$SU(3)$ Crossing Matrix

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The $SU(3)$ crossing matrix is given in a simplified form and is related to the more symmetric $6pq$ symbol.

1. $SU(3)$ CROSSING MATRIX

The $SU(3)$ crossing matrix is useful in relating the $SU(3)$ parts of amplitudes in crossed channels of two-particle-in two-particle-out reactions. Its formulation and application have been discussed by several authors.[1–3] In this section, we derive a simplified formula for its actual calculation. We follow the notation and phase conventions of de Swart[1].

According to de Swart the crossing matrix from channel II to channel I is [his Eq. (6a)]

$$(\mu\beta\gamma|\beta_{\mathrm{II}}(\mu_1,\mu_2,\mu_3,\mu_4)|\mu'\beta'\gamma') = \sum_{\nu_1\nu_2\nu_3\nu_4\nu'} (-1)^{v_2+v_3} \left(\begin{matrix}\mu_1 & \mu_2 & \mu \\ \nu_1 & \nu_2 & \nu\end{matrix}\;\gamma\right) \left(\begin{matrix}\mu_3 & \mu_4 & \mu \\ \nu_3 & \nu_4 & \nu\end{matrix}\;\beta\right) \times \left(\begin{matrix}\mu_1 & \mu_3^* & \mu' \\ \nu_1 & -\nu_3 & \nu'\end{matrix}\;\gamma'\right) \left(\begin{matrix}\mu_2^* & \mu_4 & \mu' \\ -\nu_2 & \nu_4 & \nu'\end{matrix}\;\beta'\right). \tag{1.1}$$

The μ's here label $SU(3)$ representations and the ν's are the "magnetic" quantum numbers T, M, Y which label the states of a representation. The β's and γ's are labels to distinguish the degenerate couplings of three representations. μ^* means the representation conjugate to μ and $-\nu$ means T, $-M$, $-Y$. Also,

$$v=\tfrac{2}{3}(q-p)+M+\tfrac{1}{2}Y \tag{1.2}$$

is the generalization of de Swart's (1) to an arbitrary representation; he considers the octet model where $\frac{1}{3}(q-p)$ is an integer.

We now give an equivalent definition of the crossing matrix in terms of the recoupling of three representations. For this purpose consider the state

$$\sum_{\nu\nu_1\nu_3\nu_4} \left|\begin{matrix}\mu^* \\ -\nu\end{matrix}\right\rangle \left|\begin{matrix}\mu_4 \\ \nu_4\end{matrix}\right\rangle \left|\begin{matrix}\mu_1 \\ \nu_1\end{matrix}\right\rangle (-1)^{v+v_3} \times \left(\begin{matrix}\mu_3 & \mu_4 & \mu \\ \nu_3 & \nu_4 & \nu\end{matrix}\;\beta\right) \left(\begin{matrix}\mu_1 & \mu_3^* & \mu' \\ \nu_1 & -\nu_3 & \nu'\end{matrix}\;\gamma'\right), \tag{1.3}$$

in which the representations $\mu^*\mu_4$ are combined to give μ_3^* and the result coupled to μ_1 to give a state which transforms as

$$\left|\begin{matrix}\mu' \\ \nu'\end{matrix}\right\rangle.$$

We note parenthetically that

$$(-1)^{v+v_3}\left(\begin{matrix}\mu_3 & \mu_4 & \mu \\ \nu_3 & \nu_4 & \nu\end{matrix}\;\beta\right)$$

is equal to

$$(N_\mu/N_{\mu_3})^{1/2}\left(\begin{matrix}\mu^* & \mu_4 & \mu_3^* \\ -\nu & \nu_4 & -\nu_3\end{matrix}\;\beta\right)$$

within a phase which is independent of magnetic quantum numbers. The N's are the dimensionalities of the respective representations.

