Investigation of the ρ Bootstrap and the Determinantal Approximation*

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The ρ bootstrap is considered critically by comparing the results of a δ-function and a Breit-Wigner exchange. It is found that the self-consistent solution that exists for the δ-function case is not reproduced by the Breit-Wigner input. The reason for this is traced to an abnormally large repulsive long-range contribution in the δ-function case that is absent in the Breit-Wigner exchange. It is concluded that the ρ bootstrap is in serious trouble.

1. INTRODUCTION

SINCE the present paper is but the latest addition to an already extensive literature on the ρ-meson bootstrap, a preliminary apologia and a justification of the particular viewpoint of this work may be considered desirable. While there seems to be reasonable agreement among the various partial-wave calculations that a ρ bootstrap is possible, but that it corresponds to an unphysically wide "resonance,"[1–6] there is no such unanimity as to whether this should be regarded optimistically, as suggesting the possibility of a quantitatively acceptable bootstrap in a more comprehensive calculation, or pessimistically, as indicating that the exchange of a narrow ρ resonance is manifestly incapable of bootstrapping itself.

In this report, two of the approximations that are usually made in the calculation are investigated critically. These are the determinantal approximation for the resolution of the N/D equations, and the δ-function approximation of the exchange contribution. It is found that it is possible to improve the determinantal approximation by choosing the subtraction point of the D equation in such a way that the norm of the kernel of the N equation is minimized. The determinantal approximation is then capable of much higher accuracy than might have been thought, as is shown by a comparison with a standard matrix-inversion technique. The details of this work are given in Sec. 3.

The δ-function approximation is investigated by comparing it with the exchange of a Breit-Wigner P-wave resonance. It is found that there is reasonable agreement between the two cases for the correct value of the width, viz., 120 MeV; but that there is no agreement at all for the "width" necessary to achieve self-consistency for the δ-function approximation, viz., 525 MeV. In fact, there is no self-consistent solution for the Breit-Wigner input. The reason for this can be understood in terms of the much reduced repulsion of the Breit-Wigner left-hand cut. The implications of this are discussed in the Conclusion of the report, Sec. 5. It is clear that the general import of the result is to cast doubt on the apparent success of the δ-function approximation; because, while one may not be willing to believe a Breit-Wigner formula corresponding to a width of some 500 MeV, one should be even more suspicious of a δ-function exchange, which, after all, is obtained as the *narrow-width* approximation of a pole (i.e., a Breit-Wigner term).

What, then, is the remedy? It is necessary to find a means whereby a narrow exchanged resonance can produce a narrow output resonance. The idea that exchange of the whole ρ Regge trajectory might suffice has been investigated[7] and found inadequate. The suggestion that other channels are important has been considered by several authors.[2,7] The general finding is that this does improve the situation materially although no calculation of this nature has succeeded, to the author's knowledge, in generating a ρ meson with a width less than twice the correct value. Nevertheless, this may be the correct avenue of approach: perhaps the ρ is a composite of many different particles. Indeed, in a recent report[8] it was suggested that the nucleon-antinucleon channel was most important in producing the ρ, although the uniform treatment of all the left-hand cuts in Ref. 8 is a potential source of inaccuracy that could, perhaps, be gross.

A further possibility that has been suggested recently is that the ρ is produced, after all, by the $\pi\pi$ channel, but that the singular tail of the ρ exchange should be taken seriously.[9–11] The influence of the tail does seem to reduce the required width, but the quantitative details are not yet clear.

The possibility that other exchange forces are important seems slight. It is shown in Sec. 3 that the effect of a σ meson is negligible. The effect of the f_0 meson seems hard to evaluate because of the highly singular

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[1] F. Zachariasen, Phys. Rev. Letters **7**, 112 (1961); **7**, 268 (1961).

[2] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).

[3] D. Y. Wong, Phys. Rev. **126**, 1220 (1962).

[4] M. Bander and G. Shaw, Ann. Phys. (N. Y.) **31**, 506 (1965).

[5] A. P. Contogouris, B. Diu, and M. Pusterla, Nuovo Cimento **48A**, 412 (1967).

[6] A. Morel, Saclay Report, 1965 (unpublished).

[7] M. Bander and G. Shaw, Phys. Rev. **135**, B267 (1964); W. J. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, *ibid.* **154**, 1515 (1967).

[8] J. S. Ball and Michael Parkinson, Phys. Rev. **162**, 1509 (1967).

[9] A. Bassetto and F. Paccanoni, Nuovo Cimento **44A**, 1139 (1966).

[10] D. H. Lyth, University of Lancaster Report (unpublished).

[11] D. Atkinson and A. P. Contogouris, Nuovo Cimento **39**, 1082 (1965); **39**, 1102 (1965).

tail, which leads to an extreme cutoff dependence. However, because of its relatively high mass, one might hope that the f_0 meson could be ignored in the problem of the ρ meson.

2. FORMULATION OF THE BOOTSTRAP PROBLEM

Some care will be taken to define all the quantities that arise in the calculation, despite its elementary nature, since the numerical results seem to be quite sensitive to apparently trivial changes, as a cursory comparison of the various published results will tesitfy.

