Perturbative Approach to Superconvergence Relations and Current Algebra*

K. BARDAKCI† AND M. B. HALPERN
Department of Physics, University of California, Berkeley, California

AND

G. SEGRÈ
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania

(Received 30 October 1967)

Using unitary representations of the Lorentz group, and expanding in powers of mass splitting, we introduce a perturbative approach to superconvergence relations and current algebra.

MUCH effort has been devoted recently toward representation of current algebra and/or superconvergence relations[1-3] in a one-particle subspace. It appears that the relations with a finite number of external one-particle states have an infinite number of solutions,[4] so we are encouraged to study the full problem (with nondegenerate masses).

In this paper we introduce a perturbative approach, expanding in powers of the mass splitting around various degenerate mass solutions. We begin with the set of all helicity-flip-two superconvergence relations associated with the scattering of an isospin-zero tower from ω mesons (or commutative current algebra). Solutions are exhibited to second order for this system. Models with internal symmetry are also briefly studied, and finally a scheme which introduces $SU(6)$-like solutions is discussed.

We begin by reviewing[1,3] the statement of superconvergence relations and the kinematic structure of current matrix elements in the $p_z=\infty$ frame. Having in mind scattering of a tower from a vector-meson target, we define currents $\mathfrak{F}^\alpha(\mathbf{q})$ in the $p_z=\infty$ frame as the boosted form of

$$\int d\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}}\mathfrak{F}_0{}^\alpha(0,\mathbf{x}).$$

Here α is an isospin or $SU(3)$ index, and $\mathbf{q}$ is the two-dimensional momentum, perpendicular to the third direction, carried by the current. In terms of these operators, the set of all helicity-flip-two superconvergence relations (or commutative current algebra) takes the form

$$[F^\alpha(\mathbf{q}),F^\beta(\mathbf{q}')]=0,\qquad \mathbf{q}^2=(\mathbf{q}')^2=-(\text{vector meson mass})^2. \quad (1)$$

Moreover, sandwiching the currents between states of definite momentum, spin, and mass, one has

$$\begin{aligned}\langle \mathbf{p}',J_2,J_{z2},M_2|F^\alpha(\mathbf{q})|\,\mathbf{p},J_1,J_{z1},M_1\rangle &= (2\pi)^3\delta(p_0'+p_3'-p_0-p_3)\delta^{(2)}(\mathbf{p}_1'+\mathbf{q}-\mathbf{p}_1)\\ &\quad\times\langle J_2,J_{z2},M_2|\,\mathfrak{F}_q{}^\alpha\mathcal{G}_{M_2M_1}(\mathbf{q})\,|J_1,J_{z1},M_1\rangle, \quad (2)\end{aligned}$$

where $\mathcal{G}_{M_2M_1}(\mathbf{q})$, containing the boosts from both external states (and leaving them at rest), is

$$\begin{aligned}\mathcal{G}_{M_2M_1}(\mathbf{q}) &= \exp[iK_3\ln(M_2/M_0)]\exp(iq\cdot x)\times\exp[-iK_3\ln(M_1/M_0)],\\ q\cdot x &= (1/M_0)[q_2(J_1+K_2)-q_1(J_2-K_1)];\\ x &= J_1+K_2,\quad y=J_2-K_1.\end{aligned} \quad (3)$$

$\mathbf{J}$, $\mathbf{K}$ are the usual rotations and boosts, M_0 is a number (later the degenerate mass), and $\mathfrak{F}_q{}^\alpha$ is the "reduced matrix element." $\mathfrak{F}_q{}^\alpha$ must transform like a vector plus a scalar under the rotations of the little group that leaves both $\mathbf{p}$, $\mathbf{p}'$ invariant,[1] thus giving rise to the angular condition

