These expressions can always be checked against the similar expressions obtained by using traces and projection operators. For example, in our case

$$\sum_{\lambda_f\lambda_i}|\Gamma_{\lambda_f\lambda_i}{}^{(x)}|^2 = -\mathrm{Tr}\frac{P_i+M_p}{2M_p}\pi_\nu{}^{(x)}\gamma_5\frac{P_f+M_f}{2M_f}\left\{g_{\nu\mu}-\frac{2}{3}\frac{P_{f\mu}P_{f\nu}}{M_f{}^2} + (1/3M_f)(P_{f\nu}\gamma_\mu - P_{f\mu}\gamma_\nu)-\tfrac{1}{3}\gamma_\nu\gamma_\mu\right\}\gamma_5\pi_\mu{}^{(x)}\,, \quad \text{(A12)}$$

where

$$\pi_\nu{}^{(x)}=C_3(qg_{\nu x}-q_\nu\gamma_x)+C_4(q\cdot P_f g_{\nu x}-q_\nu P_{fx}) +C_5(q\cdot P_i g_{\nu x}-q_\nu P_{ix})\,. \quad \text{(A13)}$$

It is probably worth mentioning that the method using Eqs. (A10) and (A11) takes much less effort than the one using Eq. (A12) unless the trace in the latter is taken by computer. We have used all methods checking both by hand calculations and by the computer program of Hearn.[24]

[24] A. C. Hearn, Commun. ACM 9, 573 (1966). See also "REDUCE User's Manual," Stanford Institute of Theoretical Physics Report No. ITP-247 (unpublished).

PHYSICAL REVIEW VOLUME 168, NUMBER 5 25 APRIL 1968

Free Massless Fields as Infinite-Dimensional Representations of the Lorentz Group*

CARL M. BENDER†
Harvard University, Cambridge, Massachusetts

(Received 1 September 1967; revised manuscript received 31 October 1967)

Free quantized massless field theories of arbitrary spin L are investigated. The transverse potential in the radiation gauge is shown to transform as a nonunitary infinite-dimensional representation of the Lorentz group: $(L,1)\oplus(L,-1)$ for integer spin and $(L+\frac{1}{2},\frac{3}{2})\oplus(L+\frac{1}{2},-\frac{3}{2})$ for integer $+\frac{1}{2}$ spin (Gel'fand and Shapiro's notation). Using Lorentz group theory, it is argued that free quantized massless field theories of spin >1 do not possess a stress-energy tensor $T^{\mu\nu}$.

I. INTRODUCTION

In a recent paper,[1] Strocchi showed that the A potential in free-field quantum electrodynamics cannot transform as a vector, as it does in the classical theory. What, then, is its transformation law (if any), and is it unique?

It is the main purpose of this paper to elucidate this transformation law, not only for spin 1, but also for spin L. In Sec. II, the assumptions of this paper will be stated, and the radiation gauge will be precisely defined. In Sec. III, this definition will be used to prove the radiation gauge manifestly covariant; the transverse fields will be shown to belong to infinite-dimensional, nonunitary representations of the Lorentz group.

With this established, the simpler case of integer-spin massless-field theories will be developed. Section IV contains the derivation of the transformation law and the field equations of these theories and some remarks on the field strengths. Section V discusses some applications of the transformation law, such as the construction of scalar and tensor bilinear forms. In Sec. VI (Conclusions), the Lorentz invariance of the theory and the question of the Lagrangian in massless-field theories are discussed. It is concluded that canonically quantized Lagrangian massless-field theories of spin $L>1$ do not possess a covariant stress-energy tensor $T^{\mu\nu}$.

* Supported in part by Air Force Office of Scientific Research under Contract AF 49 (638)-1380.
† Supported by a National Science Foundation Predoctoral Fellowship.

[1] F. Strocchi, Phys. Rev. 162, 1429 (1967).

II. RADIATION GAUGE

Gauge invariance occurs in massless-field theories of spin $L>\frac{1}{2}$ because the field equations that are derived from a Lagrangian do not completely determine the fields. Gauge transformations leave invariant that part of the field which is determined.

The fields in the radiation gauge are called transverse. The radiation gauge is defined by stating the properties of these transverse fields [Eq. (1)].

For Bose-Einstein field theories of spin $L\geq 1$, $A(L)^L{}_{a_1\cdots a_L}$ is a Hermitian, totally symmetric, traceless tensor field, with a_i being 3-space indices,

$$\nabla_{a_i}A(L)^L{}_{a_1\cdots a_L}=0,\quad i=1,\cdots,L\,. \quad \text{(1a)}$$

The superscript L indicates that A has L indices. The L in parentheses indicates that A describes a spin-L theory.

For Fermi-Dirac field theories of spin $L+\frac{1}{2}\geq\frac{3}{2}$, $\psi(L+\frac{1}{2})^L{}_{a_1\cdots a_L}$ is a Hermitian, totally symmetric, traceless tensor-spinor with the spinor index suppressed,

$$\nabla_{a_i}\psi(L+\tfrac{1}{2})^L{}_{a_1\cdots a_L}=\gamma_{a_i}\psi(L+\tfrac{1}{2})^L{}_{a_1\cdots a_L}=0,\qquad i=1,\cdots,L \quad (1b)$$

where a_i are 3-space indices.

A count of the indices, using Eq. (1), reveals that A and ψ contain just two degrees of freedom, which refer to the two helicity states.

