while the spatial parts of the current are

$$I^{\pm}(\mathbf{q})=\mp\frac{i}{n'n}\frac{1}{m_e}[(2l'+1)(2l+1)]^{1/2}\sum_{k'k}\begin{pmatrix}\frac{1}{2}(n'-1) & \frac{1}{2}(n'-1) & l' \\ \frac{1}{2}(m\pm1)-k' & \frac{1}{2}(m\pm1)+k' & -m\mp1\end{pmatrix}\begin{pmatrix}\frac{1}{2}(n-1) & \frac{1}{2}(n-1) & l \\ \frac{1}{2}m-k & \frac{1}{2}m+k & -m\end{pmatrix}$$
$$\times\{[(\tfrac{1}{2}(n'+m))^2-k'^2]^{1/2}h_{k',k}{}^{0,(l'-l+1)}(\alpha,\gamma)v_{\frac{1}{2}(n'\mp1)+k',\frac{1}{2}(n)+k}{}^{\frac{1}{2}(m+1)}(\beta)v_{\frac{1}{2}(n'\mp1)-k',\frac{1}{2}(n)-k}{}^{\frac{1}{2}(m+1)}(-\beta)$$
$$+[(\tfrac{1}{2}(n'-m))^2-k'^2]^{1/2}h_{k',k}{}^{0,(l'-l+1)}(\alpha,\gamma)v_{\frac{1}{2}(n'\pm1)+k',\frac{1}{2}(n)+k}{}^{\frac{1}{2}(m+1)}(\beta)v_{\frac{1}{2}(n'\pm1)-k',\frac{1}{2}(n)-k}{}^{\frac{1}{2}(m+1)}(-\beta)\} \quad \text{(A5)}$$

and

$$I^{3}(\mathbf{q})=\frac{i}{n'n}\frac{1}{m_e}[(2l'+1)(2l+1)]^{1/2}\sum_{k'k}\begin{pmatrix}\frac{1}{2}(n'-1) & \frac{1}{2}(n'-1) & l' \\ \frac{1}{2}m-k' & \frac{1}{2}m+k' & -m\end{pmatrix}\begin{pmatrix}\frac{1}{2}(n-1) & \frac{1}{2}(n-1) & l \\ \frac{1}{2}m-k & \frac{1}{2}m+k & -m\end{pmatrix}$$
$$\times\{\tfrac{1}{2}[(n'+1+2k')^2-m^2]^{1/2}h_{k',k}{}^{+,(l'-l+1)}(\alpha,\gamma)v_{\frac{1}{2}n'+k'+1,\frac{1}{2}n+k}(\beta)$$
$$-\tfrac{1}{2}[(n'-1+2k')^2-m^2]^{1/2}h_{k',k}{}^{-,(l'-l+1)}(\alpha,\gamma)v_{\frac{1}{2}n'+k'-1,\frac{1}{2}n+k}(\beta)\}v_{\frac{1}{2}n'-k',\frac{1}{2}n-k}(-\beta). \quad \text{(A6)}$$

The angles α,β,γ are defined in terms of the momentum transfer q^2 and the principal numbers n and n' of initial and final states by

$$\sinh(\tfrac{1}{2}\beta)=\frac{1}{2(n'n)^{1/2}}[(n'-n)^2+q^2n'^2n^2]^{1/2}$$

and

$$\alpha=\text{arc}\,\sin(nq/\sinh\beta)\,,$$
$$\gamma=-\text{arc}\,\sin(n'q/\sinh\beta)\,,$$

where the principal value of arc sin has to be taken for $n'\leq n$, while for $n'>n$, α starts out at $q=0$ with the value π, and γ with $-\pi$. A plot of some of these form factors is given in Ref. 10.

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Multipion Bound States and a σ Meson*

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The possible existence of multipion nonrelativistic bound states is examined with variational principles. It is found that of two likely candidates (the η meson, and a possible σ meson with $I=J=0$ and even parity), only the σ can be a bound state of 4 pions. If the η were bound, there would unavoidably be a three-pion bound state with the quantum numbers of the ω and mass $<3m_\pi$. The four-pion channel couples to the two-pion channel, which turns the σ into a resonance. The multichannel problem is investigated with an N/D method, which yields an estimate of the width of the σ resonance. A theorem of Blankenbecler is used to dispose of the difficulties of anomalous thresholds. Crude calculations show that the physical σ should lie above the four-pion threshold.

1. INTRODUCTION

THERE has been much interest recently in the idea that so-called elementary particles are, in fact, composite structures, in much the same way as a nucleus is a composite of protons and neutrons. Unfortunately, it is rare in elementary particle physics to find a composite system and a proposed set of constituent particles such that the binding energy per particle is small compared to the constituents' masses. It is important to study these weakly bound cases because they must have a dynamical structure which parallels that of ordinary nuclei (which are also weakly bound), and we certainly understand nuclear physics better than we do elementary particle physics. In particular, we can use such well-tested machinery as the Schrödinger equation to carry out part of the investigation.

There are at least two interesting weak-binding candidates for systems of a small number (four or less) of pions. These are: (1) the η meson, mass 550 MeV, with zero spin and odd parity; (2) an $IJ^P=00^+$ meson which

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we call the σ meson in the nomenclature of Brown and Singer,[1] who proposed a resonance with these quantum numbers to explain $K_{\pi 3}$ and η decay spectra. Brown and Singer suggested a mass around 400 MeV or less, but such a low mass would contradict the K_{e4} data.[2] It is possible, however, that there could be a σ meson which is a bound state of four pions, and a mass (in the absence of coupling to 2π channels) of the order of 450–550 MeV. Likewise, the η might possibly be a four-pion bound state. In both cases, the binding energy per pion would be nonrelativistic. Coupling to 2π channel shifts the σ mass and turns it into a resonance; our calculations show that the real part of the σ mass increases, even though there are attractive forces in all channels. The reason for the increase is essentially that the bound-state channel (4π) is coupled to a channel with a lower threshold, as we discuss in Sec. 4.