$$\begin{aligned}\langle J_2,J_{z2},M_2|\,[[[\mathfrak{F}_q{}^\alpha,\mathbf{J}\cdot\mathbf{Q}],\mathbf{J}\cdot\mathbf{Q}],\mathbf{J}\cdot\mathbf{Q}]e^{i\boldsymbol{\xi}\cdot\mathbf{K}}|\,J_1,J_{z1},M_1\rangle\\ =\mathbf{Q}^2\langle J_2,J_{z2},M_2|\,[\mathfrak{F}_q{}^\alpha,\mathbf{J}\cdot\mathbf{Q}]e^{i\boldsymbol{\xi}\cdot\mathbf{K}}|\,J_1,J_{z1},M_1\rangle; \quad (4)\end{aligned}$$

$\boldsymbol{\xi}$ is parallel to $\mathbf{Q}$, $\tanh\xi=[(q^2+Q_3{}^2)/(q^2+Q_3{}^2+M_1{}^2)]^{1/2}$, $Q=[\mathbf{q},(1/2M_2)(M_2{}^2-M_1{}^2-q^2)]$. We have in mind first assigning a mass to each J so that the states at rest can be labeled by J, J_z, and then solving Eqs. (1) and (4) as operator equations in the space of a unitary representation of the Lorentz group.

With these preliminaries, we mention a previously noted[1,2] degenerate mass solution (suppressing isospin): If M_0 is the degenerate mass, $\mathcal{G}_{M_2M_1}(\mathbf{q})$ collapses to its second factor and one can take

$$\mathfrak{F}^{(0)}=p_0+p_3,\quad F^{(0)}=(p_0+p_3)e^{iq\cdot x}. \quad (5)$$

This solution automatically satisfies the commutativity and degenerate mass angular condition. In a perturbation expansion around this solution, each order will be proportional to p_0+p_3, so, for simplicity, we factor this out and require a scalar condition on the remaining

$$\langle J_2,J_{z2},M_2|\,[\mathfrak{F}_q,\mathbf{J}\cdot\mathbf{Q}]e^{i\boldsymbol{\xi}\cdot\mathbf{K}}|\,J_1,J_{z1},M_1\rangle=0. \quad (6)$$

The commutativity condition is satisfied quite generally

* Research supported in part by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under Grant No. AF-AFOSR-232-66, and in part by the U. S. Atomic Energy Commission.
† Alfred P. Sloan Foundation Fellow.

[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); R. Dashen and M. Gell-Mann, Phys. Rev. Letters **17**, 340 (1966); and M. Gell-Mann, Notes for 1966 International School of Physics, Erice (unpublished).
[2] S. Fubini, Invited talk at IVth Coral Gables Conference on Symmetry Principles at High Energy, 1967. (unpublished).
[3] K. Bardakci and G. Segrè, Phys. Rev. **159**, 1263 (1967).
[4] D. Atkinson and M. B. Halpern, Phys. Rev. **166**, 1724 (1968).

by the ansatz

$$F_q = Uf(\mathbf{x}^2, x\cdot q)e^{iq\cdot x}U^{-1}, \quad (7)$$

where U is a unitary transformation independent of the direction of $\mathbf{q}$. Equation (7) does not automatically satisfy the angular condition. This we attempt to demand in perturbation theory;

$$U=e^{iH},\quad H=H_1+H_2+\cdots,\quad f=1+f_1+f_2+\cdots,$$
$$\mathfrak{F}=1+\mathfrak{F}^{(1)}+\mathfrak{F}^{(2)}+\cdots,\quad M(J)=M_0(1+\Delta_1+\Delta_2+\cdots),$$
$$\mathbf{Q}=\mathbf{Q}^{(0)}+\mathbf{Q}^{(1)}+\cdots, \quad (8)$$

where H is Hermitian. Expanding Eqs. (3) and (7) in this way to first order, we can, with attention to the ordering of the mass operator, solve for the operator $\mathfrak{F}^{(1)}$:

$$\mathfrak{F}^{(1)}=-i\Delta_1K_3+i(K_3\Delta_1)_q+f_1+iH_1-i(H_1)_q;$$
$$(O)_q\equiv e^{iq\cdot x}Oe^{-iq\cdot x}, \quad (9)$$

where H_1 and f_1 are thus far undetermined. We must use these to satisfy the angular condition expanded to first order,

$$[\mathfrak{F}^{(1)},\mathbf{J}\cdot\mathbf{Q}^{(0)}]=0,\quad \mathbf{Q}^{(0)}=(\mathbf{q},\,-q^2/2M_0). \quad (10)$$