The fields in the radiation gauge (as opposed to the Lorentz gauge, for example) are uniquely determined by the gauge condition (1) and other field equations that the fields obey. Thus, if the fields possess a transformation law, it will be unique. The transformation law of the transverse fields is assumed to be[2]

$$-i[A(L)^L{}_{a_1\cdots a_L},J^{0k}]=(x_k\partial^0-x^0\nabla_k)A(L)^L{}_{a_1\cdots a_L}+\sum_{i=1}^{L}\mathfrak{D}_{a_i}A(L)^L{}_{a_1\cdots\bar{a}_i\cdots a_Lk}, \quad (2a)$$

$$-i[\psi(L+\tfrac{1}{2})^L{}_{a_1\cdots a_L},J^{0k}]$$
$$=(x_k\partial^0-x^0\nabla_k)\psi(L+\tfrac{1}{2})^L{}_{a_1\cdots a_L}+\tfrac{1}{2}\gamma^0\gamma_k\psi(L+\tfrac{1}{2})^L{}_{a_1\cdots a_L}+\sum_{i=1}^{L}\mathfrak{D}_{a_i}\psi(L+\tfrac{1}{2})^L{}_{a_1\cdots\bar{a}_i\cdots a_Lk}. \quad (2b)$$

The notation $\bar{a}_i$ indicates that the index a_i is missing.

The transformation law preserves transversality because Eq. (2) is consistent with Eq. (1). According to Eq. (2), under Lorentz transformations the transverse fields transform as the purely spatial parts of massive fields of the same spin, but an additional term is needed to restore transversality. This additional term is nonlinear and nonlocal, and makes the transformation laws of massless-field theories less clear mathematically than those of the massive ones. For this reason it is difficult to determine whether the fields in the radiation gauge are manifestly covariant.

The simplest infinite-dimensional massless-field theory is of spin 1 (electrodynamics). In this case, $A(1)^1{}_a$ is just the transverse magnetic potential, and Eq. (2a) is its well-known transformation law. For spin 2 (the linearized gravitational theory), $A(2)^2{}_{ab}$ is usually written as h_{ab}, which is the deviation from the flat-space metric.

In other papers, Eqs. (1) and (2) have been derived from a Lagrangian for the cases of spin 1, spin $\frac{3}{2}$, and spin 2.[3] In these cases, it was found that the radiation gauge is absolutely fundamental because: (1) The generator of field variations[4] may be inverted to derive the commutation relations only in the radiation gauge. (It is singular in all other gauges.) Thus canonical commutation relations only make sense for transverse fields. (2) Furthermore, it is only in the radiation gauge that the Hilbert space has a positive-definite metric.

Equations (1) and (2) are generalizations to higher spin of what has been proved in the spin-1, spin-$\frac{3}{2}$, and spin-2 theories. In addition, it will be assumed that the transverse fields obey the massless Klein-Gordon or the massless Dirac Eqs. (3),

$$\Box^2 A(L)^L{}_{a_1\cdots a_L}=0, \quad (3a)$$

$$\partial_\mu\gamma^\mu\psi(L+\tfrac{1}{2})^L{}_{a_1\cdots a_L}=0. \quad (3b)$$

Equations (1), (2), and (3) constitute the assumptions of this paper. They will be used to explore the mathematical structure of massless-field theories.

III. REPRESENTATION OF TRANSVERSE FIELDS

The notation used in this section is taken from Gel'fand and Shapiro.[5] Their notation fully characterizes all finite- and infinite-dimensional irreducible representations of the Lorentz group. In their notation a representation is labeled by a pair of numbers (l_0,l_1). The number l_0 is the lowest spin contained in the representation. The number l_1 is the highest spin plus one if the representation has a highest-spin component (the representation is finite-dimensional), and l_1 is some other complex number if the representation is infinite-dimensional. For example, a scalar is represented by (0,1), a vector is represented by (0,2), and a spinor is represented by $(\frac{1}{2},\frac{3}{2})\oplus(\frac{1}{2},-\frac{3}{2})$.

Only the relative sign of l_0 and l_1 can distinguish between two representations having the same absolute values of l_0 and l_1. This means that (l_0,l_1) is the same as $(-l_0,-l_1)$, but different from $(l_0,-l_1)$.

It will now be shown that for every spin, the transverse fields belong to some infinite-dimensional representation of the Lorentz group. Define new fields by

$$B(L)^L{}_{a_1\cdots a_L}\equiv\mathfrak{D}_p\epsilon_{pqa_1}A(L)^L{}_{qa_2\cdots a_L},$$

$$A(L)^{L+1}{}_{a_1\cdots a_{L+1}}\equiv\sum_{i=1}^{L+1}\mathfrak{D}_{a_i}A(L)^L{}_{a_1\cdots\bar{a}_i\cdots a_{L+1}},$$

$$B(L)^{L+1}{}_{a_1\cdots a_{L+1}}\equiv\sum_{i=1}^{L+1}\mathfrak{D}_{a_i}B(L)^L{}_{a_1\cdots\bar{a}_i\cdots a_{L+1}}, \quad (4a)$$

where $B(L)^L$, $A(L)^{L+1}$, and $B(L)^{L+1}$ are totally symmetric and traceless, and $B(L)^L$ is transverse. Also

[2] $\nabla^{-2}f(x)\equiv-(1/4\pi)\int d_3x'(1/|x-x'|)f(x')$, and let $\mathfrak{D}_a\equiv\partial_0\nabla_a\nabla^{-2}$.
[3] For spin 1, see J. Schwinger, *Brandeis Lectures, 1964* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1965), p. 147. For spin $\frac{3}{2}$, see C. Bender and B. McCoy, Phys. Rev. **148**, 1375 (1966). For spin 2, see S. Deser, J. Trubatch, and S. Trubatch, Nuovo Cimento **39**, 1159 (1965); S. J. Chang, thesis, Harvard University, 1967 (unpublished).
[4] J. Schwinger, Phys. Rev. **91**, 713 (1953).
[5] I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and Their Applications* (The Macmillan Company, New York, 1963), pp. 188–197.