Now expand the state (1.3) in terms of the complete set

$$\sum_{\nu\nu_1\nu_2\nu_4} \left|\begin{matrix}\mu^* \\ -\nu\end{matrix}\right\rangle \left|\begin{matrix}\mu_4 \\ \nu_4\end{matrix}\right\rangle \left|\begin{matrix}\mu_1 \\ \nu_1\end{matrix}\right\rangle (-1)^{v+v_2} \times \left(\begin{matrix}\mu_1 & \mu_2 & \mu \\ \nu_1 & \nu_2 & \nu\end{matrix}\;\gamma\right) \left(\begin{matrix}\mu_2^* & \mu_4 & \mu' \\ -\nu_2 & \nu_4 & \nu'\end{matrix}\;\beta'\right), \tag{1.4}$$

in which $\mu^*\mu_1$ are coupled to give μ_2^* and the result coupled to μ_4 to give μ'; specifically we write

$$\sum_{\nu\nu_1\nu_3\nu_4} \left|\begin{matrix}\mu^* \\ -\nu\end{matrix}\right\rangle \left|\begin{matrix}\mu_4 \\ \nu_4\end{matrix}\right\rangle \left|\begin{matrix}\mu_1 \\ \nu_1\end{matrix}\right\rangle (-1)^{v+v_3} \left(\begin{matrix}\mu_3 & \mu_4 & \mu \\ \nu_3 & \nu_4 & \nu\end{matrix}\;\beta\right) \left(\begin{matrix}\mu_1 & \mu_3^* & \mu' \\ \nu_1 & -\nu_3 & \nu'\end{matrix}\;\gamma'\right) = \sum_{\mu_2\beta'\gamma} \frac{N_{\mu_2}}{N_{\mu'}} \sum_{\nu\nu_1\nu_2\nu_4} \left|\begin{matrix}\mu^* \\ -\nu\end{matrix}\right\rangle \left|\begin{matrix}\mu_4 \\ \nu_4\end{matrix}\right\rangle \left|\begin{matrix}\mu_1 \\ \nu_1\end{matrix}\right\rangle (-1)^{v+v_2} \left(\begin{matrix}\mu_1 & \mu_2 & \mu \\ \nu_1 & \nu_2 & \nu\end{matrix}\;\gamma\right) \left(\begin{matrix}\mu_2^* & \mu_4 & \mu' \\ -\nu_2 & \nu_4 & \nu'\end{matrix}\;\beta'\right) Z_{\mu_2\beta'\gamma}. \tag{1.5}$$

Taking the scalar product of both sides of Eq. (1.5) with the state (1.4) identifies the coefficient $Z_{\mu_2\beta'\gamma}$ as being just the crossing matrix element (1.1).

But we can evaluate $Z_{\mu_2\beta'\gamma}$ more simply, following the approach used by Jucys and Bandzaitis[4] to evaluate $SU(2)$ $6j$ and $9j$ symbols. Equate coefficients of

$$\left|\begin{matrix}\mu^* \\ -\nu\end{matrix}\right\rangle \left|\begin{matrix}\mu_4 \\ \nu_4\end{matrix}\right\rangle \left|\begin{matrix}\mu_1 \\ \nu_1\end{matrix}\right\rangle$$

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[1] J. J. de Swart, Nuovo Cimento **31**, 420 (1964).
[2] M. M. Nieto, Phys. Rev. **140**, B434 (1965).
[3] H. Mani, G. Mohan, L. Pande, and V. Singh, Ann Phys. (N. Y.) **36**, 285 (1966); D. B. Fairlie, J. Math. Phys. **7**, 811 (1966).
[4] A. Jucys and A. Bandzaitis, Lithuanian S.S.R. Science Academy, Institute of Physics and Mathematics Publication No. 6, in Russian, 1965 (unpublished).

on the two sides of Eq. (1.5); the ν_2 (ν_3) sum is over T_2 (T_3) only. The result is

$$\sum_{\mu_2\beta'\gamma}\frac{N_{\mu 2}}{N_{\mu'}}\sum_{T_2}\begin{pmatrix}\mu_1 & \mu_2 & \mu & \\ \nu_1 & \nu_2 & \nu & \gamma\end{pmatrix}\begin{pmatrix}\mu_2^* & \mu_4 & \mu' & \\ -\nu_2 & \nu_4 & \nu' & \beta'\end{pmatrix}Z_{\mu_2\beta'\gamma}$$

$$=(-1)^{v+v_3}\sum_{T_3}\begin{pmatrix}\mu_3 & \mu_4 & \mu & \\ \nu_3 & \nu_4 & \nu & \beta\end{pmatrix}$$

$$\times\begin{pmatrix}\mu_1 & \mu_3^* & \mu' & \\ \nu_1 & -\nu_3 & \nu' & \gamma'\end{pmatrix}. \quad (1.6)$$