This is done with the choice

$$\Delta_1=\epsilon V_0,\quad H_1=\tfrac{1}{2}\epsilon[K_3,V_0]_+ +\epsilon S,$$
$$f_1=\epsilon(V_0-V_3)=\epsilon(\mathbf{x}^2)^{1/2}, \quad (11)$$

where ϵ is an arbitrary constant (presumably small), $S=S(J)$ is an undetermined Hermitian scalar (under three-dimension rotations), and V_μ is the Majorana four-vector.[5] This leads to[6]

$$\mathfrak{F}^{(1)}=\tfrac{1}{2}\epsilon[V_0+(V_0)_q]+i\epsilon S-i\epsilon(S)_q. \quad (12)$$

V_0 is diagonal in J and equal to $2J+1$. We believe that the solution for the mass spectrum is unique, but the off-diagonal couplings ($\Delta J\neq 0$), sensitive to $S=S(J)$, are essentially arbitrary. For simplicity in the solution of the second-order equations, we shall temporarily set $S=0$.

Turning now to second-order mass splitting, we expand Eqs. (3) and (7) to this order, obtaining ($\epsilon=1$)

$$\mathfrak{F}^{(2)}=\tfrac{1}{2}iV_0^2K_3+\tfrac{1}{2}V_0^2K_3^2-\tfrac{1}{2}i(K_3V_0^2)_q+\tfrac{1}{2}(K_3^2V_0^2)_q$$
$$-i\Delta_2K_3+i(K_3\Delta_2)_q-\tfrac{1}{2}iV_0\mathfrak{F}^{(1)}K_3+i\mathfrak{F}^{(1)}(K_3V_0)_q$$
$$-V_0K_3(K_3V_0)_q-\tfrac{1}{8}[K_3,V_0^2]_+{}^2-\tfrac{1}{8}([K_3,V_0]_+{}^2)_q$$
$$+\tfrac{1}{2}i[K_3,V_0]_+(V_0-V_3)-\tfrac{1}{2}i(V_0-V_3)([K_3,V_0]_+)_q$$
$$+\tfrac{1}{4}[K_3,V_0]_+([K_3,V_0]_+)_q+iH_2-i(H_2)_q+f_2, \quad (13)$$

where we have used the previously obtained values of H_1, f_1, Δ_1. Doing a number of "absorptions" with Hermitian H_2's, and taking $f_2=\mathbf{x}^2$, this can be simplified to

$$\mathfrak{F}^{(2)}=-\tfrac{1}{8}V_0^2-\tfrac{1}{8}(V_0^2)_q+\tfrac{1}{4}V_0(V_0)_q-\tfrac{1}{2}[V_0,V_3]_+$$
$$-\tfrac{1}{2}([V_0,V_3]_+)_q-(q^2/4M^2)V_0^2+[1+(q^2/4M_0^2)]$$
$$\times V_0V_3+\tfrac{1}{2}i[K_3,\Delta_2]+\tfrac{1}{2}i([K_3,\Delta_2])_q+\tfrac{1}{4}[V_0,V_3]$$
$$+\tfrac{1}{4}([V_0,V_3])_q. \quad (14)$$

The angular condition at second order is seen to be[7]

$$[\mathfrak{F}^{(2)},\mathbf{J}\cdot\mathbf{Q}^{(0)}]+[M_0+(q^2/2M_0)]V_0[\mathfrak{F}^{(1)},J_3]$$
$$-M_0[\mathfrak{F}^{(1)},J_3](V_0)_q=0. \quad (15)$$