define

$$\chi(L+\tfrac{1}{2})^L{}_{a_1\cdots a_L}=\frac{2L+3}{3L+3}\mathfrak{D}_p\epsilon_{pqa_1}\psi(L+\tfrac{1}{2})^L{}_{qa_2\cdots a_L}-\gamma^0\gamma_p\epsilon_{pqa_1}\frac{L}{3L+3}\psi(L+\tfrac{1}{2})^L{}_{qa_2\cdots a_L},$$

$$\psi(L+\tfrac{1}{2})^{L+1}{}_{a_1\cdots a_{L+1}}=\frac{2L+3}{2L+2}\sum_{i=1}^{L+1}\mathfrak{D}_{a_i}\psi(L+\tfrac{1}{2})^L{}_{a_1\cdots\bar{a}_i\cdots a_{L+1}}+\frac{\gamma^0}{2L+2}\sum_{i=1}^{L+1}\gamma_{a_i}\psi(L+\tfrac{1}{2})^L{}_{a_1\cdots\bar{a}_i\cdots a_{L+1}},\tag{4b}$$

$$\chi(L+\tfrac{1}{2})^{L+1}{}_{a_1\cdots a_{L+1}}=\frac{2L+3}{2L+2}\sum_{i=1}^{L+1}\mathfrak{D}_{a_i}\chi(L+\tfrac{1}{2})^L{}_{a_1\cdots\bar{a}_i\cdots a_{L+1}}+\frac{\gamma^0}{2L+2}\sum_{i=1}^{L+1}\gamma_{a_i}\psi(L+\tfrac{1}{2})^L{}_{a_1\cdots\bar{a}_i\cdots a_{L+1}},$$

where $\chi(L+\frac{1}{2})^L$, $\psi(L+\frac{1}{2})^{L+1}$, and $\chi(L+\frac{1}{2})^{L+1}$ are totally symmetric and traceless, and $\chi(L+\frac{1}{2})^L$ is transverse.

Then Eq. (2) becomes

$$-i[A(L)^L{}_{a_1\cdots a_L},J^{0k}]=(x_k\partial^0-x^0\nabla_k)A(L)^L{}_{a_1\cdots a_L}+\frac{1}{L+1}\sum_{i=1}^{L}\epsilon_{a_ikq}B(L)^L{}_{a_1\cdots\bar{a}_i\cdots a_Lq}+\frac{L}{L+1}A(L)^{L+1}{}_{a_1\cdots a_Lk},\tag{5a}$$

$$-i[\psi(L+\tfrac{1}{2})^L{}_{a_1\cdots a_L},J^{0k}]=(x_k\partial^0-x^0\nabla_k)\psi(L+\tfrac{1}{2})^L{}_{a_1\cdots a_L}+\frac{3}{4L+6}\gamma^0\gamma_k\psi(L+\tfrac{1}{2})^L{}_{a_1\cdots a_L}+\frac{2L}{2L+3}\psi(L+\tfrac{1}{2})^{L+1}{}_{a_1\cdots a_Lk}$$
$$+\frac{3}{2L+3}\sum_{i=1}^{L}\epsilon_{a_ikq}\chi(L+\tfrac{1}{2})^L{}_{a_1\cdots\bar{a}_i\cdots a_Lq}.\tag{5b}$$

Equation (5) is not surprising, because this is just a change of notation. But Eq. (6) is most remarkable, because it suggests a closed-group representation that is not the result of any notational tricks.

$$-i[B(L)^L{}_{a_1\cdots a_L},J^{0k}]=(x_k\partial^0-x^0\nabla_k)B(L)^L{}_{a_1\cdots a_L}-\frac{1}{L+1}\sum_{i=1}^{L}\epsilon_{a_ikq}A(L)^L{}_{a_1\cdots\bar{a}_i\cdots a_Lq}+\frac{L}{L+1}B(L)^{L+1}{}_{a_1\cdots a_Lk},\tag{6a}$$

$$-i[\chi(L+\tfrac{1}{2})^L{}_{a_1\cdots a_L},J^{0k}]=(x_k\partial^0-x^0\nabla_k)\chi(L+\tfrac{1}{2})^L{}_{a_1\cdots a_L}+\frac{3}{4L+6}\gamma^0\gamma_k\chi(L+\tfrac{1}{2})^L{}_{a_1\cdots a_L}+\frac{2L}{2L+3}\chi(L+\tfrac{1}{2})^{L+1}{}_{a_1\cdots a_Lk}$$
$$-\frac{3}{2L+3}\sum_{i=1}^{L}\epsilon_{a_ikq}\psi(L+\tfrac{1}{2})^L{}_{a_1\cdots\bar{a}_i\cdots a_Lq}.\tag{6b}$$

Equations (5) and (6) may be simplified, using a decomposition:

$$C(L)^L\equiv A(L)^L+iB(L)^L,\quad C(L)^{L+1}\equiv A(L)^{L+1}+iB(L)^{L+1},$$
$$C(L)^{L*}\equiv A(L)^L-iB(L)^L,\quad C(L)^{L+1*}\equiv A(L)^{L+1}-iB(L)^{L+1},\tag{7a}$$

$$\Lambda^L\equiv\tfrac{1}{2}(1+i\gamma_5)[\psi(L+\tfrac{1}{2})^L+i\chi(L+\tfrac{1}{2})^L],\quad \Lambda^{L+1}\equiv\tfrac{1}{2}(1+i\gamma_5)[\psi(L+\tfrac{1}{2})^{L+1}+i\chi(L+\tfrac{1}{2})^{L+1}],$$
$$\Lambda^{L*}\equiv\tfrac{1}{2}(1-i\gamma_5)[\psi(L+\tfrac{1}{2})^L-i\chi(L+\tfrac{1}{2})^L],\quad \Lambda^{L+1*}\equiv\tfrac{1}{2}(1-i\gamma_5)[\psi(L+\tfrac{1}{2})^{L+1}-i\chi(L+\tfrac{1}{2})^{L+1}],\tag{7b}$$

where $\gamma_5=\gamma^0\gamma^1\gamma^2\gamma^3$, γ_5 is real, and $(\gamma_5)^2=-1$.