Our investigation begins with a study of the nonrelativistic bound states of two, three, or four pions with a given set of two-body potentials. It is important, of course, that these potentials do not form bound states where none are observed experimentally. Potentials which meet this constraint are too weak to bind the η, leaving the σ meson as the only possible weakly bound state. It is perhaps not unreasonable that pion-pion forces do not bind the η, but that they do form a σ. Forces in the s-wave nucleon-antinucleon channel can make the odd-parity η, but in p wave these forces are weakened by the centrifugal barrier, and may play only a minor role in forming the even-parity σ. We shall focus our attention on the σ, first considering it as a four-pion bound state, and then considering the coupling of the σ to the two-pion channel, in which the σ appears as a resonance.

A number of possible multipion states have been studied, principally with the aid of variational techniques. Similar problems have been dealt with by other authors, including Schiff[3] and Mitra and Ray.[4] Schiff uses a variational principle to discuss the ω meson with a square-well potential, while Mitra and Ray use separable potentials. We try to stick to more realistic potentials, mostly dominated by ρ exchange. We agree with the qualitative conclusions of these authors: the η meson cannot be a four-pion bound state,[4] and the pion-pion potential must have some repulsion in it, to keep objects like the ω from being too tightly bound.[3]

Table I lists the various multipion channels of interest. In each channel, there is an experimentally observed resonance (with the possible exception of the σ channel). Of course, not all of these resonances can be considered as nonrelativistic bound states or low-lying resonances; nonetheless, it is important to study these channels in order to make sure that no spurious bound states are made by a certain choice of a two-particle

TABLE I. Multipion bound states.

Channel			Observed states			Is the observed state a possible nonrelativistic bound state?
J^{PG}	Isospin	No. of pions	Name	Mass (MeV)	Decay mode	
0^{++}	0	2	σ	400–500	2π	no
0^{++}	0	4	σ	400–500	2π	yes
0^{-+}	0	4	η	550	3π, 2γ	yes
0^{--}	1	3	π	140	stable	no
1^{--}	0	3	ω	780	3π	no
1^{-+}	1	2	ρ	770	2π	no
1^{+-}	1	3	A^1	1070	$\rho\pi$	no

potential. Some of the states in this table have already been discussed as multipion states in various relativistic frameworks, especially the ρ meson.[5] It is traditional to think of the ρ as mainly a 2π state, but it should not be excluded that ρ has a large 4π component; we have not studied this possibility.

For each channel, we write an appropriate multiparticle Schrödinger equation and look for bound states with standard variational techniques. We assume that pions interact through two-particle potentials only. The potential consists of two parts: an attractive force coming from ρ exchange, and a repulsion, which might, for example, come from Pomeranchuk exchange as envisioned by Chew.[6] The parameters of the repulsive potential are considered adjustable, and are determined by the requirement that there be a four-pion σ bound state with small binding energy, and that there be no spurious bound states in other channels. It turns out that the parameters so determined are quite close to those given by Chew.[5] As a further check on the reasonability of this potential, we calculate the low-energy s-wave π-π scattering with a special N/D equation, in the Appendix.

In Sec. 3, we consider the more difficult problem of coupling the four-pion σ meson to 2π channels. This two-channel multiparticle system is simulated by replacing the 4π channel with two σ mesons, and gives a two-body multichannel problem with prominent anomalous thresholds. According to a theorem of Blankenbecler,[7] the determinant of the D matrix has no anomalous thresholds, which greatly simplifies the question of determining the mass shift and resonant width of the σ meson. We prove this theorem for the problem at hand.

The actual 4π scattering states are very crudely investigated by using a generalized set of N/D equations given by Blankenbecler.[8] The determinant of D yields the parameters of the physical σ meson. We find that if the 4π bound state has zero binding energy, the physical

[1] L. Brown and P. Singer, Phys. Rev. **133**, B812 (1964).
[2] R. W. Birge *et al.*, Phys. Rev. **139**, B1600 (1965).
[3] L. Schiff, Phys. Rev. **125**, 777 (1962).
[4] A. N. Mitra and S. Ray, Phys. Rev. **137**, B982 (1965).
[5] See, e.g., F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).
[6] G. Chew, Phys. Rev. **140**, B1427 (1965).
[7] R. Blankenbecler (private communication).
[8] R. Blankenbecler, Phys. Rev. **122**, 983 (1961).

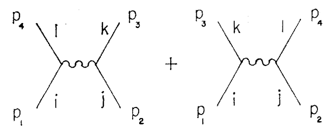

FIG. 1. Exchange of a ρ meson in π-π scattering.

σ meson has a mass of about 650 MeV and a width of 120 MeV, for a reasonable value of a cutoff parameter.

2. VARIATIONAL CALCULATIONS

A. Schrödinger Potentials

In addition to the usual long-range one-particle exchange forces, we must also consider short-range forces (of a repulsive character) which are a phenomenological representation of multiparticle exchange. Chew[6] has characterized this force by Pomeranchuk exchange, which gives a local, almost energy-independent, potential of Gaussian shape. We treat the strength and range of the Gaussian potential as semiadjustable parameters, but our final values agree rather well with Pomeranchuk exchange.

The only significant attractive force between two pions comes from ρ exchange. (Exchange of the σ meson with properties we later calculate is fairly unimportant.) The numerators of the Feynman amplitudes in Fig. 1 are to be evaluated at the ρ pole, with nonresonant terms dropped. These nonresonant terms are repulsive and of very short range (δ function in coordinate space); our phenomenological repulsive potential presumably is a more correct way of handling the repulsion.

The contribution of Fig. 1 is then

$$V_\rho(s,t,u)=\frac{g^2(2s+m_\rho^2-4m_\pi^2)}{m_\rho^2-t}\epsilon_{ilm}\epsilon_{jkm}+(t\leftrightarrow u,\,k\leftrightarrow l)\,,\quad (1)$$

where

$$s=(P_1+P_2)^2=4(m_\pi^2+q^2)\,,$$
$$t=(P_1-P_4)^2=-2q^2(1-\cos\theta)\,,$$
$$u=(P_1-P_3)^2=-2q^2(1+\cos\theta)\,,$$

and q, $\cos\theta$ are the center-of-mass momentum and scattering angle, respectively. We shall ignore the s dependence of the potential, and set $s=4m_\pi^2$ in the variational calculation. The potential (1) has both a direct and an exchange part; because of the over-all Bose symmetry of the wave functions to which V_ρ is applied, the effective potential is either twice the direct potential or zero. The value of g^2 is gotten from the decay $\rho\to 2\pi$; we take[9] $g^2/4\pi=2.4$.