Multiply Eq. (1.6) by

$$\begin{pmatrix}\mu_1 & \mu_2 & \mu & \\ \nu_1 & \nu_2 & \nu & \gamma\end{pmatrix}$$

and sum over ν_1, T with ν_2, ν_4, ν' fixed. From the orthogonality of the Clebsch-Gordan coefficients, there results

$$\sum_{\beta'}\begin{pmatrix}\mu_2^* & \mu_4 & \mu' & \\ -\nu_2 & \nu_4 & \nu' & \beta'\end{pmatrix}Z_{\beta'}$$

$$=N_{\mu'}/N_\mu\sum_{\nu_1 T T_3}(-1)^{v+v_3}\begin{pmatrix}\mu_1 & \mu_2 & \mu & \\ \nu_1 & \nu_2 & \nu & \gamma\end{pmatrix}$$

$$\times\begin{pmatrix}\mu_3 & \mu_4 & \mu & \\ \nu_3 & \nu_3 & \nu & \beta\end{pmatrix}\begin{pmatrix}\mu_1 & \mu_3^* & \mu' & \\ \nu_1 & -\nu_3 & \nu' & \gamma'\end{pmatrix}. \quad (1.7)$$

We have suppressed all but the β' dependence of the crossing matrix element $Z_{\beta'}$. It is advantageous to factor each $SU(3)$ Clebsch-Gordan coefficient in Eq. (1.7) into an isoscalar factor and an $SU(2)$ Clebsch-Gordan coefficient[5]:

$$\begin{pmatrix}\mu_1 & \mu_2 & \mu & \\ \nu_1 & \nu_2 & \nu & \gamma\end{pmatrix}=\left(\begin{matrix}\mu_1 & \mu_2 \\ T_1Y_1 & T_2Y_2\end{matrix}\middle|\begin{matrix}\mu & \\ TY & \gamma\end{matrix}\right)$$

$$\times\left\langle\begin{matrix}T_1 & T_2 \\ M_1 & M_2\end{matrix}\middle|\begin{matrix}T \\ M\end{matrix}\right\rangle. \quad (1.8)$$

Nieto[2] has pointed out that the $SU(2)$ factors in the crossing matrix yield a $6j$ symbol. In fact, by Jucys and Bandzaitis's Eq. (22.2), we see[4]

$$\left\langle\begin{matrix}T_2 & T_4 \\ -M_2 & M_4\end{matrix}\middle|\begin{matrix}T' \\ M'\end{matrix}\right\rangle^{-1}\sum_{M_1}(-1)^{T_2+M_2+T_3+M_3}\left\langle\begin{matrix}T_1 & T_2 \\ M_1 & M_2\end{matrix}\middle|\begin{matrix}T \\ M\end{matrix}\right\rangle\left\langle\begin{matrix}T_3 & T_4 \\ M_3 & M_4\end{matrix}\middle|\begin{matrix}T \\ M\end{matrix}\right\rangle\left\langle\begin{matrix}T_1 & T_3 \\ M_1 & -M_3\end{matrix}\middle|\begin{matrix}T' \\ M'\end{matrix}\right\rangle$$

$$=(-1)^{T+T_2+T'-T_3}(2T+1)\begin{Bmatrix}T_1 & T & T_2 \\ T_4 & T' & T_3\end{Bmatrix}, \quad (1.9)$$