Then Eqs. (5) and (6) become

$$-i[C(L)^La_1\cdots a_L,J^{0k}]=(x_k\partial^0-x^0\nabla_k)C(L)^L{}_{a_1\cdots a_L}+\frac{1}{i(L+1)}\sum_{i=1}^{L}\epsilon_{a_ikq}C(L)^L{}_{a_1\cdots\bar{a}_i\cdots a_Lq}+\frac{L}{L+1}C(L)^{L+1}{}_{a_1\cdots a_Lk},\tag{8a}$$

$$-i[\Lambda^La_1\cdots a_L,J^{0k}]=(x_k\partial^0-x^0\nabla_k)\Lambda^L{}_{a_1\cdots a_L}+\frac{3}{4L+6}\gamma^0\gamma_k\Lambda^L{}_{a_1\cdots a_L}$$
$$+\frac{2L}{2L+3}\Lambda^{L+1}{}_{a_1\cdots a_Lk}+\frac{3}{i(2L+3)}\sum_{i=1}^{L}\epsilon_{a_ikq}\Lambda^L{}_{a_1\cdots\bar{a}_i\cdots a_Lq},\tag{8b}$$

and the Hermitian-conjugate equations.

By induction, it may be shown that Eq. (8) is the first of an infinite set of transformation laws. That is, repeated commutation with J^{0k}, the generator of pure Lorentz transformations, generates a sequence of fields $\{C^L, C^{L+1}, C^{L+2}, \cdots$, and H.c.$\}$ and $\{\Lambda^L, \Lambda^{L+1}, \Lambda^{L+2}, \cdots$, and H.c.$\}$, which transform among themselves under a Lorentz transformation. (This sequence is derived explicitly for integer-spin fields in Sec. IV.) Furthermore, each element of the sequence is a unique spin component. Therefore C and Λ are infinite-dimensional, irreducible representations of the Lorentz group whose lowest spin component $l_0=L$ and $L+\frac{1}{2}$, respectively.

Knowing Eq. (8) is sufficient to identify l_1.[5] Then,

$$\begin{aligned} &C(L) \text{ transforms as } (L,1)\,, \\ &C(L)^* \text{ transforms as } (L,-1)\,, \end{aligned} \tag{9a}$$

$$\begin{aligned} &\Lambda \text{ transforms as } (L+\tfrac{1}{2},\tfrac{3}{2})\,, \\ &\Lambda^* \text{ transforms as } (L+\tfrac{1}{2},-\tfrac{3}{2})\,, \end{aligned} \tag{9b}$$

where $C(L)$ and $C(L)^*$, Λ and Λ^* are helicity representations. Under a parity transformation, the above representations (l_0,l_1) become the conjugate representations $(l_0,-l_1)$. This indicates that C and C^*, Λ and Λ^* represent opposite handedness.

Finally, the representation of the Hermitian fields $A(L)$ and $\psi(L+\frac{1}{2})$ is:

$$A(L) \text{ transforms as } (L,1)\oplus(L,-1)\,, \tag{10a}$$

$$\psi(L+\tfrac{1}{2}) \text{ transforms as } (L+\tfrac{1}{2},\tfrac{3}{2})\oplus(L+\tfrac{1}{2},-\tfrac{3}{2})\,. \tag{10b}$$

Although Eq. (10) was derived only for $L\geq 1$, it is most gratifying to find that it is also true for $L=0$. If $L=0$, $\psi(\frac{1}{2})$ transforms as the finite-dimensional spinor representation $(\frac{1}{2},\frac{3}{2})\oplus(\frac{1}{2},-\frac{3}{2})$. Note that this is the representation for neutrinos. The $A(0)$ transforms as $(0,1)\oplus(0,-1)$, which simplifies to $(0,1)$ (because the relative signs of l_0 and l_1 in these representations are the same). So, for spin 0 the two helicity states combine, and the usual finite-dimensional scalar representation emerges. Hence Eq. (10) is universal, and *holds for massless fields of any spin.*

IV. INTEGER-SPIN FIELD EQUATIONS AND TRANSFORMATION LAWS

The complete massless field theory will be exhibited for integer spin only. The additional complexities of the matrix algebra associated with half-odd-integer spin seem to provide no new insights or techniques, although the results are analogous.

A. Field Equations

A full set of field equations may be derived by induction on Eqs. (1), (3), and (8):

$$\Box^2 C(L)^N{}_{a_1\cdots a_N}=0\,, \tag{11}$$

$$\nabla_{a_N} C(L)^N{}_{a_1\cdots a_N}=\partial_0 C(L)^{N-1}{}_{a_1\cdots a_{N-1}}\,, \tag{12}$$

$$\nabla_b \epsilon_{bac} C(L)^N{}_{aa_2\cdots a_N}=-i\partial_0 \frac{L}{N} C(L)^N{}_{ca_2\cdots a_N}+(1/N)\partial_0 \sum_{i=2}^{N} \epsilon_{a_i q c} C(L)^{N-1}{}_{qa_2\cdots \bar{a}_i\cdots a_N}\,, \tag{13}$$

$$\begin{aligned} \nabla_k C(L)^N{}_{a_1\cdots a_N}=&\frac{(1+N)^2-L^2}{(2N+1)(N+1)}\partial_0 C(L)^{N+1}{}_{a_1\cdots a_N k}-i\partial_0\frac{L}{N(N+1)}\sum_{i=1}^{N}\epsilon_{k a_i q} C(L)^N{}_{qa_1\cdots\bar{a}_i\cdots a_N} \\ &+\frac{2N-1}{N(2N+1)}\partial_0\sum_{i=1}^{N}\delta_{a_i k} C(L)^{N-1}{}_{a_1\cdots\bar{a}_i\cdots a_N}-\frac{1}{N(2N+1)}\partial_0\sum_{\substack{i,j=1\\ i\neq j}}^{N}\delta_{a_i a_j} C(L)^{N-1}{}_{a_1\cdots\bar{a}_i\cdots\bar{a}_j\cdots a_N k}\,, \end{aligned} \tag{14}$$

and the H.c. equations. These equations are true for $N\geq L$. Remember that $C(L)^N$ for $N<L$ does not exist (is zero).

