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Dynamics in the Bronzan-Lee Model*

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The nonrelativistic Bronzan-Lee model is solved by the method of old-fashioned strong-coupling theory. The isobar physical states are derived in terms of the bare parameters, and all the renormalization constants are calculated. It is argued that taking all the bare couplings to infinity leads to a bootstrap solution.

INTRODUCTION

In a previous note,[1] the author has given the solution of the ordinary Lee model[2] in the strong-coupling limit.[3] In this paper, the same method will be used to solve an extended Lee model first introduced by Bronzan.[4] Though the model is still extremely crude it has some of the properties of a quantum field theory. In particular, the elementary U^- particle in the two-meson sector may be subject to a bootstrap mechanism similar to that which can be imposed[5] on the V (neutron in my notation), which is in the one-meson sector. The simplicity of the solutions presented here will allow easy physical interpretations of the results.

Perhaps most important is from the author's point of view that static models with extended sources are quantum-mechanics problems defined in terms of *bare* parameters. Renormalization constants are to be calculated from the solutions; they are not to be varied indiscriminately as independent variables. A well-known example of this is that the Lee-model renormalized coupling constant may not be raised past a critical value without obtaining nonsense, whereas the bare coupling may be varied from 0 to ∞. Obviously, this approach fails in a local field theory; however, some insight is gained in the simpler case of an extended-source static model.

Finally, we shall be concerned in this note with bootstrapping the baryon states; no reference will be made to a baryon field. We shall see that there is an ambiguity in defining the physical U state because of its mixing with a "bound" state having the same quantum numbers. The ambiguity would only be compounded then if we tried to work with renormalized U field operators.

SOLUTION OF THE MODEL

The nonrelativistic Bronzan-Lee model is defined by the Hamiltonian

$$H=\sum_{\mathbf{k}} \epsilon_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + g u_{\mathbf{k}} a_{\mathbf{k}} \begin{bmatrix} 0&0&0\\ 1&0&0\\ 0&0&0 \end{bmatrix} + g u_{\mathbf{k}} a_{\mathbf{k}}^\dagger \begin{bmatrix} 0&1&0\\ 0&0&0\\ 0&0&0 \end{bmatrix} + f u_{\mathbf{k}} a_{\mathbf{k}} \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&1&0 \end{bmatrix} + f u_{\mathbf{k}} a_{\mathbf{k}}^\dagger \begin{bmatrix} 0&0&0\\ 0&0&1\\ 0&0&0 \end{bmatrix} + \mathcal{E}_N \begin{bmatrix} 0&0&0\\ 0&1&0\\ 0&0&0 \end{bmatrix} + \mathcal{E}_U \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&1 \end{bmatrix}, \quad (1)$$

where $\epsilon_{\mathbf{k}}$ equals $m+\mathbf{k}^2/2m$; m is the meson mass; $a_{\mathbf{k}}$ equals the π^- meson destruction operator; $u_{\mathbf{k}}$ is the cutoff function for the sources; g equals (π^-,P,N), bare coupling; f equals (π^-,N,U), bare coupling; $\mathcal{E}_{N,U}$ may be regarded as bare N, U, masses; and the 3×3 matrices operate on the bare (P,N,U) states.

We proceed now as in I[6] by splitting the meson destruction operator into "bound" and "quasifree" parts. As shown in I, this leads to a Hamiltonian which is the sum of two commuting parts when the source radius is taken large enough. We may disregard the quasifree excitations since their frequencies are large compared to the isobar frequencies we are trying to calculate; we must then diagonalize only

$$H^{\text{bound}}=\begin{bmatrix} wa^\dagger a & Ga^\dagger & 0\\ Ga & \mathcal{E}_N+wa^\dagger a & Fa\\ 0 & Fa & \mathcal{E}_U+wa^\dagger a \end{bmatrix}, \quad (2)$$

where the bound-meson desctruction operator a is given by inverting

$$a_{\mathbf{k}}=u_{\mathbf{k}}a/\lambda\,, \quad (3)$$

with

$$\lambda^2\equiv\sum_{\mathbf{k}} u_{\mathbf{k}}^2,\quad [a,a^\dagger]=1\,,$$

and

$$w\equiv\sum_{\mathbf{k}} \epsilon_{\mathbf{k}} u_{\mathbf{k}}^2\lambda^{-2},\quad G\equiv g\lambda,\quad F\equiv f\lambda.$$

Fortunately, the eigenstates of H^{bound} are easily

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[1] G. North, Phys. Rev. **164**, 2056 (1967), hereafter referred to as I.

