Fixed-Angle Dispersion Approach to the Induction of Strong-Interaction Symmetries*

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Dispersion relations for forward and backward MM and MB scattering amplitudes are considered, where the M and B are hypothetical, separately degenerate sets of mesons and baryons. Self-consistency conditions for the quantum numbers and three-particle interaction constants for the particles are obtained from the hypothesis that the dispersion integrals are saturated with poles associated with the M and B states in the s, t, and u channels. The spin components are treated on the same footing as the internal quantum numbers. If a subset of the mesons is connected with interactions antisymmetric in the exchange of two mesons, this subset must correspond to the regular representation of a compact, simple Lie group, as in Cutkosky's model of vector mesons. On the other hand, mesons and baryons of both parities must be present in this model. If all the helicity states of the odd-parity mesons have nonzero interactions, the mesons must be vector and pseudoscalar mesons, and the Lie group must contain $SU(2)$ as a noninvariant subgroup applied to the spins according to the W-spin prescription. An $SU(6)_W$-symmetric solution to the model is discussed briefly. This solution is similar to that obtained previously from partial-wave dispersion relations, and is in agreement with the experimental hadron spectrum.

I. INTRODUCTION

GENERALLY, the self-consistency relations of bootstrap models have been formulated in terms of approximate partial-wave dispersion relations. Crossing is complicated in the partial-wave representation, since a pole in one channel is a branch cut in a crossed channel, but this complication is compensated by the simplicity of the unitarity condition. However, if several different partial waves are present and connected by crossing, the spherical wave representation is far from simple. Therefore, it is not surprising that fixed momentum-transfer and fixed-angle dispersion relations have been used with increasing frequency in recent years. Superconvergence relations are one popular type of fixed-momentum-transfer equations.[1]

Several years ago, Cutkosky used partial-wave dispersion relations to derive self-consistency conditions for a system of degenerate V (vector) mesons.[2] Later Polkinghorne extended the model to include spin-$\frac{1}{2}$ baryons of one parity.[3] Consistency in this model implies that if the particles cannot be divided into subsets with no interactions between them, the V mesons correspond to the regular representation of a compact, simple Lie group, and that the baryons correspond to some irreducible representation of the group. The model is not consistent with respect to external quantum numbers since potentials in states of spins and parities other than those corresponding to the V and B particles are neglected, even though some of them are as strong as the potentials considered. Last spring, the author extended this partial-wave model, and showed that the difficulty of neglected spins and parities can be remedied if the interaction group is $SU(6)_W$, and if certain meson and baryon multiplets of both parities exist.[4] The multiplets are classified with the group $U(6)\otimes U(6)\otimes O(3)$ in this $SU(6)_W$ model.

In the present paper, the consistency conditions of this general model are reformulated in terms of fixed-angle dispersion relations. The forward and backward angles are chosen, because the crossing of helicity states is particularly simple for collinear amplitudes. The bootstrap assumptions are that the dispersion integrals are saturated with pole terms that result from composites, and that the set of composites is identical to the set of external particles. This formulation is much cleaner than the partial-wave formulation. In the partial-wave formulation, there is a great deal of ambiguity concerning the relation between the potentials in different partial waves. On the other hand, there is no such ambiguity concerning the analogous quantities of the present formulation, the residues of poles.

Using the fixed-angle treatment, we prove several new results. These are summarized in Sec. VI. The consistency condition for meson-meson scattering and for forward meson-baryon scattering is derived in Sec. II of the paper and applied to the meson-meson states in Sec. III. The meson-baryon states are discussed in Sec. IV. The implications of the model concering the manner in which the spin-dependence of the interaction corresponds to a Lie group is discussed in Sec. V.

II. FORWARD-ANGLE CONSISTENCY CONDITION

We consider MB (meson-baryon) scattering in the forward direction. The mesons and baryons are assumed separately degenerate, with masses μ and m,

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[1] V. deAlfaro, S. Fubini, C. Rossetti, and G. Furlan, Phys. Letters **21**, 576 (1966).

[2] R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963).

[3] J. C. Polkinghorne, Ann. Phys. (N. Y.) **34**, 153 (1965).

[4] R. H. Capps, Phys. Rev. **161**, 1538 (1967).

respectively. The condition derived here may be applied to MM scattering also, if one sets μ equal to m. The energy variable to be used is $\nu=\frac{1}{2}(s-u)$, where s and u are the Mandelstam variables. Along the forward ($t=0$) curve, ν is equal to $s-m^2-\mu^2$, so ν is simply s translated in order to be odd under $s\leftrightarrow u$ crossing.

The crossing properties of helicity amplitudes are derived in the literature.[5] Since we are concerned only with collinear scattering, we define all spin components with respect to the positive z axis (direction of the incoming meson). Crossing the two mesons reverses the signs of their spin components and the signs of their additive, internal quantum numbers. Amplitudes that vanish in the collinear directions, but whose derivatives exist, are not considered here.