Commutating the fields with J^{0k}, we find that these field equations constitute a manifestly covariant, infinite-dimensional set. Thus, contrary to popular opinion, the radiation gauge is manifestly covariant.

Equations (14) and (11) or Eqs. (14) and (12) are sufficient to imply the other field equations. These are a full set of equations because all of the higher fields $C(L)^{L+1}, C(L)^{L+2}, C(L)^{L+3}, \cdots$, are determined in terms of the lowest field $C(L)^L$. This is a peculiar field theory, because there are an infinite number of fields with an infinite number of constraint equations. This leaves only a finite number (two) of independent degrees of freedom.

Equations (13) and (14) may be made totally symmetric and traceless [Eqs. (11) and (12) already are]:

$$\sum_{i=1}^{N} \nabla_p \epsilon_{pqa_i} C(L)^N{}_{a_1\cdots\bar{a}_i\cdots a_N q}=-iL\partial_0 C(L)^N{}_{a_1\cdots a_N}\,, \tag{13'}$$

$$\sum_{i=1}^{N+1}\nabla_{a_i} C(L)^N{}_{a_1\cdots\bar{a}_i\cdots a_{N+1}}-\frac{1}{2N+1}\sum_{\substack{i,j=1\\ i\neq j}}^{N+1}\nabla_q\delta_{a_i a_j} C(L)^N{}_{qa_1\cdots\bar{a}_i\cdots\bar{a}_j\cdots a_{N+1}}=\frac{(N+1)^2-L^2}{2N+1}\partial_0 C(L)^{N+1}{}_{a_1\cdots a_{N+1}}\,. \tag{14'}$$

This new set of field equations contains no less information than the previous set.

B. Transformation Law

The transformation law for the fields is explicitly

$$-i[C(L)^N{}_{a_1\cdots a_N},J^{0k}]=(x^k\partial^0-x^0\nabla_k)C(L)^N{}_{a_1\cdots a_N}+N\frac{(N+1)^2-L^2}{(N+1)(2N+1)}C(L)^{N+1}{}_{a_1\cdots a_Nk}$$
$$+\frac{L}{iN(N+1)}\sum_{i=1}^{N}\epsilon_{a_ikq}C(L)^N{}_{a_1\cdots\bar{a}_i\cdots a_Nq}+\frac{N+1}{N(2N+1)}\sum_{\substack{i,j=1\\i\neq j}}^{N}\delta_{a_ia_j}C(L)^{N-1}{}_{a_1\cdots\bar{a}_i\cdots\bar{a}_j\cdots a_Nk}$$
$$-\frac{(N+1)(2N-1)}{N(2N+1)}\sum_{i=1}^{N}\delta_{a_ik}C(L)^{N-1}{}_{a_1\cdots\bar{a}_i\cdots a_N}. \quad (15)$$

From Eqs. (14) and (15) the transformation law for the Pth time derivative of the $C(L)$ field [Eq. (16)] may be computed:

$$-i[(\partial_0)^PC(L)^N{}_{a_1\cdots a_N},J^{0k}]=(x^k\partial^0-x^0\nabla_k)(\partial_0)^PC(L)^N{}_{a_1\cdots a_N}+\frac{(N-P)[(N+1)^2-L^2]}{(2N+1)(N+1)}(\partial_0)^PC(L)^{N+1}{}_{a_1\cdots a_Nk}$$
$$+\frac{L(P+1)}{iN(N+1)}\sum_{i=1}^{N}\epsilon_{a_ikq}(\partial_0)^PC(L)^N{}_{a_1\cdots\bar{a}_i\cdots a_Nq}$$
$$+\frac{(N+1+P)}{N(2N+1)}\sum_{\substack{i,j=1\\i\neq j}}^{N}\delta_{a_ia_j}(\partial_0)^PC(L)^{N-1}{}_{a_1\cdots\bar{a}_i\cdots\bar{a}_j\cdots a_Nk}$$
$$-\frac{(2N-1)(N+1+P)}{N(2N+1)}\sum_{i=1}^{N}\delta_{a_ik}(\partial_0)^PC(L)^{N-1}{}_{a_i\cdots\bar{a}_1\cdots a_N}. \quad (16)$$

From this equation it may be determined that

$$(\partial_0)^PC(L)\text{ transforms as }(L,1+P),\quad(\partial_0)^PC(L)^*\text{ transforms as }(L,-1-P). \quad (17)$$

The smallest integer P for which the representation $(L, 1+P)$ becomes finite-dimensional is L. For $L=P$, the transformation law [Eq. (16)] for the lowest spin component $N=L$ is simply

$$-i[(\partial_0)^LC(L)^L{}_{a_1\cdots a_L},J^{0k}]=(x^k\partial^0-x^0\nabla_k)(\partial_0)^LC(L)^L{}_{a_1\cdots a_L}+1/i\sum_{i=1}^{L}\epsilon_{a_ikq}(\partial_0)^LC(L)^L{}_{a_1\cdots\bar{a}_i\cdots a_Lq}. \quad (18)$$

This implies that the Hermitian fields $(\partial_0)^LA^L(L)$, $(\partial_0)^LB^L(L)$ belong to the finite-dimensional tensor representation $(L, L+1)\oplus(L, -L-1)$. This tensor will be called a *field strength*, and is written as $F^{\mu_1\nu_1,\mu_2\nu_2,\mu_3\nu_3,\cdots,\mu_L\nu_L}$. The tensor F is antisymmetric under exchange of any μ_i and ν_i, symmetric under interchange of any pair $\mu_i\nu_i$ and $\mu_j\nu_j$, and traceless under summation of any two indices. Explicitly,

$$F^{0a_1,0a_2,0a_3,\cdots,0a_L}=-(\partial_0)^LA(L)^L{}_{a_1\cdots a_L},$$
$$F^{0a_1,0a_2,0a_3,\cdots,0a_{L-1}nm}=(\partial_0)^L\epsilon_{nmi}B(L)^L{}_{a_1\cdots a_{L-1}i},$$
$$F^{0a_1,0a_2,0a_3,\cdots,0a_{L-2}nm,pq}=(\partial_0)^L\epsilon_{nmi}\epsilon_{pqj}B(L)^L{}_{a_1\cdots a_{L-2}ij},\text{ etc.} \quad (19)$$