[2] T. D. Lee, Phys. Rev. **95**, 1329 (1954).

[3] G. Wentzel, Helv. Phys. Acta **13**, 169 (1940).

[4] J. B. Bronzan, Phys. Rev. **139**, B751 (1965).

[5] M. Vaughn, R. D. Amado, and R. Aaron, Phys. Rev. 124, 1258 (1961).

[6] The limitations of the method are (1) that Eq. (3) be valid when taken between the discrete isobar states, i.e., that the isobar separations be much less than the meson mass; (2) that terms coupling the bound and quasifree states be small, i.e., that the source radius be large. Both conditions will be fulfilled if $1\ll mg_0^2/(mR)^3\ll(mR)^4$, where R is the source radius.

found to be[7]

$$v_n=\beta_n\begin{bmatrix}\psi_n\\ \alpha_n\psi_{n-1}\\ \gamma_n\psi_{n-2}\end{bmatrix} \tag{4a}$$

with eigenvalue(s) E_n defined by

$$H^{\text{bound}}v_n=E_nv_n\,, \tag{4b}$$

and the ψ_n are harmonic-oscillator wave functions, such that

$$a^\dagger a\psi_n=n\psi_n\,,$$

$$[(a^\dagger)^n/\sqrt{(n!)}]\psi_0=\psi_n. \tag{5}$$

For general n, the conditions on the coefficients α_n, γ_n, β_n, and the energy E_n are

$$wn+\alpha_nG\sqrt{n}=E_n\,, \tag{6a}$$

$$G\sqrt{n}+\alpha_n[\mathcal{E}_N+w(n-1)]+F(n-1)^{1/2}\gamma_n=\alpha_nE_n\,, \tag{6b}$$

$$F(n-1)^{1/2}\alpha_n+[\mathcal{E}_U+w(n-2)]\gamma_n=\gamma_nE_n\,, \tag{6c}$$

and

$$\beta_n{}^2=(1+\alpha_n{}^2+\gamma_n{}^2)^{-1}. \tag{6d}$$

For $n=1$, the solution is trivial and equivalent to the ordinary Lee-model results given in I: $\gamma_1=0$, $\alpha_1=-w/G$, and the choice $\mathcal{E}_N=G^2/w$ causes the neutron to have zero energy along with the proton.

For $n=2$ the situation is more complicated since we must solve

$$2w+\alpha_2G\sqrt{2}=E_2\,, \tag{7a}$$

$$G\sqrt{2}+\alpha_2(G^2/w+w)+F\gamma_2=\alpha_2E_2\,, \tag{7b}$$

$$F\alpha_2+\mathcal{E}_U\gamma_2=\gamma_2E_2. \tag{7c}$$

Now α_2 and γ_2 may be eliminated to give a cubic equation for E_2 with the free parameters F, G, $\mathcal{E}_U$ appearing in the coefficients. The three parameters are constrained by the field-splitting hypothesis [validity of Eq. (3)]. We may satisfy this requirement by causing the roots of Eqs. (7) to cluster about $E_2=0$ and $E_2=\infty$. The roots near the origin will correspond to strong-coupling isobar states, whereas the large roots correspond to fictitious states which are decoupled from the isobar subspace because of their large energy separation. For this discussion we shall always take $G/w\gg1$, and we shall choose $\mathcal{E}_U$ so that one root is at $E_2=2w^3/G^2$. This last facilitates comparison with Gerstein's paper,[8] since this is the value that we would obtain for the bound state in the absence of the U particle; c.f. I. We shall find that for either small or large F the strong-coupling requirements on the roots will be met, and these two cases will require separate consideration.