MacDowell has shown that many meson-baryon amplitudes contain a piece that is odd in $W=s^{1/2}$, so that a branch cut occurs at zero if s is chosen as the energy variable.[6] However, it has been shown that forward MB amplitudes are even functions of W, so no such branch cut occurs in the cases considered in this section.[7]

The Lorentz-invariant $MB\rightarrow MB$ amplitude T is related to the differential cross section for states of definite helicities by the formula, $d\sigma/d\Omega=|T|^2/s$. If the high-energy behavior of T is dominated by the leading Regge trajectory, this behavior is given by $T\sim s^{\alpha(t)}$. In general, T is not sufficiently convergent to enable one to write an unsubtracted dispersion relation. Therefore, we define a subtracted amplitude U by the equation

$$U=T/[\nu^2-(2m\mu)^2], \qquad (1)$$

where the subtraction points ($\pm 2m\mu$) are the s- and u-channel thresholds, respectively. The denominator is related to conventional s- or u-channel variables by the equations

$$\nu^2-(2m\mu)^2=4k_s^2s=4k_u^2u, \qquad (2)$$

where k_i is the magnitude of the particle momentum in the center-of-mass system in the i channel.

We next make the pole-saturation assumption that U is saturated with poles representing the s- and u-channel composites, and the poles resulting from the subtractions. At this point it is convenient to write U in the form $U=U_{\text{even}}+U_{\text{odd}}$, where the real parts of the terms are even and odd in ν (in $s\leftrightarrow u$ crossing). The high-s (or high-u) behavior of U_{odd} is $s^{\alpha(0)-2}$. The Pomeranchuk trajectory does not contribute to odd amplitudes, so we are justified in assuming that $\alpha(0)<1$. This implies that in the pole approximation, the sum of the residues of the poles in U_{odd} must vanish. The sum of the residues of U_{even} vanishes automatically, when summed over the positive and negative ν range.

Thus, we need no longer make the separation into even and odd parts; the self-consistency condition is the condition that the sum of the residues of all the poles in U vanish.

We consider the process $a+b\rightarrow c+d$ in the s channel, where a and c are meson states with momenta in the positive z direction, and b and d are either both meson states or both baryon states. We want to relate the residues to physically meaningful quantities. The residue sum R may be written as R^s+R^u, where the s-channel part R^s includes all the s-channel composite poles and the s-threshold pole at $\nu=2m\mu$, and R^u includes the corresponding u-channel poles. Since R^u may be obtained from R^s by crossing, we need consider only R^s at present. It is convenient to write this s-channel part $(R^s)_{cd,ab}$ as a sum of parts that are even and odd in the exchange of the momentum directions of the two final particles. For meson-meson amplitudes, these two parts are just $\frac{1}{2}[(R^s)_{cd,ab}\pm(R^s)_{dc,ab}]$. All composites of odd orbital angular momenta occur only in the odd part R^{so}, while all even-l composites and the threshold term at $\nu=2m\mu$ occur only in the even part R^{se}.

We now focus attention on the odd-l residues. A dimensionless coupling constant associated with the vertex $r\rightarrow a+b$ is denoted by G_{abr}, where the momentum directions of the a and b are positive and negative (along the z axis), respectively, in the r rest system. The coupling constant for the opposite vertex $a+b\rightarrow r$ is then G_{abr}^*, where the a-particle momentum is again in the positive direction. All odd-l amplitudes vanish at least as rapidly as the momentum squared at threshold, so the factor k_s^2 may be removed and the constants G defined so as to satisfy the equation

$$\lim_{s\to m_{r2}}[(s-m_r^2)k_s^{-2}T_{cd,ab}]=-\sum_r G_{cdr}G_{abr}^*, \qquad (3)$$

where m_r is the mass of the composite r and the sum is included since more than one degenerate composite may occur. The negative sign is required, since the residue of an elastic amplitude must be negative. If a composite exists in P states, the G are conventional coupling constants (multiplied by appropriate Clebsch-Gordan coefficients). If a composite corresponds to higher l, the G are conventional coupling constants multiplied by some positive power of k_r^2/m_r^2, where k_r is the value of k_s at $s=m_r^2$.

The relation of the residue sum R^{so} to the G's follows immediately from the definition of U, Eq. (1), if use is made of Eq. (2). The result is

$$(R^{so})_{cd,ab}=-\sum_r G_{cdr}G_{abr}^*/(4m_r^2).$$

We define dimensional coupling constants (γ^o) for the odd-l states by the equation $\gamma^o=G/(2m_r)$. The residue sum is simply related to the γ's, i.e.,

$$(R^{so})_{cd,ab}=-\sum_r \gamma_{cdr}{}^o\gamma_{abr}{}^{o*}. \qquad (4)$$

[5] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) 26, 322 (1964).

[6] S. W. MacDowell, Phys. Rev. 116, 774 (1959).

[7] This result has been given by several authors, among them Yasuo Hara, Phys. Rev. 136, B507 (1964).

We now turn to the even-l amplitudes. These are not proportional to k^2 at threshold, so no threshold factor may be removed. Dimensionless coupling constants F for a set of even-l composites of mass m_r may be defined so as to satisfy the equation

$$\lim_{s\to m_r^2} [(s-m_r^2)T_{cd,ab}] = -\sum_r F_{cdr}F_{abr}^* m_r^2. \quad (5)$$

If a composite is an S-wave composite, the F are conventional interaction constants. If a higher orbital-angular momentum is involved, the F are conventional constants multiplied by some positive power of (k_r^2/m_r^2). It may be shown from the definition of U that R^{se} is given by

$$(R^{se})_{cd,ab} = -\sum_r \frac{F_{cdr}F_{abr}^*}{4k_r^2} + \frac{(m+\mu)}{4m\mu} A_{cd,ab}, \quad (6)$$

where A is the scattering length.