The form of the repulsive potential is obtained[6] from the s-wave projection of the Pomeranchuk exchange amplitude in π-π scattering. In our parameterization of this potential, we neglect the energy dependence and set

$$V_P(t,u)=-G(m_\pi/M)^2(2\pi)^{1/2}e^{t/M^2}\delta_{il}\delta_{jk}+(t\leftrightarrow u,\,k\leftrightarrow l)\,,\quad (2)$$

where G and M are adjustable. The total potential is just $V=V_\rho+V_P$, and the configuration-space potential is related to $V(t,u)$ by

$$\frac{V(t,u)}{32\pi}=-\frac{m_\pi^2}{4\pi}\int d\mathbf{x}[e^{i\mathbf{t}\cdot\mathbf{x}}V_{\rm dir}(\mathbf{x})+e^{+i\mathbf{u}\cdot\mathbf{x}}V_{\rm ex}(\mathbf{x})]\,,\quad (3)$$

where $V_{\rm dir}$, $V_{\rm ex}$ are the direct and exchange potentials, and $\mathbf{t}=\mathbf{P}_1-\mathbf{P}_4$, $\mathbf{u}=\mathbf{P}_1-\mathbf{P}_3$.

B. Trial Wave Functions

For all of the states listed in Table I, the trial wave functions are of the form (angular momentum wave function)$\times$(isospin wave function)$\times\psi(W)$. Here $\psi(W)$ is a scalar wave function of the type

$$\psi(W)=W^\beta\exp(-\tfrac{1}{2}\sigma W^{1/2})\,,\quad (4)$$

$$W=\sum_{i<j}^{N}(\mathbf{r}_i-\mathbf{r}_j)^2\,,\quad N=2,3,4\,.\quad (5)$$

The variational parameters are β and σ; of course, $\sigma>0$, and β is such that the kinetic energy of the wave function is finite. $\psi(W)$ is totally symmetric, hence so is the product of angular momentum and isospin wave functions. It is easy to construct the total wave function; we list the result only for the η and ω cases:

$$\begin{aligned}\psi_\omega&=\mathbf{r}\times\boldsymbol{\rho}(\mathbf{A}_1\times\mathbf{A}_2\cdot\mathbf{A}_3)\psi(W)\,,\\ \boldsymbol{\rho}&=2^{-1/2}(\mathbf{r}_1-\mathbf{r}_2)\,,\\ \mathbf{r}&=6^{-1/2}(\mathbf{r}_1+\mathbf{r}_2-2\mathbf{r}_3)\,,\\ W&=3(\rho^2+r^2)\,,\end{aligned}\quad (6)$$

$$\begin{aligned}\psi_\eta&=[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})\tfrac{3}{2}(2u^2-v^2-w^2)(\mathbf{A}_1\times\mathbf{A}_2)\cdot(\mathbf{A}_3\times\mathbf{A}_4)\\ &\qquad-6\mathbf{v}\cdot\mathbf{w}(\mathbf{A}_1\times\mathbf{A}_4)\cdot(\mathbf{A}_2\times\mathbf{A}_3)]\psi(W)\,,\\ \mathbf{u}&=\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2-\mathbf{r}_3-\mathbf{r}_4)\,,\\ \mathbf{v}&=2^{-1/2}(\mathbf{r}_1-\mathbf{r}_2)\,,\\ \mathbf{w}&=2^{-1/2}(\mathbf{r}_3-\mathbf{r}_4)\,,\\ W&=4(u^2+v^2+w^2)\,.\end{aligned}\quad (7)$$

The $\mathbf{A}_i$ are isospin wave functions. The particular choice of spatial variables in (6) and (7) diagonalizes the kinetic energy, and it is relatively easy (but tedious) to calculate the normalized expectation value of the Hamiltonian

$$H=-\frac{1}{2m_\pi}\sum\nabla_i^2+\sum_{i<j}V(\mathbf{r}_i-\mathbf{r}_j)\,.\quad (8)$$

All integrals in $\langle H\rangle$ can be done in closed form, but the length of the calculation makes it convenient to evaluate the potential energy on a computer, which can search

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[9] J. J. Sakurai, Phys. Rev. Letters **17**, 1021 (1966).

for minima as the parameters β and σ are varied. The calculations were done twice, once with Pomeranchuk repulsion included, and once with only ρ-exchange forces, which are predominantly attractive.

C. Results

The calculations show at once that a repulsive potential is necessary, as well as the attractive ρ-exchange potential of Fig. 1. With no repulsion, the s-wave multipion bound states (which have the quantum numbers of σ and π) would have binding energy greater than the rest mass of the constituents. Even with this unpleasant feature ignored, the purely attractive ρ-exchange force is unable to bind the η. (The state with quantum numbers of ω, ρ, and A^1 are also unbound.) When repulsion is added, with parameters chosen somewhere near those given by Chew,[6] the result is that the only possible candidate for a multipion bound state is a σ meson made up out of four pions. These facts are summarized in Table II.

To the extent that these calculations do not predict unobserved bound states, they are in satisfactory accordance with experience. One might have hoped, however, that the π and η could be multipion bound states. Our methods have nothing to say about the π, which would have to be a relativistic bound state, but we can draw a very definite conclusion about the η: *There is no physically reasonable π-π potential which binds the η, which does not at the same time form a 3π bound state with the quantum numbers of ω.* The words "physically reasonable" mean that the potential is not too singular, and not too long range. This conclusion obviates the hope that a clever combination of attraction and repulsion, or exchanged isospin, could lead to an η made of four pions with no other unphysical consequences.

The reason is to be found in an examination of the η and ω wave functions [Eqs. (6) and (7)]. The first term in square brackets of the η wave function (7) has pion pairs in relative p wave, just as for the ω; hence also these pairs are in relative $I=1$ states. Thus the expectation value of the potential energy from this term is the

TABLE II. Variational binding energies without repulsion. Results of variational calculation with repulsion ($G=45$, $M=M_\rho$).