The field-strength tensor is gauge invariant. It obeys Maxwell's equations, which are

$$\partial_{\mu_1}F^{\mu_1\nu_1,\cdots,\mu_L\nu_L}=0. \quad (20)$$

For spin 1 (electrodynamics), the field-strength tensor is the usual $F^{\mu\nu}$. For spin 2 (linearized gravitational field), $F^{\mu_1\nu_1,\mu_2\nu_2}$ is the Riemann-curvature tensor. The contracted Riemann tensor $F^{\mu\nu}$ vanishes as expected, because there is no coupling to an external energy density. Although $F^{\mu_1\nu_1,\mu_2\nu_2}$ generally contains 20 independent quantities, there are only 10 in this case, because the contracted tensor $F^{\mu\nu}$ is 0. These ten quantities are just the five components of the $A(2)^2{}_{ab}$ and of the $B(2)^2{}_{ab}$ fields.

Equations (16) and (17) are the transformation law for multiple-time derivatives of the fields. It is also possible to construct the time-integral fields and their transformation laws. Since all fields obey the massless Klein-Gordon equation, $\partial_0\nabla^{-2}$ is effectively a time-integral operator which contains its own boundary conditions. Formally,

$$(\partial_0)^{-P}C(L)\equiv(\partial_0/\nabla^2)^P C(L). \tag{21}$$

Now, Eqs. (16) and (17) are true for multiple-time integrals of the fields when $P\to -P$. For example, the first time integral of the fields $(\partial_0)^{-1}L$ transforms as $(L,0)$, and so does its Hermitian conjugate. The first time integral of the fields is interesting because it is the only case in which there is a *real* transformation law (no imaginary coefficients):

$$-i[(\partial_0)^{-1}C(L)^N{}_{a_1\cdots a_N},J^{0k}]=(x^k\partial^0-x^0\nabla_k)(\partial_0)^{-1}C(L)^N{}_{a_1\cdots a_N}+\frac{(N+1)^2-L^2}{2N+1}(\partial_0)^{-1}C(L)^{N+1}{}_{a_1\cdots a_N k}$$
$$+\frac{1}{2N+1}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\delta_{a_ia_j}(\partial_0)^{-1}C(L)^{N-1}{}_{a_1\cdots\bar{a}_i\cdots\bar{a}_j\cdots a_N k}-\frac{2N-1}{2N+1}\sum_{i=1}^{N}\delta_{a_ik}(\partial_0)^{-1}C(L)^{N-1}{}_{a_1\cdots\bar{a}_i\cdots a_N}. \tag{22}$$

V. BILINEAR FORMS

It has been established that the $C(L)$ and $C(L)^*$ fields are irreducible representations of the Lorentz group. What, if any, simple Hermitian-covariant objects can be constructed out of bilinear forms of these fields? That is, what are the first few terms in the Clebsch-Gordon series? The answer to this problem can be obtained by the application of difference equations to Eqs. (11)–(15):

$$\begin{aligned}(L,1)\otimes(L,1)\oplus(L,-1)\otimes(L,-1)=&\ (0,1)\ \text{(a Hermitian scalar } S)\\ &\oplus(0,3)\ \text{(a Hermitian traceless symmetric tensor } R^{\mu\nu})\\ &\oplus\cdots,\end{aligned} \tag{23}$$

$$\begin{aligned}(L,1)\otimes(L,-1)\oplus(L,-1)\otimes(L,1)=&\ (0,3)\ \text{(a Hermitian traceless symmetric tensory } Q^{\mu\nu})\\ &\oplus\cdots.\end{aligned} \tag{24}$$

The higher terms in these series are many-index tensors and perhaps new infinite-dimensional representations.

The structure of $Q^{\mu\nu}$, $R^{\mu\nu}$, and S is as follows. Let

$$D(N+1)=\frac{(N+2)^2-L^2}{(2N+1)(N+3)}D(N), \tag{25}$$

where $D(L)$ is an undetermined constant. Then

$$S=\sum_{N=L}^{\infty}D(N)\frac{(N+2)(2N+1)L}{N[(N+1)^2-L^2]}[C(L)^N{}_{a_1\cdots a_N}C(L)^N{}_{a_1\cdots a_N}+\text{H.c.}], \tag{26}$$

$$Q^{00}=\sum_{N=L}^{\infty}D(N)\frac{(2N+1)(N+2)(N+1)}{L[(N+1)^2-L^2]}[C(L)^N{}_{a_1\cdots a_N}.C(L)^{N*}{}_{a_1\cdots a_N}],$$

$$Q^{0n}=\sum_{N=L}^{\infty}-D(N)\frac{(N+2)}{L}[C(L)^{N+1}{}_{na_1\cdots a_N}.C(L)^{N*}{}_{a_1\cdots a_N}+\text{H.c.}]$$
$$+\sum_{N=L}^{\infty}-iD(N)\frac{(2N+1)(N+2)}{(N+1)^2-L^2}\epsilon_{ABn}[C(L)^N{}_{a_1\cdots a_{N-1}A}.C(L)^{N*}{}_{a_1\cdots a_{N-1}B}], \tag{27}$$