This is solved by[8]

$$\mathfrak{F}^{(2)}=\mathfrak{F}_0^{(2)}+\tfrac{1}{2}(1+q^2/2M_0^2)[V_0,V_3]-\tfrac{1}{2}V_3\mathbf{V}\cdot\mathbf{Q}^{(0)}, \quad (16)$$

where $\mathfrak{F}_0^{(2)}$ satisfies the "simple" angular condition, Eq. (10). Taking Eqs. (14) and (16) together, we can solve for $\mathfrak{F}_0^{(2)}$. The only terms which do not commute with $\mathbf{J}\cdot\mathbf{Q}^{(0)}$ are

$$\mathfrak{F}_0^{(2)}\approx-\tfrac{1}{4}[V_0,V_3]_+-\tfrac{1}{4}([V_0,V_3]_+)_q+\tfrac{1}{2}i[K_3,\Delta_2]_+$$
$$+\tfrac{1}{2}i([K_3,\Delta_2]_+)_q. \quad (17)$$

The only way to eliminate the bad terms is to take $\Delta_2=\frac{1}{2}V_0^2$, or, with the parameter ϵ back in,

$$M(J)=M(1+\epsilon V_0+\tfrac{1}{2}\epsilon^2V_0^2). \quad (18)$$

Again, although the mass spectrum appears unique, the second-order couplings can be changed by adding a Hermitian rotational scalar to H_2.

A mass spectrum corresponding to a linear Regge trajectory would have $\Delta_2=-\frac{1}{2}\epsilon^2V_0^2$. Is there any way we could find such a spectrum? We have studied the effect of the arbitrary S (in first order) on the second order, and find that the second-order mass shift is not changed. The only hope would appear to be a different zeroth-order solution. As a simple modification, consider Eq. (5), but without the factor p_0+p_3. Then we could allow vector plus scalar $\mathfrak{F}$. At each order in the expansion, the reader can easily convince himself, the mass operator is not determined,[9] although the ability to solve the $(n+1)$ order depends on the nth-order splitting. It is trivial to observe that, taking the first-order solution as above, we can solve the second order (as above) with $\Delta_2=-\frac{1}{2}\epsilon^2V_0^2$, this time absorbing extra terms of the form $[K_3,\Delta_2]$ into the vector part of $\mathfrak{F}_q^{(2)}$. Moreover, we have not yet been able to solve the second-order equations for any but $\Delta_1=\epsilon V_0$. On the other hand, we have not studied the solubility of the third order with this (physical) Δ_2.

We now consider $SU(3)\otimes SU(3)$ current algebra for which we must satisfy

$$[F^\alpha(\mathbf{q}),F^\beta(\mathbf{q}')]=if^{\alpha\beta\gamma}F^\gamma(\mathbf{q}+\mathbf{q}'),\text{ etc.} \quad (19)$$

[5] Thus we find a solution only in the space of the two Majorana representations. See, e.g., M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon Press, New York, 1964).

[6] Here we have used the commutation relations $[V_0,K_3]=iV_3$, $[V_3,K_3]=iV_0$. Note that

$$(V_0)_q=\left(1+\frac{q^2}{2M_0^2}\right)V_0+\frac{\mathbf{q}\cdot\mathbf{V}}{M_0}-\frac{q^2}{2M_0^2}V_3$$

indeed commutes with $\mathbf{J}\cdot\mathbf{Q}^{(0)}$.

[7] A useful identity here is $\exp(i\boldsymbol{\xi}^{(0)}\cdot\mathbf{K})V_0\exp(-i\boldsymbol{\xi}^{(0)}\cdot\mathbf{K})=(V_0)_q$.