Channel	Binding energy	No. of pions
	(a)	
π	$B>3m_\pi$	3
ω	not bound	3
A^1	not bound	3
σ	$B>4m_\pi$	4
η	not bound	4
σ	$B>2m_\pi$	2
ρ	not bound	2
	(b)	
π	not bound	3
σ	$B\sim 0$	4
σ	not bound	2

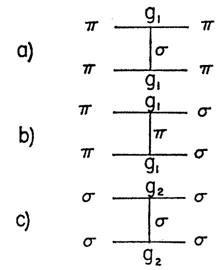

FIG. 2. Force graphs from t-channel exchange.

same, per pion pair, as for the ω; this is also essentially true for the kinetic energy of this term. The second term has two pion pairs in d waves, hence it contributes very little to the potential energy, but a great deal to the kinetic energy. It is clearly impossible to overcome this large kinetic energy without a potential which would bind three pions into a state with ω quantum numbers. We are not, of course, at a loss for mechanisms to form the odd-parity π and η; perhaps they are $\bar{N}N$ bound states.[10,11]

In the next sections, we focus our attention on the σ meson. We do not know for sure, of course, whether we have not used too weak a repulsive potential, and nature consequently has no four-pion bound state. We shall assume that the forces are such that they can bind a σ meson, with small binding energy. With the values $G=45$, $M=M_\rho$, the variational binding energy is zero. These should be compared to the values $G=50$, $M=0.92M_\rho$ given by Chew.[6] In what follows, we will ignore the binding energy, so the 4π bound state has a mass of $\sim$550 MeV. This will be changed by forces coming from the coupling to 2π channels.

3. COUPLED 2π AND 4π CHANNELS

A. Effective Two-Body Problem

By virtue of coupling to 2π channels, the 4π bound state σ of Sec. 2 becomes a resonance, with a primary decay channel into 2π. It is, of course, very difficult to handle scattering states with more than two particles. Consequently, we make a crude attempt to simulate the s-wave coupled channel problem with the two-body reactions $2\pi\to 2\pi$, $2\pi\to 2\sigma$, and $2\sigma\to 2\sigma$. We need not at this point fix the mass M of the σ, since eventually we make an analytic continuation in this mass, but for the physical problem $M>2m_\pi$. As is well known, this means that there are additional singularities in the S matrix. In this section, we state and prove a theorem of Blankenbecler's[7] to the effect that the determinant of the D matrix does not have such singularities.

The scattering amplitude is a matrix:

$$T=T_{ij}=ND^{-1},\quad i,j=1,2 \tag{9}$$

where the index 1 refers to the 2π channel and 2 refers

[10] E. Fermi and C. N. Yang, Phys. Rev. **76**, 1739 (1949).
[11] J. S. Ball, A. Scotti, and D. Y. Wong, Phys. Rev. **142**, 1000 (1966).

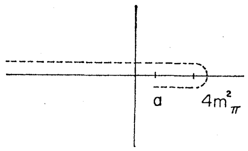

FIG. 3. Path of left-hand cut in $T_{12}^B(s)$ as M becomes greater than $2m_\pi$.

to the 2σ channel. Let us call the $\sigma\pi\pi$ coupling constant g_1, the $\sigma\sigma\sigma$ coupling constant g_2, and consider the graphs of Fig. 2. The s-wave Born amplitudes are

$$T_{11}^B(s)=(g_1^2/4q^2)\ln(1+4q^2/M^2)\,, \tag{10}$$

$$T_{12}^B(s)=T_{21}^B(s)=(g_1^2/4pq)\times\ln\left(\frac{p^2+q^2+2pq+m_\pi^2}{p^2+q^2-2pq+m_\pi^2}\right), \tag{11}$$

$$T_{22}^B(s)=(g_2^2/4p^2)\ln(1+4p^2/M^2)\,, \tag{12}$$

where

$$q^2=\tfrac{1}{4}s-m_\pi^2\,,\quad p^2=\tfrac{1}{4}s-M^2\,. \tag{13}$$

Examination of $T_{11}^B(s)$ and $T_{22}^B(s)$ shows that they have left-hand cuts for $s<4m_\pi^2-M^2$ and $s<3M^2$, respectively, where both cuts extend to $-\infty$. The left-hand cut of $T_{12}^B(s)=T_{21}^B(s)$ begins at

$$a=4M^2(1-M^2/4m_\pi^2)\,. \tag{14}$$

If one gives the σ mass a slightly positive imaginary part ($M\to M+i\epsilon$) and allows M to increase through the values $\sqrt{2}m_\pi$ to $M>2m_\pi$, the end point (a) of the left-hand cut circles the $2m_\pi$ threshold as pictured in Fig. 3. This encircling of the $2m_\pi$ thresholds gives rise to anomalous cuts in the $T_{2\pi\to2\sigma}$ and $T_{2\sigma\to2\pi}$ amplitudes. When $M<\sqrt{2}m_\pi$, there are no anomalous thresholds. Notice also that the anomalous cuts do not encircle the threshold at $4M^2$. When $M>\sqrt{2}m_\pi$ the right-hand unitarity cuts beginning at $2m_\pi$ must be distorted (see Fig. 4) to avoid crossing the anomalous left-hand cuts as was first indicated by Mandelstam.[12]

The unitarity equations are (for $M<\sqrt{2}m_\pi$)

$$D=I-\frac{1}{\pi}\int\frac{\rho N ds'}{(s'-s)}\,,\quad N=t_B+\frac{1}{\pi}\int\frac{[t_B(s')-t_B(s)]\rho N ds'}{(s'-s)}\,, \tag{15}$$

$$\rho_{ij}(s)=\begin{vmatrix}\dfrac{q}{\sqrt{s}}\theta(s-4m_\pi^2) & 0 \\ 0 & \dfrac{p}{\sqrt{s}}\theta(s-4M^2)\end{vmatrix}\,, \tag{16, 17}$$

and $T_{ij}^B(s)$ is given by Eqs. (10)–(12). When $M>2m_\pi$ the equations for the D_{ij} become

$$D_{11}=1-\frac{1}{\pi}\int_{4m_\pi^2+C}\frac{\rho_{11}N_{11}ds'}{(s'-s)}\,, \tag{18}$$

$$D_{12}=-\frac{1}{\pi}\int_{4m_\pi^2+C}\frac{\rho_{12}N_{12}ds'}{(s'-s)}\,, \tag{19}$$