$$Q^{mn}=\sum_{N=L}^{\infty}\delta_{mn}D(N)\frac{(N+2)(2N+1)(N+1+2L^2)}{L(2N+3)[(N+1)^2-L^2]}[C(L)^N{}_{a_1\cdots a_N}\cdot C(L)^{N*}{}_{a_1\cdots a_N}]$$
$$+\sum_{N=L}^{\infty}D(N)\frac{(N+2)(2N+1)(N^2+N-3L^2)}{L(2N+3)[(N+1)^2-L^2]}[C(L)^N{}_{a_1\cdots a_{N-1}m}\cdot C(L)^{N*}{}_{a_1\cdots a_{N-1}n}+\text{H.c.}]$$
$$+\sum_{N=L}^{\infty}D(N)\frac{(N+2)^2-L^2}{L(2N+3)}[C(L)^{N+2}{}_{a_1\cdots a_N mn}\cdot C(L)^{N*}{}_{a_1\cdots a_N}+\text{H.c.}]$$
$$+\sum_{N=L}^{\infty}D(N)\{\epsilon_{ABm}[iC(L)^{N+1}{}_{a_1\cdots a_{N-1}An}\cdot C(L)^{N*}{}_{a_1\cdots a_{N-1}B}+\text{H.c.}]$$
$$+\epsilon_{ABn}[iC(L)^{N+1}{}_{a_1\cdots a_{N-1}Am}\cdot C(L)^{N*}{}_{a_1\cdots a_{N-1}B}+\text{H.c.}]\}\,,$$
$$R^{00}=\sum_{N=L}^{\infty}-D(N)\frac{(2N+1)(N+2)(2N^2+2N-L^2)}{2LN[(N+1)^2-L^2]}[C(L)^N{}_{a_1\cdots a_N}C(L)^N{}_{a_1\cdots a_N}+\text{H.c.}],$$
$$R^{0n}=\sum_{N=L}^{\infty}D(N)\frac{2(N+2)}{L}[C(L)^{N+1}{}_{na_1\cdots a_N}\cdot C(L)^N{}_{a_1\cdots a_N}+\text{H.c.}],$$
$$R^{mn}=\sum_{N=L}^{\infty}\delta_{mn}D(N)\frac{(2N+1)(N+2)[L^2(2N^2+3N-3)-2N(N+1)^2]}{LN(2N+3)(N+1)[(N+1)^2-L^2]}[C(L)^N{}_{a_1\cdots a_N}C(L)^N{}_{a_1\cdots a_N}+\text{H.c.}]$$
$$+\sum_{N=L}^{\infty}\frac{-D(N)2(2N+1)(N+2)[N^2(N+1)^2+L^2(N^2+N-3)]}{LN(N+1)(2N+3)[(N+1)^2-L^2]}[C(L)^N{}_{a_1\cdots a_{N-1}m}\cdot C(L)^N{}_{a_1\cdots a_{N-1}n}+\text{H.c.}]$$
$$+\sum_{N=L}^{\infty}-D(N)\frac{2[(N+2)^2-L^2]}{L(2N+3)}[C(L)^{N+2}{}_{a_1\cdots a_N mn}\cdot C(L)^N{}_{a_1\cdots a_N}+\text{H.c.}]$$
$$+\sum_{N=L}^{\infty}D(N)\frac{2}{N+1}\{\epsilon_{ABm}[iC(L)^{N+1}{}_{a_1\cdots a_{N-1}An}\cdot C(L)^N{}_{a_1\cdots a_{N-1}B}+\text{H.c.}]$$
$$+\epsilon_{ABm}[iC(L)^{N+1}{}_{a_1\cdots a_{N-1}Am}\cdot C(L)^N{}_{a_1\cdots a_{N-1}B}+\text{H.c.}]\}\,. \tag{28}$$

The dot on the line indicates Bose operator symmetrization.

The properties of $Q^{\mu\nu}$, $R^{\mu\nu}$, and S may be obtained by using the field equations (11)–(14):

$$\partial_\mu Q^{\mu\nu}=0\,, \tag{29}$$
$$\partial_\mu R^{\mu\nu}\neq 0\,, \tag{30}$$
$$\partial_\mu\partial_\nu R^{\mu\nu}=\Box^2 S\,, \tag{31}$$
$$\Box^2 Q^{\mu\nu}=0\,, \tag{32}$$
$$(\Box^2)^P S=\sum_{N=L}^{\infty}\left\{\frac{2^P[P^2-L^2][(P-1)^2-L^2]\cdots[1-L^2]}{(N+P+1)(N+P)\cdots(N+2)(N-P)(N-P+1)\cdots(N-1)}\times\frac{D(N)(N+2)(2N+1)L}{N[(N+1)^2-L^2]}[(\partial_0)^P L^N{}_{a_1\cdots a_N}(\partial_0)^P L^N{}_{a_1\cdots a_N}+\text{H.c.}]\right\}. \tag{33}$$

In Eq. (33), if $P=L$, then all terms in the infinite series vanish except for the first term ($N=L$) because of the factors (P^2-L^2) in the numerator and ($N-P$) in the denominator. Thus $(\Box^2)^L S$ is the unique simple (no longer an infinite series, but a single term) scalar. It is constructed from elements of the field-strength tensor. In fact, it is the square of the tensor. In electromagnetic theory this unique simple scalar is E^2-H^2. There is a similar construction for the pseudoscalar $E\cdot H$.

VI. CONCLUSIONS

Although the selection of the radiation gauge was motivated by quantum mechanics, the infinite-dimensional fields have been treated classically. That is, only their field equations and transformation laws have been used. Is it possible to quantize these fields?

The most general Hermitian scalar Lagrangian is of the form

$$\mathcal{L}=a\vec{\partial}_\mu S\overleftarrow{\partial}^\mu+b\vec{\partial}_\mu R^{\mu\nu}\overleftarrow{\partial}_\nu+c\vec{\partial}_\mu Q^{\mu\nu}\overleftarrow{\partial}_\nu\,, \tag{34}$$

where a, b, and c are adjustable coefficients, and ∂ operates on the first field and $\overleftarrow{\partial}$ on the second field in each term of the bilinear forms Q, R, and S. Varying this Lagrangian gives the correct field equations for every choice of a, b, and c. However, there are some field equations missing; that is, there is additional gauge freedom.[6] The radiation gauge again must be chosen by restating the condition [Eq. (1)]. This eliminates all gauge ambiguity.