[8] Useful identities are $[(V_0)_q,J_3]=(i/M_0)(V_1q_2-V_2q_1)$, $[V_3,\mathbf{J}\cdot\mathbf{Q}^{(0)}]=i(V_2q_1-V_1q_2)$.

[9] The mass term at nth order is of the form $[K_3,\Delta_n]$, and can always be absorbed into the vector part of $\mathfrak{F}^{(n)}$. Only in the case of vector zeroth-order solutions, e.g., Eq. (5) or $\mathfrak{F}^{(1)}=V_0-V_3$, is the splitting determined. In this latter case, $\Delta_1=\epsilon V_0$ again.

An already noted[1,2] zeroth-order solution for the vector current alone is Eq. (5) multiplied by $\lambda^\alpha[SU(3)]$. The only necessary change in our formalism for this starting point is that $f(\mathbf{x}^2, x\cdot q)=1$ in Eq. (7). This tells us immediately that there is no solution to first order for this starting point, as we needed $f_1=V_0-V_3$ to solve above. Instead we begin with the zeroth-order solution $\mathfrak{F}^{(0)\alpha}=\lambda^\alpha$. In first order, there is no restriction on the mass spectrum just as discussed above. However, as above, the solubility of second order puts restrictions on the first order. A first-order solution which makes the second order soluble is

$$\mathfrak{F}_{(1)}{}^\alpha=\tfrac{1}{2}\epsilon\lambda^\alpha[V_3+(V_3)_q]+i\epsilon S^\alpha-i\epsilon(S^\alpha)_q,\ \Delta_1=\epsilon V_0. \quad (20)$$

This solution predicts (unphysically) magnetic moments (etc.) proportional to charge. Also, the saturating states stay within one representation of $SU(3)$.[10] The following model seems to avoid some of these unpleasant features.

Define the generators of the Lorentz group by

$$\mathbf{J}=\mathbf{r}\times\mathbf{p}+\mathbf{S},\quad \mathbf{K}=\mathbf{S}\times\hat{p}+(\mathbf{r}\cdot\mathbf{p})\hat{p}-\mathbf{r}p. \quad (21)$$

Here $p^2=\text{const}$, $r_i=i\partial/\partial p_i$, and $\mathbf{S}$ is a finite representation of $SU(2)$ (spin matrix). Further, S_i, λ^α, together with some additional operators $S_i{}^\alpha$, are supposed to form an $SU(6)$ algebra

$$[S_i,S_j{}^\alpha]=i\epsilon_{ijk}S_k{}^\alpha,$$
$$[S_i{}^\alpha,S_j{}^\beta]=if_{\alpha\beta\gamma}\delta_{ij}\lambda^\gamma+id_{\alpha\beta\gamma}\epsilon_{ijk}S_k{}^\gamma,\ \text{etc.} \quad (22)$$

This gives rises to a reducible representation of the Lorentz group. We take a particular representation of this algebra, **56** for baryons and **35** for mesons. The states of the space in which Eqs. (21) and (22) are realized will be characterized by J^2, S^2, L^2 ($\mathbf{L}=\mathbf{J}-\mathbf{S}$), plus hypercharge and isospin. The multiplet structure is therefore $SU(6)\otimes O(3)$. A zeroth-order ansatz for the vector and axial-vector currents is $\mathfrak{F}_{(0)}{}^\alpha=\lambda^\alpha$, $\mathfrak{F}_{(0)}{}^{\alpha5}=\mathbf{S}^\alpha\cdot\hat{p}$, where $\mathfrak{F}^{\alpha5}$ is the axial current. A first-order solution that allows solution at second order is

$$\mathfrak{F}_{(1)}{}^\alpha=\tfrac{1}{2}\epsilon\lambda^\alpha[V_3+(V_3)_q],$$
$$\mathfrak{F}_{(1)}{}^{\alpha5}=\tfrac{1}{2}\epsilon\mathbf{S}^\alpha\cdot\hat{p}[V_3+(V_3)_q],\quad \Delta_1=\epsilon V_0, \quad (23)$$