$$D_{21}=-\frac{1}{\pi}\int_{4M^2}\frac{\rho_{22}N_{21}ds'}{(s'-s)}\,, \tag{20}$$

$$D_{22}=1-\frac{1}{\pi}\int_{4M^2}\frac{\rho_{22}N_{22}ds'}{(s'-s)}\,, \tag{21}$$

where $\theta(s-4M_\pi^2)$ is to be ignored in $\rho_{11}(s)$ and the contour C is given in Fig. 4. Only $D_{11}(s)$ and $D_{12}(s)$ have anomalous cuts. We can obtain the contributions to these anomalous cuts by examining N_{11} and N_{12}:

$$N_{11}=T_{11}^B+\int_{4m_\pi^2+C}\frac{[T_{11}^B(s')-T_{11}^B(s)]\rho_{11}(s')N_{11}(s')}{(s'-s)}+\int_{4M^2}\frac{[T_{12}^B(s')-T_{12}^B(s)]\rho_{22}(s')N_{21}(s')}{(s'-s)}\,, \tag{22}$$

$$N_{12}(s)=T_{12}^B(s)+\frac{1}{\pi}\times\int_{4m_\pi^2+C}\frac{[T_{11}^B(s')-T_{11}^B(s)]\rho_{11}(s')N_{12}(s')}{(s'-s)}+\frac{1}{\pi}\times\int_{4M^2}\frac{[T_{12}^B(s')-T_{12}^B(s)]\rho_{22}(s')N_{22}(s')}{(s'-s)}\,. \tag{23}$$

There are two types of terms in these N functions; the first may be written in the form

$$N(s')=\int_a^\infty\frac{g(s'')ds''}{s''-s}\,, \tag{24}$$

whereas the second may be written in the form

$$N(s')=T_{12}^B(s')\int_a^\infty\frac{g(s'')ds''}{s''-s}\,. \tag{25}$$

We consider first the type

$$N(s')=\int_a^\infty\frac{g(s'')ds''}{(s''-s)}\,,$$

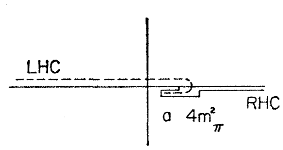

FIG. 4. Distortion of the unitarity cut to avoid the anomalous threshold.

[12] S. Mandelstam, Phys. Rev. Letters 9, 84 (1960).

which when inserted into the equation for $D(s)$,

$$D(s)=1-\frac{1}{\pi}\int_{4m_\pi^2+C}^{\infty}\frac{N(s')\rho(s')ds'}{(s'-s)}, \tag{26}$$

yields

$$D(s)=1-\frac{1}{\pi}\int_{4m_\pi^2+C}^{\infty}ds'\int_{a}^{\infty}\frac{ds''g(s'')\rho(s')}{(s'-s-i\epsilon)(s''-s'-i\epsilon)}. \tag{27}$$

By examining Fig. 5, which shows both the s'' and s' integrations used for $D(s)$, one ascertains that no cuts of the function $1/(s''-s'-i\epsilon)(s'-s-i\epsilon)$ are crossed when s is in the anomalous region. Therefore this part of the N function does not contribute to the D function's anomalous singularities.

The second type of term in $N(s)$ is of the form

$$N(s')=T_{12}{}^B(s')\int_{4M^2}^{\infty}\frac{\tilde{g}(s'')ds''}{(s''-s')}. \tag{25'}$$

This term does contribute to the anomalous cut in the D functions, because the left-hand cut of $T_{12}{}^B(s)$ is crossed by the contour C (see Fig. 4). From Eqs. (25) and (27) the anomalous discontinuity in D is

$$\Delta D(s)=\Delta T_{12}{}^B(s)\int_{4M^2}^{\infty}\frac{\tilde{g}(s'')ds''}{(s''-s)}, \tag{28}$$

where, from Eq. (11),

$$\Delta T_{12}{}^B(s)=2\pi i/pq. \tag{29}$$

From our preceding discussion and the equation for N_{11}, the part of N_{11} which contributes to the anomalous cut in $D_{11}(s)$ is

$$N_{11}{}^A(s)=-\frac{1}{\pi}T_{12}{}^B(s)\int_{4M^2}^{\infty}\frac{\rho_{22}(s')ds'N_{21}(s')}{(s'-s)}, \tag{30}$$

which, recalling our equations for D_{ij}, becomes

$$N_{11}{}^A(s)=T_{12}{}^B(s)D_{21}(s). \tag{31}$$

Similarly for $N_{12}(s)$, the anomalous part is

$$N_{12}{}^A(s)=T_{12}{}^B(s)D_{22}(s). \tag{32}$$

We may now rewrite our Eqs. (18) and (19) for $D_{11}(s)$ and $D_{12}(s)$ as

$$D_{11}(s)=1-\frac{1}{\pi}\int_{a}^{4m_\pi^2}\frac{\Delta T_{12}{}^B(s')D_{21}(s')ds'}{(s'-s)} -\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{N_{22}(s')ds'}{(s'-s)}, \tag{33}$$

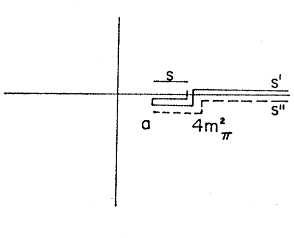

FIG. 5. Integration contours of s', s'' used in $D(s)$.

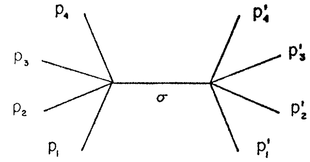

FIG. 6. σ pole in 4π scattering amplitude.

$$D_{12}(s)=-\frac{1}{\pi}\int_{a}^{\infty}\frac{\Delta T_{12}{}^B(s')D_{22}(s')ds'}{(s'-s)} -\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{N_{12}(s')\rho_{11}(s')ds'}{(s'-s)}. \tag{34}$$

We now show that the determinant of $D_{ij}(s)$ has no anomalous cuts. Its discontinuity in the region $a<s<4m_\pi^2$ is

$$\Delta|D|=\Delta D_{11}D_{22}-\Delta D_{12}D_{21}, \tag{35}$$

but from Eqs. (32) and (33), the discontinuity becomes

$$\Delta|D|=\Delta T_{12}{}^B(s)[D_{22}D_{21}-D_{22}D_{21}]=0. \tag{36}$$

The absence of anomalous cuts in the determinant of $D_{ij}(s)$ even though the individual $D_{ij}(s)$ may have them, is the main feature of this coupled-channel model.