To compute the canonical stress tensor $T^{\mu\nu}$, space-time variations of the Lagrangian are performed.[4] For arbitrary choice of a, b, and c, the symmetric, traceless, divergenceless $T^{\mu\nu}$ is a bilinear form in the Clebsch-Gordan expansion of $(L,2)\otimes(L,2)\oplus(L,-2)\otimes(L,-2)$. But when the total energy P^0 is computed from a 3-space integral of T^{00}, it is found *not* to be positive definite. (It can be written as a difference of squares.) For this reason the Lagrangian [Eq. (34)] is unacceptable.

If it were possible to find a symmetric traceless tensor in the Clebsch-Gordan expansion of $(L,2)\otimes(L,-2)\oplus(L,-2)\otimes(L,2)$, the associated P^0 might be positive definite (sum of squares). But there is no symmetric tensor in this expansion except for the case of spin 1 (electrodynamics), where $T^{00}=\frac{1}{2}(E^2+H^2)$. This tensor exists for spin 1 because $(L,\pm 2)$ for spin 1 is $(1,\pm 2)$, which is the finite-dimensional field-strength tensor.[7] However, this tensor cannot be derived from the infinite-dimensional spin-1 theory.

It is most important to note that a *covariant energy-density tensor* for conventional massless field theories of spin greater than 1 does not exist. (If it did exist, it would appear in the Clebsch-Gordan expansion.) This is the reason for the remarkable anomalies of gauge invariance and noncausality in massless spin-$\frac{3}{2}$ and spin-2 theories.[3] There, "pseudo" (noncovariant) energy densities were substituted for the unavailable covariant energy densities.[8]

[6] It was originally hoped that in this infinite-dimensional field theory the whole concept of gauge invariance would be eliminated. However, gauge invariance seems to be fundamentally connected with masslessness and cannot be suppressed even in this infinite-dimensional framework. Lagrange functions with only first derivatives, but with auxiliary fields, were also tested. These Lagrangians yield results identical to those obtained from the Lagrangian (34).

[7] The quantization of electrodynamics is successful because the field-strength tensor is the first time derivative of the transverse potential. No other massless-field theory has this advantage.

[8] These pseudotensors are called noncovariant because they do not transform as finite-dimensional representations of the Lorentz group. However, they might transform as covariant infinite-dimensional tensors. The possibility that the stress-energy tensor of massless field theories is infinite-dimensional is not considered in this paper.

Although a covariant field-theoretic energy density cannot exist, there is nothing preventing the existence of a Lorentz-invariant theory describing massless particles of arbitrary spin traveling through empty space. In mathematical terms, covariant generators P^μ, $J^{\mu\nu}$, which have the correct positiveness and commutation properties of the Lorentz group, may be constructed using a noncovariant pseudo-stress-tensor "T":

$$\begin{aligned}``T^{00}"&=\tfrac{1}{2}[(\partial_0 A)^2+(\partial_0 B)^2]\,,\\ ``T^{0i}"&=-\epsilon_{ijk}\partial_0 A\,(L)^L{}_{a_1\cdots a_{L-1}j}\partial_0 B(L)^L{}_{a_1\cdots a_{L-1}k}\,,\end{aligned} \tag{35}$$

where A obeys the canonical commutation relations[9]

$$\begin{aligned}[A\,(L)^L{}_{a_1\cdots a_L}(x),\partial_0 A\,(L)^L{}_{a_1'\cdots a_L'}(x')]_{x^0=x^{0\prime}}\\ =i[\delta^3(x-x')\delta_{a_1a_1'}\cdots\delta_{a_La_L'}]^T.\end{aligned} \tag{36}$$

Then,

$$\begin{aligned}P^0&=\int d_3x\,``T^{00}"\,,\\ P^k&=\int d_3x\,``T^{0k}"\,,\\ J^{0k}&=\int d_3x\,(x^0``T^{0k}"-x^k``T^{00}")\,,\\ J^{ik}&=\int d_3x\,(x^i``T^{0k}"-x^k``T^{0i}")\,.\end{aligned} \tag{37}$$

When the 3-space integrals are performed, these global operators become covariant and induce the correct transformations on the fields. Hence the noncovariance problems in massless field theories occur only for operator densities such as the stress tensor.

An interesting open question is whether the experimental nonexistence of massless particles of spin>1 is a direct consequence of the nonexistence of a covariant stress tensor for conventionally quantized fundamental massless field theories of higher spin.[10]

ACKNOWLEDGMENTS

The author wishes to thank Professor S. Coleman and Dr. B. M. McCoy for many fruitful discussions.

[9] For an arbitrary 3-tensor test function F with no symmetry properties, the generalized transverse δ function is defined by $\int d_3x'\,F_{a_1'\cdots a_n'}{}^{(x')}[\delta^3(x-x')\delta_{a_1a_1'}\cdots\delta_{a_na_n'}]^T=F^T{}_{a_1\cdots a_n}(x)$, where F^T is defined in Eq. (1), and is totally symmetric and traceless.

[10] These arguments have been given for flat Minkowski space. Hence the existence of the graviton (spin 2) necessarily requires an additional physical idea such as the curvature of space; that is, the possibility of performing unobservable general coordinate transformstions. By performing unobservable coordinate transformations, the noncovariant stress tensor "$T^{\mu\nu}$" becomes covariant (the noncovariant part of "$T^{\mu\nu}$" may be transformed away), and T^{00} satisfies the Schwinger energy-density commutation relations. Apparently, for 4-space to support a fundamental spin-2 quantized field, the new physical concept of curvature is required. See J. Schwinger, Phys. Rev. **130**, 800 (1963); **130**, 1253 (1963); **132**, 1317 (1963).