The bootstrap hypothesis that has been used already is that U is dominated by poles corresponding to composites and the scattering length terms. We now extend this hypothesis and assume that the scattering length terms either vanish or arise as corrections to S-waves composite poles. The first (and less realistic) of these possibilities, that all the A vanish, would occur if S-wave–S-wave transitions were absent, for example. In such a case one could rewrite Eq. (6) in a form similar to Eq. (4), i.e.,

$$(R^{se})_{cd,ab} = \sum_r \gamma_{cdr}{}^e \gamma_{abr}{}^{e*}. \quad (7)$$

The positive sign is present in this equation because k_r^2 is negative for bound states. Thus $\gamma_{abr}{}^e$ is given by $F_{abr}/(2|k_r|)$. The second, and physically more realistic, possibility exists if the even-l residues are dominated by amplitudes for which at least one of the vertices is an S-wave vertex. (The vertices could be of the types S-S, S-D, S-G, etc.) One assumes that the scattering length is proportional to the S-S part of the amplitude, so that Eq. (6) can be written

$$(R^{se})_{cd,ab} = -\sum_r \frac{F_{cdr}F_{abr}^*}{4k_r^2}(1+\eta\mathcal{P}_{SS}),$$

where $\mathcal{P}_{SS}$ is the projection operator of the S-S amplitude, and η is a constant that does not change the sign of R, i.e., $(1+\eta)>0$. One may again obtain Eq. (7), by defining the S-wave coupling constants γ^e to be equal to $F(1+\eta)^{1/2}/(2|k_r|)$ and the higher-l coupling constants to be equal to $F(1+\eta)^{-1/2}/(2|k_r|)$.[8]

If mass differences were taken into account, the scattering length correction terms would be important. The coefficient of A in Eq. (6) is such that if one varies the mass of a hypothetical composite so that k_r^2 approaches zero, the leading parts of the terms in the equation cancel, and the result remains finite in the limit. In this paper, we are interested only in Eq. (7), and not in the magnitude of the η correction.

We take the mesons a and c to correspond to Hermitian fields so that $s\leftrightarrow u$ crossing involves the simple transposition $a\leftrightarrow c$. Since ν changes sign under the interchange $s\leftrightarrow u$, the u-channel residues involve an extra minus sign. The self-consistency condition, $R^s+R^u=0$, is

$$R_{cd,ab} = \sum_r (-\gamma_{cdr}{}^o\gamma_{abr}{}^{o*} + \gamma_{cdr}{}^e\gamma_{abr}{}^{e*} + \gamma_{adr}{}^o\gamma_{cbr}{}^{o*} - \gamma_{adr}{}^e\gamma_{cbr}{}^{e*}) = 0. \quad (8)$$

Frequently, it is useful to write this expression in a form where no complex conjugations appear. Since the γ's are proportional to the physical coupling constants F and G, with real proportionality constants, we treat the γ's as coupling constants here. Since the first index of γ^o or γ^e refers to a Hermitian meson field, our definition is such that $\gamma_{abr}{}^o$ or $\gamma_{abr}{}^e$ is proportional to the matrix element

$$\langle b(-k)|M_a|r(0)\rangle = \langle r(0)|M_a|b(-k)\rangle^*,$$

where M_a is a Hermitian operator associated with a and the quantities in parentheses are the momenta of the r and b particles along the z axis. The vertices are assumed invariant to Lorentz transformations along the z axis; if b is taken into its rest system, the vertex is $\langle r(k)|M_a|b(0)\rangle^*$. The quantity in brackets differs from the γ_{arb} vertex only in that the momenta of the final particle r is positive. However, since γ^o and γ^e are odd and even, respectively, in the exchange of the momentum directions of the final particles, one may write

$$\gamma_{abr}{}^o = -\gamma_{arb}{}^{o*}, \quad (9a)$$

$$\gamma_{abr}{}^e = \gamma_{arb}{}^{e*}. \quad (9b)$$

These equations will be used in conjunction with Eq. (8) in the next sections.

III. MESON SYSTEM

The consistency condition will be applied now to a finite system of degenerate mesons. Transposing the two mesons in the final state is equivalent to transforming from the forward to the backward direction. Therefore, application of Eq. (8) to all $a+b\to c+d$ combinations exhausts the collinear amplitudes. For convenience, we take all the meson states to correspond to Hermitian fields. Then $\gamma_{ijk}{}^o$ and $\gamma_{ijk}{}^e$ are odd and even, respectively, with respect to transposition of the first two indices (which refer to the two external particles of opposite momenta). These conditions, when combined with Eqs. (9a) and (9b), imply that all the γ are real, and that γ^o and γ^e are completely antisymmetric and completely symmetric, respectively. Since

[8] This assumption can lead to difficulty, because the classification of a vertex by the orbital angular momentum may depend upon which of the three particles is regarded as the composite. It is pointed out in Sec. VI that the saturation assumption is likely to be most accurate when applied to MM and MB scattering. For these amplitudes, there is no ambiguity concerning which particle is regarded as the composite for the even-l vertices.

the state labels describe spin as well as internal-symmetry quantum numbers, the odd and even interactions describe vertices in which the product of the intrinsic parities are odd and even, respectively.