[7] A few other versions of the Lee model may be solved by harmonic-oscillator wave functions; to name two: the Lee model with several neutrons (V's); the Lee model with an elementary U^{--}, or U^{---}, etc. These models probably lead to no new physical results.

[8] I. S. Gerstein, Phys. Rev. **142**, 1047 (1966).

We may rearrange Eqs. (7) to obtain

$$\alpha_2=(E_2-2w)/\sqrt{2}G\,, \tag{8}$$

$$\gamma_2=[E_2{}^2-E_2(G^2/w+3w)+2w^2]/\sqrt{2}GF\,, \tag{9}$$

and E_2 is to be obtained from the cubic

$$-E_2{}^3+E_2{}^2(G^2/w+3w+\mathcal{E}_U)+E_2(F^2-2w^2-3w\mathcal{E}_U-\mathcal{E}_UG^2/w)-2wF^2+2w^2\mathcal{E}_U=0. \tag{10}$$

If we force one of the roots to be at $E_2=2w^3/G^2$, we obtain $\mathcal{E}_U=F^2G^2/3w^3$. Qualitative examination of Eq. (10) shows that for $\mathcal{E}_U$ very small there is one large root (of order G^2) and two small roots, whereas, for $\mathcal{E}_U$ very large there is one low-lying root with two that are very large. In each of these cases we may work within the subspace of low-lying states since the higher states are assumed to be only weakly coupled to these. We proceed then with a separate discussion of the two cases.

Case (1): $F^2G^2/3w^3\ll m$, $G/w\gg1$

There are two low-lying states in the $n=2$ sector; their energies are given by

$$E_2{}^A=F^2G^2/3w^3\,, \tag{11a}$$

$$E_2{}^B=2w^3/G^2\,, \tag{11b}$$

where A and B are used to label the states. Neither of these states may be called the physical U state since this "state" mixes with the π^-, N bound state. We may define, therefore, renormalization constants for *each* of the states. The probability of finding the bare U in state A, $Z_U{}^A$ is given by

$$(Z_U{}^A)^{1/2}=\beta_2{}^A\gamma_2{}^A\approx\frac{\sqrt{2}w^2}{FG}\left(1+\frac{2w^4}{F^2G^2}\right)^{-1/2}, \tag{12}$$

which approaches unity as we decouple the bare U particle. For the state B we obtain

$$(Z_U{}^B)^{1/2}=-\frac{FG}{\sqrt{2}w^2}\left(1+\frac{F^2G^2}{2w^4}\right)^{-1/2}, \tag{13}$$

which tends to zero as the bare U is decoupled. Obviously, as the bare coupling F is increased, the bare U particle moves from state A into state B.

Note that in calculating the Z's we have ignored the overlap integral taken between the bound-field wave function, ψ_n, and the wave function representing the ground state of the free-meson field. In the Lee-type models this integral goes to 1 for large enough source radius. The integral goes to 0 for small radii (Haag's theorem) and gives rise to the characteristic $e^{-g_0{}^2I}$ (I is a cutoff integral) dependence found in most other strong-coupling models.

There will be poles in the π^-, N scattering amplitude at the states A and B. The residues of these poles define the *renormalized* coupling to states A and B. We calculate these parameters by taking matrix elements of the bare meson current between the physical states A, B, and the $n=1$ isobar. For example, taking the current form Eq. (1),

$$g_{A,1}{}^A \equiv f\left(v_2{}^A, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} v_1\right) + g\left(v_2{}^A, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v_1\right) \quad (14)$$

$$= \beta_2{}^A \beta_1 (\alpha_2{}^A g + \alpha_1 \gamma_2{}^A f), \quad (15)$$

which for very small F becomes

$$g_{A,1}{}^R \approx \alpha_2 G/\gamma_2 \lambda = FG/w\lambda. \quad (16)$$

Equation (16) merely says that as the bare U state is decoupled $(F \to 0)$; the physical A state is also decoupled. The corresponding calculation for state B leads to

$$g_{B,1}{}^R \approx -\frac{\sqrt{2}w}{\lambda} + \frac{F^2}{\sqrt{2}w\lambda}. \quad (17)$$

The first term of Eq. (17) is the result from the ordinary Lee model.

