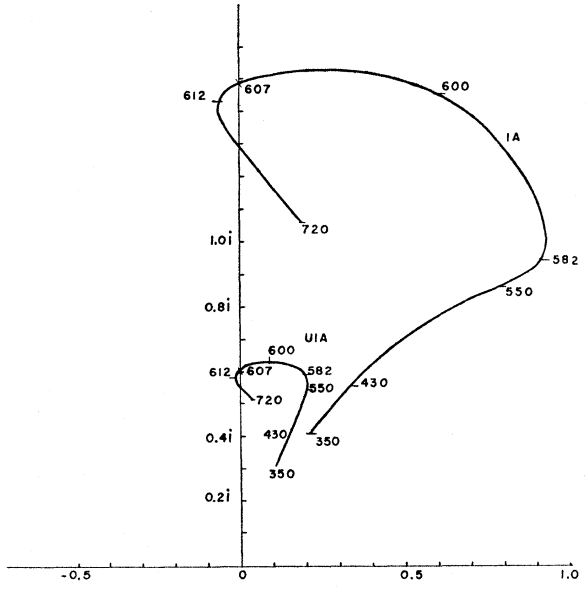

FIG. 2. The d_{13} amplitude in the impulse approximation (IA) and the unitary impulse approximation (UIA). The parameter is pion laboratory energy in MeV.

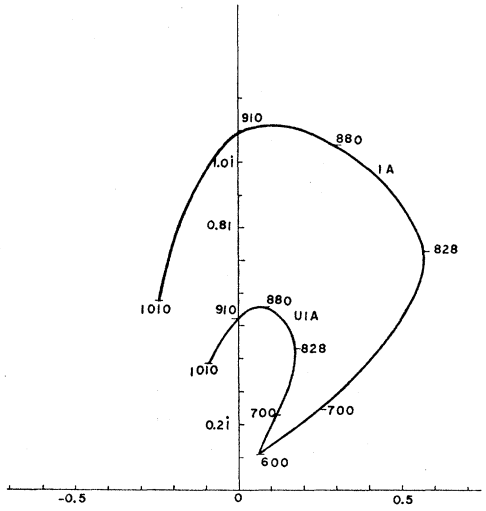

FIG. 3. The f_{15} amplitude in the impulse approximation and the unitary impulse approximation.

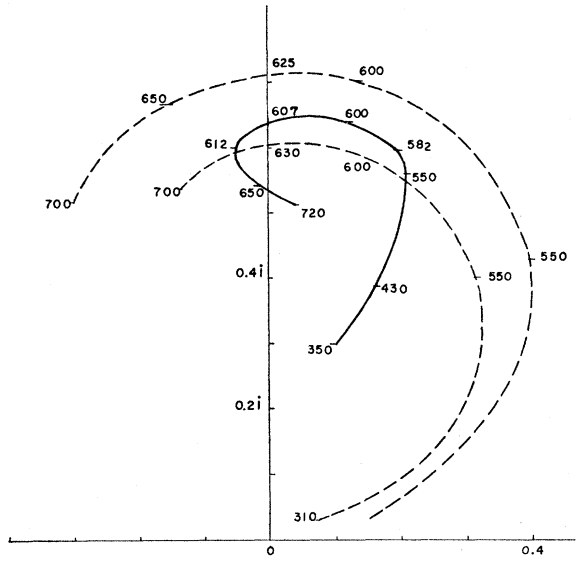

FIG. 4. The d_{13} amplitude. Dashed curves are fits to the experimental data due to Bransden, O'Donnell, and Moorhouse. The solid curve is the unitary impulse approximation.

Note added in proof. After submission of this manuscript we learned of the work of I. H. Sloan [Phys. Rev. **165**, 1587 (1968)], who proposed the same multichannel generalization of the unitary impulse approximation as that given in Sec. 2 of the present paper.

[10] B. H. Bransden, P. J. O'Donnell, and R. G. Moorhouse, Phys. Rev. **139**, B1566 (1965).