A simple solution to Eq. (8) exists, in which a single scalar meson state interacts with a symmetric cubic interaction. In order to eliminate this trivial solution, we impose the requirement that at least one antisymmetric interaction must exist.

We next consider the possibility of a Cutkosky-type solution, with only antisymmetric interactions. If one sets all γ^e equal to zero, and sets $a=b$, and $c=d$, Eq. (8) reduces to

$$\gamma_{acr}{}^o\gamma_{car}{}^o=-(\gamma_{acr}{}^o)^2=0.$$

Thus, all coupling constants vanish; no completely antisymmetric solution exists. The reason for this phenomenon is simple. The consistency condition implies that a solution must be an eigenfunction of $s\leftrightarrow u$ crossing. However, crossing mixes symmetric and antisymmetric states. In Ref. 2 the symmetric states were neglected; a solution was possible because crossing does not mix the antisymmetric regular-representation state corresponding to a compact, simple Lie group with other antisymmetric states.

Thus, interactions of both symmetries exist. We may obtain a condition involving the γ^o interactions alone by carrying out the permutation sum $\sum_P(-1)^P$ on Eq. (8), where the sum is over all permutations of *abcd* and the minus sign is included in all odd permutations. If one uses the condition $\gamma_{ijr}{}^o=-\gamma_{jir}{}^o$ to rearrange terms, the result may be written

$$\sum_r(\gamma_{abr}{}^o\gamma_{cdr}{}^o+\gamma_{acr}{}^o\gamma_{dbr}{}^o+\gamma_{adr}{}^o\gamma_{bcr}{}^o)=0. \quad (10)$$

This is just the Jacobi identity condition of Ref. 2, and implies that the γ are proportional to the structure constants of a Lie group. In order to restrict the group further, one considers the matrix $F_{rs}=\sum_{ac}\gamma_{acr}{}^o\gamma_{cas}{}^o$. Since the γ^o are real and antisymmetric, F is a real symmetric matrix. It is clear that F is nonpositive definite. A zero eigenvalue of F would imply that at least one eigenstate is not coupled to any of the orthogonal states by the antisymmetric interaction. If no state is decoupled in this manner, F is negative definite. This implies that the Lie group is compact and semisimple.[9] If the set cannot be separated into subsets with no interaction between them, the group is simple. Thus, any subset of the meson states with mutual-antisymmetric interactions connecting all states of the subset is a Cutkosky set, and corresponds to the regular representation of a compact, simple Lie group. A mutually interacting set of odd-parity mesons must be such a set, of course.

We now consider the symmetric and antisymmetric interactions together. Mesons of even and odd parity must both be present; they are denoted by M and N, respectively. We may apply the self-consistency equation to all two-meson scattering processes for which the number of external N mesons is even but not to those for which the number of N is odd. For the latter processes, the product of the intrinsic parities is odd, so that the amplitude is odd in k_s and in k_u. Therefore, there are important branch cuts at thresholds even in the Born approximation. This does not mean that no self-consistency condition could be applied to these amplitudes. However, it does mean that theoretical consistency does not require us to consider these amplitudes, and so we neglect them.

We will display a solution to the conditions for the $MM\to MM$, $NN\to NN$, $MN\to MN$, and $MM\to NN$ processes. The question of the uniqueness of this solution has not been investigated as yet.

The solution involves the group $SU(n)$, and includes $(n^2)M$ and $(n^2)N$ mesons, corresponding to the regular and identity representations. If we denote the fundamental (quark) representation of $SU(n)$ by Q, and the conjugate (antiquark) representation by $\bar{Q}$, then the transformation properties of the mesons are those of linear combinations of quark-antiquark pairs. The transformation properties of the meson A may be expressed in the form $\sum_{ij}A_{ij}Q_i\bar{Q}_j$. The meson may be represented by the matrix A_{ij}. The trace of any product of general M and N matrices transforms like a singlet. In a Hermitian representation, we may regard the meson-state r associated with the coupling constant γ_{abr} as being in the final state, along with the a and b meson states. Invariant antisymmetric and symmetric couplings may be defined by the formulas

$$\gamma_{abr}{}^o=f\,\mathrm{Tr}[(AB-BA)R], \quad (11a)$$

$$\gamma_{abr}{}^e=d\,\mathrm{Tr}[(AB+BA)R], \quad (11b)$$

where the capital letters represent the matrices associated with the mesons, and d and f are interaction constants. One could add the other permutations of the three mesons to the expressions, but this would be redundant, since a trace is invariant to cyclic permutations. The singlet states do not participate in the f-type interactions, but their total d-type couplings (sums of squares of Clebsch-Gordan coefficients) are greater than those of the regular representation states.