It is possible to define a vertex renormalization constant $Z_1{}^{A,B}$ for the states A and B using the following relation from field theory[9]:

$$(g^R)^2 = \frac{Z_N Z_U}{Z_1{}^2} f^2. \quad (18)$$

We obtain

$$Z_1{}^i = \frac{\gamma_2{}^i \alpha_1}{\alpha_2{}^i G/F + \alpha_1 \gamma_2{}^i}, \quad i = A, B. \quad (19)$$

Substituting, we obtain

$$Z_1{}^A = w^2/G^2, \quad (20)$$

which is independent of F and goes to zero as $G \to \infty$. We also obtain

$$Z_1{}^B = -\frac{F^2}{\sqrt{2}w^2}, \quad (21)$$

which is independent of G and tends to zero as $F \to 0$.

Case (2): $F/w \gg 1$, $G/w \gg 1$

There is only one low-lying state in the $n=2$ sector. We may arbitrarily interpret the state at $E_2 = 2w^3/G^2$ as the physical U particle. Equations (8) and (9) give for the physical U

$$\alpha_2 = -\sqrt{2}w/G, \quad (22)$$

$$(Z_U)^{1/2} \approx \gamma_2 = \sqrt{2}w^2/FG. \quad (23)$$

The renormalized coupling is given by

$$g_{U,1}{}^R = -\frac{\sqrt{2}w}{\lambda}\left(1 + \frac{w^2}{FG}\right), \quad (24)$$

which is very close to the ordinary Lee-model result for the coupling to the $n=2$ bound state. Finally,

$$Z_1 = w^2/G^2. \quad (25)$$

In the higher sectors we may find the energies for Cases (1) and (2) by solving Eqs. (6), using $\mathcal{E}_U = G^2F^2/3w^3$, $\mathcal{E}_N = G^2/w$. This leads to a cubic which always $(n>2)$ has two large roots and one low-lying isobar. The small root is given by

$$E_n \approx w^3 n(n-1)/G^2, \quad (n \neq 2). \quad (26)$$

Couplings can also be easily calculated using the methods given in this paper. Except for small correction terms which vanish in the limit, the results in both cases agree with the ordinary Lee model.

DISCUSSION

It remains then to ask what is a bootstrapped U^- particle? One definition consists of taking the system of states which correspond to poles in the scattering amplitude and whose residues (couplings) satisfy crossing symmetry and whose pole positions (masses) are constrained by the unitarity requirements; also the solution should have as few free parameters as possible. Goebel and his associates[10] have solved this problem for the static models with crossing symmetry, and the solutions agree with the results of the old-fashioned strong-coupling theory of Wentzel. This suggests that we may obtain solutions satisfying the usual bootstrap requirements $(Z_U \to 0,\ Z_1 \to 0)$ by letting all the bare couplings get very large. Turning the argument around, we might say that strong-coupling solutions are the bootstrap solutions of S-matrix theory. Although the Lee model does not have crossing symmetry, we can study its strong-coupling limit and argue that this is somehow analogous to the bootstrap solutions of S-matrix theory.

Therefore, our criterion for a bootstrap solution is that the low-energy parameters obtained from it (masses, couplings) agree with those from the strong-coupling limit of the ordinary Lee model. One reason for such a roundabout bootstrap definition is that one might expect the very same phenomena in charged scalar theory, and in that case the criterion is much more transparent.

We have seen in the simple model treated here that we can recover the Lee-model spectrum and renormalized couplings in either of two ways; (1) decoupling the U^- altogether, (2) taking the strong-coupling limit of the U^- bare Yakawa coupling.