with M_2M_4M' fixed. From Eqs. (1.7)–(1.9) we get

$$\sum_{\beta'}\left(\begin{matrix}\mu_2^* & \mu_4 \\ T_2-Y_2 & T_4Y_4\end{matrix}\middle|\begin{matrix}\mu' & \\ T'Y' & \beta'\end{matrix}\right)Z_{\beta'}=N_{\mu'}/N_\mu\sum_{Y_1T_1TT_3}(-1)^{v_2'+v_3'+T+T_2+T'-T_3}(2T+1)\left(\begin{matrix}\mu_1 & \mu_2 \\ T_1Y_1 & T_2Y_2\end{matrix}\middle|\begin{matrix}\mu & \\ TY & \gamma\end{matrix}\right)$$

$$\times\left(\begin{matrix}\mu_3 & \mu_4 \\ T_3Y_3 & T_4Y_4\end{matrix}\middle|\begin{matrix}\mu & \\ TY & \beta\end{matrix}\right)\left(\begin{matrix}\mu_1 & \mu_3^* \\ T_1Y_1 & T_3-Y_3\end{matrix}\middle|\begin{matrix}\mu' & \\ T'Y' & \gamma'\end{matrix}\right)\begin{Bmatrix}T_1 & T & T_2 \\ T_4 & T' & T_3\end{Bmatrix}, \quad (1.10)$$

where

$$v'=\tfrac{2}{3}(q-p)-T+\tfrac{1}{2}Y. \quad (1.11)$$

We pick D sets of values of the free variables $T_2Y_2T_4Y_4T'Y'$, where D is the degeneracy of the coupling β', in order to have D equations to solve for the D crossing matrix elements $Z_{\beta'}$. The values of the free variables used should be chosen judiciously to facilitate evaluation of the isoscalar factors and $6j$ symbol in Eq. (1.10). In practice D is very small (1 or 2); by using the symmetries of the crossing matrix we can let the β or γ with the smallest D play the role of β' [see Eq. (2.5)].

Finally we get for the crossing matrix

$$Z_{\beta'}=\sum_F A_{\beta'F}B_F. \quad (1.12)$$

Here $A_{\beta'F}$ is the reciprocal of the matrix

$$(A^{-1})_{F\beta'}=\left(\begin{matrix}\mu_2^* & \mu_4 \\ T_2-Y_2 & T_4Y_4\end{matrix}\middle|\begin{matrix}\mu' & \\ T'Y' & \beta'\end{matrix}\right),$$

whose columns are labeled by β' and whose rows are labeled by the judiciously chosen free variables F; B_F is the right-hand side of Eq. (1.10).

The crossing matrix connecting other pairs of channels, e.g., de Swart's[1] Eq. (6b), can be treated in a similar manner.

This method of evaluating the crossing matrix is not special to the group $SU(3)$.

[5] J. J. de Swart, Rev. Mod. Phys. 35, 916 (1963).

2. RELATION TO 6μ SYMBOL

The 3μ or $SU(3)$ Wigner coefficient and its relation to the $SU(3)$ Clebsch-Gordan coefficient are discussed by Ponzano[6] and by Chew and Sharp.[7] Briefly, it is defined by

$$S_\gamma(123)=\sum_{\nu_1\nu_2\nu_3}\left|\begin{matrix}\mu_1 & \mu_2 & \mu_3\\ \nu_1 & \nu_2 & \nu_3\end{matrix}\right\rangle\begin{pmatrix}\mu_1 & \mu_2 & \mu_3 & \\ \nu_1 & \nu_2 & \nu_3 & ;\ \gamma\end{pmatrix}. \quad (2.1)$$

Here $S_\gamma(123)$ is a certain normalized invariant constructed from the representations $\mu_1\mu_2\mu_3$. On interchange of two of the three representations, the invariant and hence the Wigner coefficient are multiplied by $(-1)^s$, where

$$s=\tfrac{1}{3}\left|\sum_i (p_i-q_i)\right|. \quad (2.2)$$

The Wigner coefficient is invariant under the "reversal" $\mu\rightarrow\mu^*$, $\nu\rightarrow-\nu$ applied to all three spaces. It factors into a symmetric isoscalar factor and an $SU(2)$ Wigner coefficient.