In a Hermitian representation of the mesons, the meson matrices are Hermitian. From this it follows that the traces in Eqs. (11a) and (11b) are imaginary and real, respectively. The requirement that the γ be real, established earlier, implies that f is imaginary and d real. In our solution, the magnitudes of the antisymmetric MMM and MNN couplings are equal and equal to the magnitudes of the d-type MNN and NNN couplings, i.e.,

$$f_{MMM}=f_{MNN}=id_{MMN}=id_{NNN}, \quad (12)$$

[9] A lucid discussion of the classification of Lie groups according to the properties of the matrix F is given by Morton Hamermesh, *Group Theory* (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1962), Chap. 8, particularly Sec. 10.

where $d=d^*$. We may drop the subscripts of the d and f.

We now consider $MM\to MM$ scattering, for definiteness. One more property of the mesons is useful. In the matrix representation, the virtual mesons R of either parity are a complete set and satisfy the closure property

$$\sum_R R_{ab}R_{cd}^*=\delta_{ac}\delta_{bd}. \tag{13}$$

If Eqs. (11a) and (11b) are substituted into the consistency condition of Eq. (8), and Eqs. (12) and (13) are used, the resulting expression may be written

$$R_{cd,ab}=d^2\mathrm{Tr}[(CD-DC)(AB-BA)+(CD+DC)\times(AB+BA)-(AD-DA)(CB-BC)-(AD+DA)(CB+BC)]. \tag{14}$$

It follows immediately from the fact that the trace is invariant to cyclic permutations that this expression vanishes, so that our proposed coupling scheme is self-consistent for the $MM\to MM$ processes.

The other processes to be considered are $NN\to NN$, $MN\to MN$ (with crossed process also $MN\to MN$), and $MN\to NM$ (with crossed process $MM\to NN$). However, the interaction constants of Eq. (12) are such that the s- or u-channel residues for any of these amplitudes is the same as for the $MM\to MM$ amplitude with external particles of the same $SU(n)$ character. For any (s or u) channel, the f- and d-type interactions are associated with the intermediate mesons of such parities that the products of the intrinsic parities in the vertices are odd and even, respectively. The solution is completely consistent.

IV. BARYON SYSTEM

Many types of amplitudes of baryon number one are possible in the model, so we will start by limiting attention to the process $MB\to MB$, where M are the odd-parity mesons, and B represents a multiplet of baryons of even parity. The meson and baryon masses are μ and m. Virtual mesons and baryons of both parities are allowed, the odd-parity baryons being denoted by the symbol D. Six types of composite pole diagrams are possible, corresponding to virtual B or D particles in the s and u channels, and to virtual M or N particles in the t channel.

The t-channel singularities do not contribute if one applies the saturation assumption to the curve $t=0$. For this curve, the consistency condition is Eq. (8), with only the first indices of the γ referring to the mesons. The γ^o and γ^e refer to the MBB and MBR interactions, respectively. In order to avoid confusion, we use Greek subscripts to refer to baryon states now. It is seen from Eqs. (9a) and (9b) that the γ_j^o and γ_j^e are anti-Hermitian and Hermitian matrices in the space of the baryon states, respectively. We define Hermitian-matrices Γ_j by the equations

$$\gamma_{j\alpha\beta}{}^o=i\Gamma_{j\alpha\beta}{}^o,\quad \gamma_{j\alpha\beta}{}^e=\Gamma_{j\alpha\beta}{}^e.$$

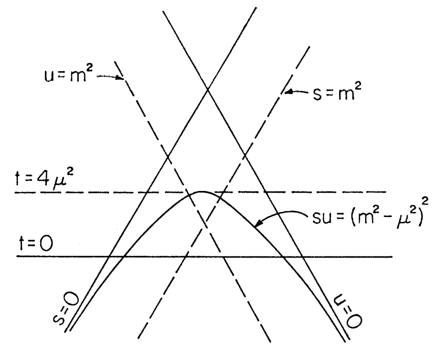

FIG. 1. The back-scattering curve for the case $\mu=\frac{1}{2}m$. The dashed lines are the s- and u-channel baryon poles, and the MM branch cut at $t=4\mu^2$.

If one uses Eqs. (9a) and (9b) to remove the complex-conjugation symbols in the consistency condition of Eq. (8), this condition for the process $\alpha+\beta\to c+\delta$ may be written in terms of the Γ's, i.e.,

$$-X^o+X^e=0, \tag{15}$$

where

$$X^i=\sum_\rho (\Gamma_{c\delta\rho}{}^i\Gamma_{a\rho\beta}{}^i-\Gamma_{a\delta\rho}{}^i\Gamma_{c\rho\beta}{}^i). \tag{16}$$

Because of the difference in the meson and baryon masses, one cannot transform from forward to backward MB scattering by transposing the two final particles. On the other hand, if composites of both parities exist, the forward-angle consistency condition is not sufficient to determine the nature of the composites. A further condition is needed. Clearly, it is reasonable to apply the saturation assumption to some curve that is parallel to, or asymptotic to, $u=0$ in the limit $s\to\infty$. One may then use the Regge argument, that the invariant amplitudes T behave as $T\sim s^{\beta(u=0)-\frac{1}{2}}$ as $s\to\infty$, where β is the leading baryon Regge trajectory.[10] However, it is not obvious which curve should be chosen. We choose the backward scattering curve. This is the branch of the hyperbola

$$su=(m^2-\mu^2)^2, \tag{17}$$

corresponding to positive s and u. This curve is shown in Fig. 1, for the case $\mu=\frac{1}{2}m$.