[9] The case of more than one state with the same quantum numbers has been treated by G. Feldman and P. T. Matthews, Phys. Rev. **132**, 823 (1963). More recent references will be found in K. Kang, *ibid.* **152**, 1234 (1966).

[10] C. J. Goebel, Phys. Rev. **109**, 1846 (1958); T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965).

The first case has been emphasized by Gerstein and Deshpande[11,8] and corresponds to the value zero for $\mathcal{E}_U$, the level shift of the U particle. Of course, if state B is interpreted as the physical U particle then the level shift is finite. The author tends to reject this weak-coupling case on the physical grounds that it is merely an expression of the reversibility of the theory to turning on and off the bare coupling.

On the other hand, Case (2) treated in this paper[12] has all the correct properties and is furthermore consistent with the usual bootstrap of the neutron.[5] That is to say, the single criterion of letting all bare couplings go to infinity is sufficient to bootstrap all the core states.

One point not discussed in this paper is what is the connection between bootstraps in extended-source static models, and local field theory (point source), where g_0 probably turns out to be infinite automatically.

[11] I. S. Gerstein and N. G. Deshpande, Phys. Rev. **140**, B1643 (1965).

[12] Cases (1) and (2) may also differ in the high-energy limit of the scattering phase shifts in the $n=2$ channel.

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New Superconvergence Relations for Meson-Baryon and Baryon-Baryon Systems

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New superconvergence relations, using Regge asymptotic behavior, have been written down for the meson-baryon and the baryon-baryon systems. We discuss the πN, $\bar{K}N\,(KN)$, $NN\,(N\bar{N})$ processes for which enough experimental data on total cross sections are available to evaluate the integrals occurring in the sum rules. The results obtained are in agreement with experiments.

1. INTRODUCTION

SUPERCONVERGENCE relations[1] have lately attracted considerable attention. Superconvergence sum rules for the pseudoscalar-meson–baryon and baryon-baryon systems have been discussed by many authors.[2–5] The major obstacle in writing the superconvergence relations in the case of meson-baryon systems arises from the fact that the spin structure of the scattering amplitude is such that none of the invariant amplitudes is superconvergent. There have been various attempts to write down the superconvergence relations for this system.[2–4] One has been to exploit the absence of Regge trajectories in states of specific quantum numbers in the crossed channel, and thereby write down the superconvergence relations.[2] A second approach is to subtract out the high-energy contribution from the invariant amplitudes and to write the superconvergence relations for the new amplitudes.[3] Yet another approach has been suggested by Costa and Zimerman[4] (specifically for meson-meson scattering, but their argument can be generalized to other systems as well), who consider the asymptotic behavior in the forward direction of an invariant amplitude which may be supposed to be governed by a single Regge trajectory corresponding to a definite isospin in the t channel. They then relate asymptotically, by $SU(3)$, the invariant amplitudes for different particle systems in which the same Regge trajectory in the cross channel dominates in such a way that the leading terms in the $P_\alpha(z)$ expansions are equal. The asymptotic behavior of the difference of these amplitudes is then governed by the next term in the expansion, which is two powers lower in ν and is therefore superconvergent.

In the case of the baryon-baryon scattering, the problem is not so involved. For this system too the kinematics is such that none of the usual invariant amplitudes[6] is superconvergent. However, one can define[5] a linear combination of the invariant amplitudes which corresponds to a definite helicity flip in the t channel such that one can write the superconvergence relation for the linear combination.

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[3] A. A. Logunov, L. D. Soloviev, and A. N. Tavkhelidze, Phys. Letters **24B**, 18 (1967); R. Gatto, Phys. Rev. Letters **18**, 803 (1967); L. A. P. Balázs and J. M. Cornwall, Phys. Rev. **160**, 1313 (1967).

[4] G. Costa and A. H. Zimerman, Nuovo Cimento **46A**, 198 (1967).

[5] T. L. Trueman, Phys. Rev. Letters **17**, 1198 (1966).

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