Even though such cancellation is unlikely in the actual 2π-4π coupled-channel problem, we may hope that the influence of the anomalous cuts on the determinant of D is small. The virtue of the theorem, of course, is that it is easy to calculate the shifted mass and width of the σ meson without having to worry about anomalous cuts.

B. Estimation of σ Mass and Width

We must make some contact between the four-body wave function of Sec. 2, and tractable approximation schemes such as that of Sec. 3 A above, in order to be able to calculate anything. For example, we know how to calculate the graph of Fig. 6 from the Schrödinger wave function, but this is of little use, since the graph depends on so many variables (four-particle scattering depends on 14 variables, in general).

Blankenbecler[8,13] has given the unitarity equations in N/D form for multiparticle scattering, and used them to estimate the influence of the ω meson on binding the ρ. We shall follow this procedure and eventually be able to approximate true four-particle graphs like that of Fig. 6 by effective two-particle amplitudes, as in Sec. 3 A. These effective two-particle amplitudes depend only on s, the squared energy, so that part of the task is to eliminate the dependence on the unwanted variables. We write the S matrix in terms of invariant amplitudes A by

$$S_{ij}=\delta_{ij}-i(2\pi)^4\delta(p_i-p_j)A_{ij}/(\prod 2E)^{1/2}, \tag{37}$$

[13] R. Blankenbecler, Phys. Rev. **125**, 755 (1962).

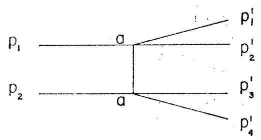

FIG. 7. One-pion-exchange contribution to N_{24} and N_{42}.

so that unitarity reads

$$\mathrm{Im}A_{ij}=-\tfrac{1}{2}\sum_k A_{ik}A_{kj}^*, \tag{38}$$

with

$$\sum\nolimits_k=\frac{1}{N!}(2\pi)^{4-3N}\int\frac{d^3p_1}{2E_1}\cdots\times\frac{d^3p_N}{2E_N}\delta(\sum\mathbf{p}_i)\delta(s^{1/2}-\sum E_i). \tag{39}$$

In (39), the channel k has N pions. The channel labels i, j, k can simply be chosen as the number of pions in the channel. No isotopic spin labels are needed, since we are dealing only with $I=0$ states, although of course isospin wave functions must be supplied in the actual calculations. The sum in (39) is understood to include isospin summation.

Unitarity, as expressed by Eq. (38), is automatically satisfied[8,13] by solving the following linear integral equations, as long as the N_{ij} are real in the physical region:

$$\sum_k A_{ik}\hat{\rho}_k^{-1}D_{kj}=N_{ij}, \tag{40}$$

$$\mathrm{Im}D_{ij}=\tfrac{1}{2}\hat{\rho}_iN_{ij}. \tag{41}$$

The N_{ij} are taken as input, and D_{ij} and A_{ij} are solved for. The quantities $\hat{\rho}$ are phase-space densities, to be defined below. The D_{ij}'s have imaginary parts coming from anomalous thresholds, and (41) is not a totally adequate specification. However, we are only interested in the determinant of D, and, in view of the theorem of Sec. 3 A, we ignore anomalous cuts.

Total two- and four-body phase space, as a function of s, is gotten by carrying out the integrals in (39). This total phase space is defined as $\rho_k(s)$, $k=2, 4$:

$$\rho_k(s)=\sum_k 1.$$

The phase-space densities are defined by

$$\rho_2(s)=\int d\Omega\,\hat{\rho}_2, \tag{42}$$

$$\rho_4(s)=\int d\Omega\,d\Omega_{12}d\Omega_{34}dM_{12}^2dM_{34}^2\hat{\rho}_4. \tag{43}$$

In (42), Ω refers to the center-of-mass angle variables. In (43), we consider a four-pion state as composed of two dipions, with masses M_{12}, M_{34}; Ω_{12} gives the angle variables for the momentum $\mathbf{p}_1-\mathbf{p}_2$ in the 12-dipion center-of-mass frame, and Ω can be taken as the angle variables for either dipion in the over-all center-of-mass frame. We have

$$\hat{\rho}_2=P/32\pi^2s^{1/2},\quad P=(\tfrac{1}{4}s-m_\pi^2)^{1/2}, \tag{44}$$

$$\hat{\rho}_4=(1/24)(2\pi)^{-8}P_{12}/4M_{12}\times P_{34}/4M_{34}\times Q/4s^{1/2}, \tag{45}$$

with

$$P_{ij}=(\tfrac{1}{4}M_{ij}^2-m_\pi^2)^{1/2},\quad M_{ij}^2=(p_i+p_j)^2,$$
$$Q^2=(4s)^{-1}[s-(M_{12}+M_{34})^2][s-(M_{12}-M_{34})^2]. \tag{46}$$

It is understood that $\hat{\rho}_2$, $\hat{\rho}_4$ vanish whenever any of the momenta P, P_{ij}, Q vanish or become imaginary.

Equation (41) for the D functions is formally solved by

$$D_{ij}=\hat{\delta}_{ij}+\frac{1}{2\pi}\int\frac{\hat{\rho}_i(s')N_{ij}(s')ds'}{s'-s}, \tag{47}$$

where $\hat{\delta}_{ij}$ is a δ function normalized so that

$$\sum_k F_{ik}\hat{\rho}_k^{-1}\hat{\delta}_{kj}=F_{ij}. \tag{48}$$

For example,

$$\hat{\delta}_{22}=\delta(\Omega_f-\Omega_i), \tag{49}$$

where Ω_f, Ω_i are the final and initial angle variables for two-pion scattering. The integral over N_{ij} in (47) is to be taken at fixed scattering angles, and not at fixed momentum transfers, or D will have a left-hand cut as well as a right-hand cut, when partial waves are projected out.

At this point, all the equations are exact (except for possible anomalous cuts), and the N's and D's depend on all the kinematic variables. Our key approximation is to assume that the scattering amplitudes A depend only on the total squared energy s; in part, this amounts to projecting out the s-wave parts of the N's and D's, and it also amounts to ignoring the dependence on the subenergy variables such as M_{ij}^2 [see Eq. (46)].