This curve is chosen because the crossing of helicity amplitudes is simple only for collinear processes. However, a partial justification of the choice may be made by appealing to the static limit. In the static limit (limit as the M mass, N mass, and D-B mass difference become small compared to m), that part of the back-scattering curve on which the virtual B and D singularities lie approaches a straight line. Therefore, any static model of MB states of only one parity that satisfies the forward-consistency condition will also satisfy

[10] V. N. Gribov, Zh. Experim. i Teor. Fiz. 43, 1529 (1963) [English transl.: Soviet Phys.—JETP 16, 1080 (1963)].

the backward condition, if t-channel singularities are neglected.[11]

It has been shown that backward MB scattering amplitudes of fixed helicities are odd in $W=s^{1/2}$, so a branch point occurs at $s=0$.[7] However, this branch point occurs at infinity on the back-scattering curve, where it does no damage.[12] The curve is tangent to the branch cut at $t=4\mu^2$; however, neglecting this cut is an approximation of the same nature as neglecting the s- and u-channel threshold cuts on both the forward- and back-scattering curves. The validity of the saturation assumption for any curve must be tested by experiment, eventually.

The variable $\nu=\frac{1}{2}(s-u)$ will be used again as the energy variable. Since we are concerned with systems that may be consistent theoretically, rather than with the known system of particles, we cannot appeal to experiment to determine the intercept of the leading baryon-Regge trajectory. However, it is safe to assume that the back amplitude diverges no more rapidly than the forward amplitude, so that sufficient convergence will be obtained if we again use the amplitude U of Eq. (1). The consistency condition is that the sum of the residues of the poles of U along the back-scattering curve must vanish. The spin components of the mesons along the z axis again behave as do internal quantum numbers under $s\leftrightarrow u$ crossing.

The back-scattering consistency equation differs from the forward equation in three respects. First, corresponding odd-l composite contributions to the two curves are of opposite sign. Second, the magnitudes of corresponding residues are different, because of the different relation between s (or u) and ν along the two curves. The virtual baryon residues involve the X^i factors of Eq. (16), multiplied by positive constants Z. It is easy to show that if a composite pole occurs at $s=m_r^2$, the multiplicative factor is

$$Z(m_r^2)=\frac{2[m_r^4+(m^2-\mu^2)^2]}{m_r^4+(m^2-\mu^2)^2+2m_r^2(m^2+\mu^2)}. \tag{18}$$

The value $Z(m^2)$, that is associated with B poles, is denoted by Z_B. The even-l composites involve the factor $Z(m_D^2)$, while the scattering-length contributions involve $Z[(m+\mu)^2]=1$. A simple solution can exist only if these two Z factors are equal. This implies either that the scattering-length terms vanish, or that m_D is equal to one of the two values $(m\pm\mu)^2$. It is assumed here that the latter condition applies, so that the Z factor for all even-l poles is unity.

The third difference between the forward- and back-scattering conditions is that the t-channel poles may contribute to the latter. It is seen from Fig. 1 that an unbound MM composite would not lead to a contribution. In our model of degenerate mesons, all mesons are bound, and the back-scattering curve crosses each t-channel pole twice. In order to understand the effect of the two intersections, we note the relation of ν to t-channel variables, $\nu=2k_\mu k_m\cos\theta_t$, where the k factors are the momenta in the MM and $B\bar{B}$ states. It is seen that a t-channel MM composite of even-orbital parity, corresponding to a virtual N meson, can make no contribution, because the two residues cancel. On the other hand, the two residues corresponding to a virtual M state add.

It follows from the above discussion that the backward-scattering consistency condition for the process $a+\beta\rightarrow c+\delta$ is given by

$$Z_BX^o+X^e-2\kappa\sum_k\Gamma_{k\delta\beta}{}^o\gamma_{kca}{}^o=0. \tag{19}$$

The last term is the M-exchange contribution; κ is a real constant that depends on the mass ratio (μ/m). The γ^o is the MMM interaction constant of Sec. II. Elimination of the D-coupling term X^e from Eqs. (15) and (19) yields

$$(Z_B+1)X^o=2\kappa\sum_k\Gamma_{k\delta\beta}{}^o\gamma_{kca}{}^o. \tag{20}$$

This and Eq. (15) are the consistency conditions for MB scattering.

It was pointed out in Sec. III that the M may correspond to the regular or identity representation. We first consider the case in which both a and c correspond to the regular representation. The above equation, with X^o given by Eq. (16), is of the form of a matrix equation for Γ^o matrices associated with the mesons in the vector space of the baryon states. Since the $\gamma_{kca}{}^o$ are proportional to the structure constants of the group (See Sec. III), Eq. (20) is the usual commutator condition on the generators, and so implies that the B correspond to a representation of the group. If the B set cannot be separated into subsets not connected by the MBB interactions, the representation is irreducible. This is the condition derived by Polkinghorne in the vector-meson–even-parity baryon model.[3]

In contrast to the Polkinghorne condition, our conditions cannot be solved if baryons of only one parity exist. This follows from Eq. (15) and the fact that the t-channel contributions to Eq. (20) do not all vanish. The consistency equation for the D multiplet, Eq. (15), has been derived previously from a potential model, and is the statement that the D and B columns of the

[11] Such a model is the strong-coupling model discussed by C. J. Goebel, Phys. Rev. Letters **16**, 1130 (1966). In this model, the mesons are not bootstrapped, and an infinite number of baryon states is necessary.