The asymmetric $SU(3)$ Clebsch-Gordan coefficient is related to the symmetric Wigner coefficient by[7]

$$\begin{pmatrix}\mu_1 & \mu_2 & \mu_3 & \\ \nu_1 & \nu_2 & \nu_3 & \gamma\end{pmatrix}=(-1)^{\nu_3}\epsilon_{123\gamma}N_3^{1/2}\begin{pmatrix}\mu_1 & \mu_2 & \mu_3^* & \\ \nu_1 & \nu_2 & -\nu_3 & ;\ \gamma\end{pmatrix}.$$

Here $\epsilon_{123\gamma}$ is a phase factor independent of magnetic quantum numbers chosen to make positive the particular asymmetric isoscalar factor in which M_3, M_1 have the largest values in their respective representations and T_2 the largest value consistent with that; in practice ϵ can be determined only after carrying out the orthogonalization of degenerate product states.[8]

We define the 6μ, or $6pq$, symbol as

$$\langle S_{\gamma_1}(124^*)S_{\gamma_2}(436^*)S_{\gamma_3}(1^*3^*5)S_{\gamma_4}(5^*2^*6)\rangle. \quad (2.3)$$

[6] G. Ponzano, Nuovo Cimento **41A**, 142 (1966).

[7] C.-K. Chew and R. T. Sharp (unpublished).

[8] It would seem more convenient to define $\epsilon_{123\gamma}=1$, thus achieving a simpler relation between Clebsch-Gordan and Wigner coefficients; the old convention (choosing a certain Clebsch-Gordan coefficient positive) has no special advantage anyway.

To evaluate the expression (2.3) it is understood that when one substitutes for the S's from Eq. (2.1), a starred state

$$\left|\begin{matrix}\mu^*\\ -\nu\end{matrix}\right\rangle$$

appears as

$$(-1)^\nu\left\langle\begin{matrix}\mu\\ \nu\end{matrix}\right|$$

with all bras on the left, kets on the right. Thus the 6μ symbol appears as a sum over a product of four $SU(3)$ Wigner coefficients [if the Wigner coefficients are factored, the M sums over $SU(2)$ coefficients yield a $6j$ symbol].

The symmetries of the 6μ symbol are apparent from formula (2.3): It is multiplied by the appropriate $(-1)^s$ on interchange of two representations in the same invariant S and is unaffected when all representations are replaced by their conjugates. A much simpler but less symmetric expression for the 6μ symbol can be obtained by methods similar to those used in Sec. 1 and is implicit in Eqs. (2.5) and (1.12).

Let $|\mu_1\mu_2(\mu_4\gamma_1)\mu_3(\mu_6\gamma_2)\rangle$ be the state formed by coupling $\mu_1\mu_2$ to give μ_4, and μ_3 with the result to give μ_6. Then it turns out that the recoupling coefficient is given in terms of the more symmetric 6μ symbol by

$$\begin{aligned}\langle\mu_1\mu_2(\mu_4\gamma_1)\mu_3(\mu_6\gamma_2)|\mu_1\mu_3(\mu_5\gamma_3)\mu_2(\mu_6\gamma_4)\rangle\\ =\epsilon_{124\gamma_1}\epsilon_{436\gamma_2}\epsilon_{135\gamma_3}\epsilon_{526\gamma_4}(N_4N_5)^{1/2}\langle S_{\gamma_1}(124^*)S_{\gamma_2}(436^*)\\ \times S_{\gamma_3}(1^*3^*5)S_{\gamma_4}(5^*2^*6)\rangle. \quad (2.4)\end{aligned}$$

Similarly the crossing matrix element (1.1) can be expressed in terms of the 6μ symbol:

$$\begin{aligned}(\mu_1\gamma_1\gamma_3|\beta_{\rm II}(\mu_3\mu_5{}^*\mu_4{}^*\mu_2)|\mu_6{}^*\gamma_4\gamma_2)\\ =N_{\mu_6}\epsilon_{35^*1\gamma_3}\epsilon_{4^*21\gamma_1}\epsilon_{346^*\gamma_2}\epsilon_{526^*\gamma_4}\langle S_{\gamma_1}(4^*21^*)S_{\gamma_2}(346)\\ \times S_{\gamma_3}(3^*51)S_{\gamma_4}(5^*2^*6^*)\rangle. \quad (2.5)\end{aligned}$$

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