Our purpose is only to give a crude estimate of the σ mass and width, so we take the N functions from Born approximation. The s-wave projection of N_{22} contains an essential singularity in the s plane, coming from the repulsive Pomeranchuk potential, which prevents the usual application of the N/D formalism; this problem and our resolution of it are discussed in the Appendix. Before projecting out the s wave, N_{22} is given (aside from a change of sign) by the $I=0$ projection of the potentials (1) and (2):

$$N_{22}=\frac{2g^2(m_\rho^2+4m_\pi^2)}{t-m_\rho^2}+G(m_\pi/m_\rho)^2(2\pi)^{1/2}\times\exp(t/m_\rho^2)+(t\leftrightarrow u). \tag{50}$$

We take $G=45$, and have set $s=4m_\pi^2$ in the ρ-exchange potential. The D function for π-π scattering (without coupling to 4π channels) as calculated from N_{22} (see the Appendix) has no resonances or ghosts.

In Born approximation, $N_{24}=N_{42}$ is given by the graph of Fig. 7, which depends on π-π scattering amplitudes. For our purposes, it is sufficient to replace these amplitudes by their value at threshold, namely $16\pi am_\pi$, where a is the $I=0$ s-wave π-π scattering length. This scattering length is estimated to be $\sim 0.5m_\pi^{-1}$, in the Appendix. Then

$$N_{24}=N_{42}=\sum_{i<j}\frac{4(16\pi am_\pi)^2}{t_{ij}-m_\pi^2}, \qquad (51)$$

$$t_{ij}=(P_1-P_i'-P_j')^2. \qquad (52)$$

Note that N_{24} does not depend on the dipion center-of-mass angle variables Ω_{12}, Ω_{34} [see (43)] but only on the over-all center-of-mass angle variables Ω, as well as the M_{ij}^2. After projecting out s waves in the variables Ω, we set all $M_{ij}^2=4m_\pi^2$, consistent with our threshold approximation for the π-π scattering amplitudes.

Near threshold, we know the entire A_{44} amplitude, from the graph of Fig. 6, in the $I=0$ s-wave channel. Ordinary nonrelativistic scattering theory gives

$$(2\pi)^3\delta(\mathbf{p}_f-\mathbf{p}_i)(2m_\pi)^{-4}A_{44} \simeq \int \frac{d^3x\,\langle 4\pi_f|V|\sigma\rangle\langle\sigma|V|4\pi_i\rangle}{E-E_B}, \qquad (53)$$

where $|4\pi\rangle$ is an $I=0$ phase-wave state, $|\sigma\rangle$ is the variational wave function of Sec. 2, and V is the π-π potential. The integral in (53) is over the center-of-mass coordinates, with the rest of the integrations subsumed in the Dirac notation. After carrying out some tedious integrations, we get the result that (near threshold)

$$A_{44}\simeq(4\pi)^4(24)(0.16m_\pi^{-2})/(s-s_B), \qquad (54)$$

where $s_B\lesssim 16m_\pi^2$.

Unfortunately, we do not know much about N_{44}; it is a very complicated object involving (for example) ρ exchange between various pion pairs. This is not a serious handicap, for (if the 2π-4π coupling is not too strong) all we need to know about the 44 channel is the position s_B of the bound state, and the residue of the s-wave part of D_{44}. The procedure we adopt is tantamount to an effective range calculation in two-body scattering. Assume that A_{44} itself approximately obeys elastic unitarity, and write $A_{44}(s)=N_{44}(s)/D_{44}(s)$, with

$$\mathrm{Im}D_{44}(s)=\tfrac{1}{2}\rho_4(s)N_{44}(s), \qquad (55)$$

where ρ_4 is the total four-body phase space [see (43)]. Choose various smooth forms for $N_{44}(s)$, such that A_{44} has the correct pole and residue as in (54). From these, the residue

$$R\equiv\partial D_{44}(s_B)/\partial s \qquad (56)$$

is evaluated. If N_{44} is chosen to damp toward zero somewhere in the range $50m_\pi^2<s<150m_\pi^2$, we find R in the range $-0.03<Rm_\pi^2<-0.01$.

The rest of the calculation is routine. The s-wave parts of N and D matrices are gotten by integrating over all angular variables and dividing by the total angular phase space. We denote by overbars all such s-wave projections; thus,

$$\bar{N}_{24}=\bar{N}_{42}=-24(16\pi am_\pi)^2 \times\frac{1}{4pq}\ln\left(\frac{p^2+q^2+2pq+m_\pi^2}{p^2+q^2-2pq+m_\pi^2}\right), \qquad (57)$$

$$p^2=\tfrac{1}{4}s-4m_\pi^2,\quad q^2=\tfrac{1}{4}s-m_\pi^2 \qquad (58)$$

(it will be recalled that, after doing the angular integrations, we are to set all $M_{ij}^2=4m_\pi^2$). Equations (40) and (41) reduce to purely algebraic ones ($\bar{N}_{44}=N_{44}$, $\bar{D}_{44}=D_{44}$) which we write for 2×2 matrices:

$$\bar{A}\rho\bar{D}=\bar{N},\quad \mathrm{Im}\bar{D}=\tfrac{1}{2}\rho\bar{N}, \qquad (59)$$

where the diagonal matrix ρ is the total phase-space matrix. The $\bar{D}$ equations are integrated with cutoffs in the range 50–150m_π^2, for $\bar{D}_{42}$. With the approximation of an energy-independent ρ-exchange potential, $\bar{D}_{22}$ needs no cutoff, and with Eq. (57) for $\bar{N}_{24}$, neither does $\bar{D}_{24}$.