[12] The Regge argument shows that the high-energy behavior of the amplitude T or U is not that of an integral power of s (unless the leading trajectory crosses the $t=0$ or $u=0$ axis at just the right place), so that almost every amplitude has a branch cut at infinity. This seems at odds with the pole-saturation technique, which leads to amplitudes with integral power behavior in the high-energy limit. However, there is no contradiction; the apparent paradox arises because the pole terms do not dominate the high-energy behavior, even though they may dominate the integrals of the sum rules.

matrix $1-C$ are proportional, with a positive proportionality constant.[4] Here, C is the $s\leftrightarrow u$ crossing matrix. It has been shown that if the group is $SU(n)$ and B is represented by a rectangular Young tableau, all columns of $1-C$ are proportional, and the proportionality constant is positive for a wide choice of D representations.[13] Thus, many $SU(n)$-symmetric solutions exist.

If one or both of the external M mesons is a singlet, the t-channel terms vanish in Eq. (20), and the equation becomes $[\Gamma_a{}^o,\Gamma_c{}^o]=0$, where $\Gamma_i{}^o$ are matrices in the space of the baryon states. The matrix representing the singlet is the identity matrix, so the condition is satisfied for any choice of the magnitude of the singlet coupling. The Γ^e equation (15) also is satisfied automatically if one of the external mesons is a singlet. Thus, solutions exist, but the ratio of the singlet-M and regular-representation-M couplings to the baryons is not specified.

The solution may be extended to some processes involving other sets of external particles. If B and D correspond to the same rectangular representation, the $MD\to MD$ conditions may be satisfied also. Furthermore, since the meson solution of Sec. III involves proportional MMM and MNN interactions, the baryon solution may be extended to include NB and ND elastic processes.

V. IDENTIFICATION OF THE SPIN SUBGROUP WITH MESON STATES

It was shown in Sec. III that if a subset of the mesons is of odd intrinsic parity, and has trilinear interactions (and cannot be separated into smaller, noninteracting subsets), the subset must correspond to the regular representation of a simple Lie group. In order that antisymmetric interactions be present, the mesons may not all be spinless. Furthermore, the spin components are treated on the same footing as internal quantum numbers, and thus must be described by the group.

In this section we investigate the consequences of the requirement that all spin components (along the interaction axis) of the odd-parity mesons have nonzero interactions. Other types of interactions that couple all the particles are possible. An example is a VVV interaction of the type

$$(G^2/m^2)e_a\cdot(k_b-k_c)e_b\cdot(k_c-k_a)e_c\cdot(k_a-k_b),$$

where the e are 4-polarization vectors, and the signs of all 4-momenta are chosen as if the particle is coming to the vertex. Only the zero-helicity components interact in this example, so the Lie group can apply to the internal symmetry alone. On the other hand, this interaction is artificial because it implies that f-wave VV states are as important as P-wave states. Our requirement that all spin components of the odd-parity meson set interact is a natural one, although not necessary for consistency.

The spin component S_z is the single additive, conserved quantum number describing the spin, and so the spin must be described by an $SU(2)$ subgroup of the total Lie group. Since the over-all group is simple, it cannot be a direct product of this $SU(2)$ group and an internal symmetry group. Rather, a group of the type of $SU(6)$ is required, although it is clear that some other groups [such as $SU(4)$] would do just as well. The generators and Casimir operator of the $SU(2)$, spin group are denoted by $\mathfrak{S}_x$, $\mathfrak{S}_y$, $\mathfrak{S}_z$, and $\mathfrak{S}^2$. We must investigate the possible physical meaning of these operators. Since the M correspond to the regular representation of the over-all group, the states involved are of the types M_{11}, M_{10}, M_{1-1}, and M_{00}, where the subscripts are the $\mathfrak{S}$-spin and $\mathfrak{S}_z$, respectively. We may identify $\mathfrak{S}_z$ with the conserved physical-spin component S_z. The mesons must be V and P mesons, and the M_{11} states may be identified with V_1 states.

The assignment of the V_0 and P states may be made by considering the behavior of these particles under the particle-antiparticle conjugation operation C. The antisymmetry of the MMM interaction implies that the interactions of the states corresponding to Hermitian fields are odd under the group-conjugation operation that reverses both internal symmetry quantum numbers and the spin component. The spin states M_{00} and M_{10} are even and odd under $\mathfrak{S}$-spin reflection, and thus must be odd and even under C. On the other hand, P mesons with self-conjugate internal quantum numbers are even under C, while the corresponding zero-helicity states of V mesons are odd under C. Thus, P and V_0 states must be identified with M_{10} and M_{00} states. Furthermore, just as the C behavior of the $V_0=V_z$ states is opposite to that required by the identification of this state with an $\mathfrak{S}$-spin triplet, so also the C behaviors of the V_x and V_y states are of this wrong sign. This requires a negative phase in the identification of V_{-1} with M_{1-1} states. In summary, if the phase relations between the up and down V and M_1 states are defined in the same way, the identification is as follows:

$$M_{11}=V_1,\quad M_{10}=P,\quad M_{1-1}=-V_{-1},\quad M_{00}=V_0.$$

This is the W-spin assignment; in fact, this assignment is a complete definition of W spin for the V and P mesons.[14] Therefore, the $SU(2)$ subgroup that applies to the spins must be assigned by the $SU(2)_W$ scheme.