where the four-vector V_μ (a natural Majorana vector) is now (for mesons)

$$V_0=\Lambda^+[\mathbf{S}\cdot\mathbf{J}-(\mathbf{S}\cdot\hat{p})^2]\Lambda^++\Lambda^-[\mathbf{S}\cdot\mathbf{J}-(\mathbf{S}\cdot\hat{p})^2]\Lambda^-,$$
$$\mathbf{V}=\Lambda^+[\mathbf{K}\times\mathbf{S}+(\mathbf{S}\cdot\hat{p})\mathbf{L}]\Lambda^++\Lambda^-[\mathbf{K}\times\mathbf{S}+(\mathbf{S}\cdot\hat{p})\mathbf{L}]\Lambda^-,$$
$$\Lambda^\pm=\tfrac{1}{2}(1\pm\mathbf{S}_1\cdot\hat{p})(1\pm\mathbf{S}_2\cdot\hat{p}), \quad (24)$$

where $\mathbf{S}_1$, $\mathbf{S}_2$ are the quark spins of the two quarks in the **35** representation, and $\mathbf{S}=\mathbf{S}_1+\mathbf{S}_2$. For the **56**, one has similar formulas[11] involving three quarks. The ability to solve to second order depends crucially on the fact that this V_μ commutes with $\mathbf{S}^\alpha\cdot\hat{p}$.

This ansatz has the virtue of $SU(6)$-type representations for low-lying states, but, on the other hand, it still generates a vector current proportional to charge. Moreover, it still gives rise to exchange degenerate trajectories. Relative to these two problems, it appears promising to introduce a parity doubling of the group $[SU(6)\otimes SU(6)]$. The helicity-flip-two superconvergence relations can be included in the scheme by contracting the $SU(6)$ algebra to $[SU(2)\otimes SU(3)]\times T_{24}$ as in the strong-coupling theory.[12]

We understand that M. Gell-Mann, D. Horn, and J. Weyers have obtained similar mass spectra in a similar perturbative approach.

Note added in proof. All the currents given in this paper are proportional to the internal symmetry generators. Since submitting the paper, we have discovered representations for which the internal symmetry does *not* factor out, but the ones we have all contain proliferating isospin [or $SU(3)$ quantum numbers]. For example, the form

$$F^\alpha=U[I^\alpha+iq\cdot xA^\alpha f(x^2)]e^{iq\cdot x}U^{-1},$$

where $f(x^2)=U=1$ defines the degenerate mass starting point, satisfies $SU(3)$ current algebra if

$$[I^\alpha,I^\beta]=if^{\alpha\beta\gamma}I^\gamma,\ [I^\alpha,A^\beta]=if^{\alpha\beta\gamma}A^\gamma,\ [A^\alpha,A^\beta]=0.$$

We recognize this as the strong-coupling algebra of Goebel and Sakita. In the degenerate limit, one has then a degenerate tower of higher and higher $SU(3)$ multiplets (as well as higher spin). We have solved the perturbation theory to second order and find that the $SU(3)$ degeneracy can be split; within the limits of the perturbative approach, it appears that the large multiplets can be pushed into the high mass region. The spin structure remains as discussed in the paper. Similar models can be constructed with $SL(6,C)$. We understand that, in the degenerate mass limit, similar models have also been discovered by M. Gell-Mann.

We would like to thank J. Harte and R. Hermann for stimulating conversations; we are particularly grateful to J. Weyers for emphasizing the possible value of the Majorana representation.

[10] That is, the solution does not mix **8** and **10** representations, in contrast to present saturation schemes. See, e.g., F. Gilman and H. Harari, Phys. Rev. **165**, 1803 (1968).

[11] Although V_0 is not explicitly $SU(3)$-dependent, the multiplets **8** and **10** will be split because of different spin content.

[12] See, e.g., T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965). This will give the necessary Abelian isospin structures.