All that we need to calculate the mass and width of the physical σ is the determinant of the matrix $\bar{D}$. The real part of this determinant vanishes at $s=s_0$, where s_0 is the real part of the physical σ mass. We make the approximation $\bar{D}_{44}(s)\simeq R(s-s_B)$, $s_B\simeq16m_\pi^2$, and find

$$s_0-s_B\simeq\left.\frac{\bar{D}_{42}\,\mathrm{Re}\bar{D}_{24}}{R(\mathrm{Re}\bar{D}_{22})}\right|_{s=s_0}. \qquad (60)$$

If s_0 is not too far above s_B, $\bar{D}_{42}$ can be considered as real. The width is given by

$$s_0^{1/2}\Gamma=(s_0-s_B)\left[\frac{\mathrm{Im}\bar{D}_{22}}{\mathrm{Re}\bar{D}_{22}}-\frac{\mathrm{Im}\bar{D}_{24}}{\mathrm{Re}\bar{D}_{24}}\right]\Bigg|_{s=s_0}. \qquad (61)$$

Observe that all components of $\bar{N}$ are negative (attractive). It is easy to check, under these circumstances, that R and $\bar{D}_{42}$ are negative and $\mathrm{Re}\bar{D}_{22}$ is positive. As for $\bar{D}_{24}$, we have [as usual, anomalous cuts are ignored in evaluating (60)]:

$$\mathrm{Re}\bar{D}_{24}(s_0)=+\frac{P}{2\pi}\int_{4m_\pi^2}^{\infty}ds'\frac{\rho_2(s')\bar{N}_{24}(s')}{s'-s_0}. \qquad (62)$$

Although $\rho_2\bar{N}_{24}<0$ [see (57)], the sign of $\mathrm{Re}\bar{D}_{24}$ is indeterminate since s_0 is above the 2π threshold. If the 2π channel were above the 4π channel, then $\mathrm{Re}\bar{D}_{24}$ would be negative; as it is, the integral (62) turns out to give a positive $\mathrm{Re}\bar{D}_{24}$, since the low-energy region dominates. This statement should be fairly trustworthy, since no cutoffs are needed in $\bar{D}_{24}$. We then see that $s_0-s_B>0$, so that the σ mass is raised.

With cutoffs in $\bar{D}_{44}$ and $\bar{D}_{42}$ in the range $50\text{–}150 m_\pi^2$, we find

$$s_0 - s_B \simeq (2\text{–}6) m_\pi^2, \qquad \Gamma \simeq (0.5\text{–}1.5) m_\pi. \tag{63}$$

Presumably, $s_B \simeq 16 m_\pi^2$. It might be reasonable to say that our calculations indicate the possibility of a σ of mass $\simeq 650$ MeV, width $\simeq 120$ MeV. Of course, these numbers should not be taken very literally.

4. DISCUSSION

The authors, of course, would be much more comfortable if there were unambiguous experimental evidence for a resonance in the $I=J=0$ 2π channel. Unfortunately, the evidence is vague and confusing, but there appears to be no doubt that the relevant phase shift is large and possibly resonant somewhere between 600 and 800 MeV (see, e.g., Ref. 14, which contains further references). This energy range is somewhat higher than we predict; if there turns out to be a resonance at about the ρ mass, one should probably take channels like 6π or $K\bar{K}$ into account, which would complicate the theory enormously. In any event, the calculations do not lead us to expect a σ meson around 400 MeV, as proposed by Brown and Singer,[1] but this is just as well, in view of experimental data on K_{e4} spectra[2] which make such a low-lying state most unlikely. If it turns out that there is no σ meson, a likely explanation would be that repulsive forces are too strong to allow a 4π bound state or low-lying resonance. We note that repulsive forces must be present in substantial amounts to account for the absence of certain multipion bound states, and that our knowledge of the exact strength of this repulsion is very uncertain.

ACKNOWLEDGMENTS

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APPENDIX: N/D EQUATIONS WITH ESSENTIAL SINGULARITIES

Here we describe an approximate, practical method of using the N/D formalism for potentials like a square well or Gaussian, whose partial waves have essential singularities as $|s| \to \infty$, but the potentials vanish in one of the limits $s \to \pm\infty$. We describe the Gaussian case, where the potential vanishes as $s \to +\infty$. Instead of using the equation

$$\operatorname{Im} N_l(s) = D_l(s) \operatorname{Im} t_l(s) \tag{A1}$$

on the left-hand cut, which cannot be integrated in the determinantal approximation ($D_l = 1$, $t_l = t_l^B =$ Born approximation), we use the well-known integral equation for N_l, with $t_l(s)$ set equal to $t_l^B(s)$:

$$N_l(s) = t_l^B(s) + \frac{1}{\pi} \int ds' N_l(s') \times \frac{[t_l^B(s) - t_l^B(s')]\rho_l(s')}{s'-s}. \tag{A2}$$

The integral is over the unitarity cut, on which $t_l^B(s)$ is well behaved for a Gaussian potential $V(r) \sim \exp(-r^2M^2)$:

$$t_l^B(s) \simeq\sim (1/q^2)[1 - \exp(-q^2/M^2)], \tag{A3}$$

where $s = 4(q^2 + m_\pi^2)$. The D function is given by the usual expression:

$$D_l(s) = 1 - \frac{1}{\pi} \int ds' \frac{\rho_l(s') N_l(s')}{s'-s} \tag{A4}$$

(we assume that no subtractions are necessary). Of course, (A2) and (A4) may be used for a potential which is the sum of a Gaussian and a well-behaved potential such as a Yukawa potential.

We have checked the accuracy of (A2) and (A4) in nonrelativistic scattering, by using an attractive Gaussian potential in π-π scattering, with $M^2 = M_\rho^2$. In the s wave, computer calculations show that the coupling constant needed to give a zero-energy bound state, as computed in our N/D method, is about twice the exact value. This relative accuracy is as good as the same equations when used with a Yukawa potential.

We calculated π-π s-wave N and D functions, for use in Sec. 3, by this method, using the ρ-exchange potential and Pomeranchuk repulsion of Eqs. (1) and (2). Energy dependence of the ρ potential was ignored. The result was a scattering length $a = 0.5 m_\pi^{-1}$, which is used in Sec. 3. As a further check, we calculated the p-wave scattering amplitude. This amplitude resonates in the ρ region ($s \simeq 30 m_\pi^2$), with an output width comparable to the input width (i.e., strength of the ρ potential). This bears out Chew's[6] contention that the ρ might bootstrap itself given some repulsion to narrow the output width. The s-wave scattering amplitude had no resonances, ghosts, or bound states.

[14] E. Malamud and P. Schlein, Phys. Rev. Letters **19**, 1056 (1967).