We now turn our attention briefly to the even-l MM and MB composites, i.e., to the N and D multiplets. It has been shown in previous references that the assumption that the even-l states are dominated by S waves and S-D transitions implies that the N and D may be classified according to the group $U(6)\otimes U(6)$

[13] R. H. Capps, Ann. Phys. (N. Y.) 43, 428 (1967).

[14] W-spin is defined and discussed by H. J. Lipkin and S. Meshkov, Phys. Rev. 143, 1269 (1966).

$\otimes O(3)$, and correspond to the triplet of $O(3)$.[15,16] This is the quark-model classification scheme. The N and D states that participate in the interaction vertices are superpositions of states with different spins and the same z component. The S-S and S-D amplitudes are distinguished by the spin exchange of the collinear amplitudes.

VI. CONCLUDING REMARKS

Some of the consistency conditions given here have been derived in Refs. 4 and 16 on the basis of partial-wave dispersion relations. It is shown in these references that assignment of the particles to $U(6)\otimes U(6)\otimes O(3)$ multiplets according to the scheme $M=(6,\bar{6},1)$, $N=(6,\bar{6},3)$, $B=(56,1,1)$, $D=(70,1,3)$, with interaction constants consistent with $SU(6)_W$, leads to a solution to the conditions (except for the conditions with external D particles, discussed below). This solution is in accord with the experimental hadron spectrum. The present derivation contains fewer parameters than the partial-wave derivation, and so leads to some new results. Chief among these are the following: (1) If the mutual interactions of a subset of the mesons are antisymmetric, they must correspond to the regular representation of a compact semisimple Lie group; (2) mesons and baryons of both parities must be present; (3) if all the spin components of the odd-parity mesons have nonzero interactions, they must be pseudoscalar and vector mesons, with spin components interacting according to the $SU(2)_W$ scheme; (4) consistency is obtained for all meson-meson scattering amplitudes for which the condition may be applied.

Physically, the odd-parity mesons are more prominent than the even-parity mesons. They are also more prominent in the model given here. This results from the fact that the consistency condition involves amplitudes odd under $s\leftrightarrow u$ channel crossing, and these amplitudes transform in the t channel as the odd-parity mesons. For this reason, the meson-exchange term in the baryon consistency conditions of Sec. IV involve the M mesons. For this reason also we were able to obtain a consistency condition involving the M alone carrying out the permutation sum $\sum_P (-1)^P$ on the basic condition of Eq. (8). [Had we omitted the $(-1)^P$ factor from this sum, all the terms would have cancelled.]

The baryon condition of Sec. IV implies one of the mass relations, $m_D=m+\mu$ or $m_D=m-\mu$. Physically, the mass splittings of the multiplets are large. However, since the odd-parity baryon resonances are stable for many MB decays, and unstable for many others, the condition $m_D=m+\mu$ is a reasonable approximation to reality when mass-splitting is neglected. The physical assignment of the D to the multiplet **70** of $SU(6)_W$ prevents satisfaction of the conditions for MD scattering, although the $MB\to MB$ conditions may be satisfied. This follows because the D representation is not rectangular (see Sec. IV). Physically, it is reasonable that the saturation assumption should be particularly accurate for $MM\to MM$ and $MB\to MB$ processes, since there are no two-particle branch cuts below threshold for these two processes in the degeneracy approximation.

It may be shown that if baryon-antibaryon scattering is considered, and saturation with M and N poles is assumed, there is no solution when the baryons correspond to the $SU(6)_W$ representation **56**, and the mesons to the representations **1** and **35**. The basic reason for this is that the singlet-exchange forces in the **405** and **2695** $B\bar{B}$ states are not proportional to the **35**-exchange forces, so that all crossed-channel poles in both these $B\bar{B}$ states cannot be cancelled. This is no more serious than the failure of the condition for the $MD\to MD$ processes. Our basic point of view is not that a simple pole-saturation approximation is applicable always, but rather that it is a good approximation for scattering states without branch cuts below threshold, and this fact may be one of the reasons that a symmetry of the type $SU(6)_W$ is observed to apply to MMM and MBB interactions.

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[15] R. H. Capps, Phys. Rev. **158**, 1433 (1967); **165**, 1899 (1968).

[16] P. Freund, R. Oehme, and P. Rotelli, Nuovo Cimento **51A**, 217 (1967) have considered sets of mesons classifiable by this group, and have discussed solutions to a set of $t=0$ superconvergence relations. The solution they obtain by considering nonsuperconvergent as well as superconvergent amplitudes is basically the same as that of Sec. III of the present paper, although the two approaches are quite different.