A resonance in either the isospin 0 and angular momentum 0 state, or in that for $I=1, J=1$, will be written in the unitary, Breit-Wigner form

$$f_{IJ}(s)=\frac{\Gamma_{IJ}\rho^{2J}(s)}{m_{IJ}^2-s-i\Gamma_{IJ}\rho^{2J+1}(s)}, \qquad (2.1)$$

where

$$\rho(s)\equiv[(s-4)/s]^{1/2} \qquad (2.2)$$

is the usual phase-space factor, s being the invariant square of the total four-momentum. Here, m_{IJ} is the mass of the resonance in units of the pion mass, and Γ_{IJ} is proportional to the width. If Δ_{IJ} is the full width of the resonance in the center-of-mass (c.m.) system, in units of the pion mass, then

$$\Delta_{IJ}=(1/m_{IJ})\rho^{2J+1}(m_{IJ}^2)\Gamma_{IJ}. \qquad (2.3)$$

The phase-space factor in Eq. (2.1) is not unique, but the particular choice seems to be a good one since, in addition to its simplicity, it incorporates the correct threshold behavior, has a good asymptotic behavior, and is fairly symmetrical in shape. (Experimental data seem to suggest a symmetrical shape in the s variable for resonances.) For the δ-function approximation (see below), the choice of phase-space factors is immaterial.

The discontinuity on the left-hand cut is given in terms of the discontinuity on the right-hand cut by the relation[12]

$$\mathrm{Im}f_{IJ}(s)=-\frac{1}{4-s}\int_4^{4-s}\sum_{I',J'}(2J'+1)\alpha_{II'} \times\mathrm{Im}f_{I'J'}(s')P_J\left(1+\frac{2s}{s'-4}\right) \times P_{J'}\left(1+\frac{2s'}{s-4}\right)ds', \quad s\leqq 0 \qquad (2.4)$$

where $\alpha_{II'}$ is the crossing matrix

$$\alpha=\begin{bmatrix}\frac{2}{3} & 2 & 10/3\\ \frac{2}{3} & 1 & -5/3\\ \frac{2}{3} & -1 & 1/3\end{bmatrix}. \qquad (2.5)$$

[12] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960); Nuovo Cimento **19**, 752 (1961).

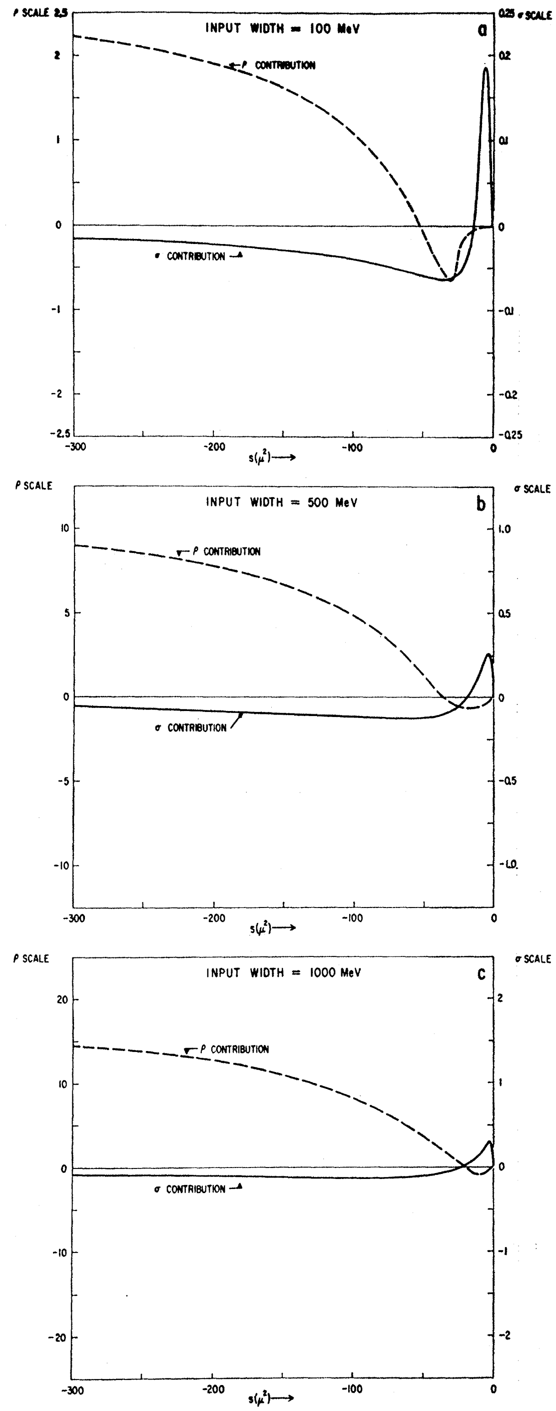

FIG. 1. The discontinuity on the left-hand cut from σ and ρ exchange in the Breit-Wigner approximation, with the same width for both mesons. Note that the ρ scale is ten times the σ scale.

It must be pointed out that the full crossing relation, Eq. (2.4), involves an infinite series of Legendre polynomials, which is convergent only for certain restricted values of s determined by the Lehmann ellipse. However, in the present simple model, it is supposed that

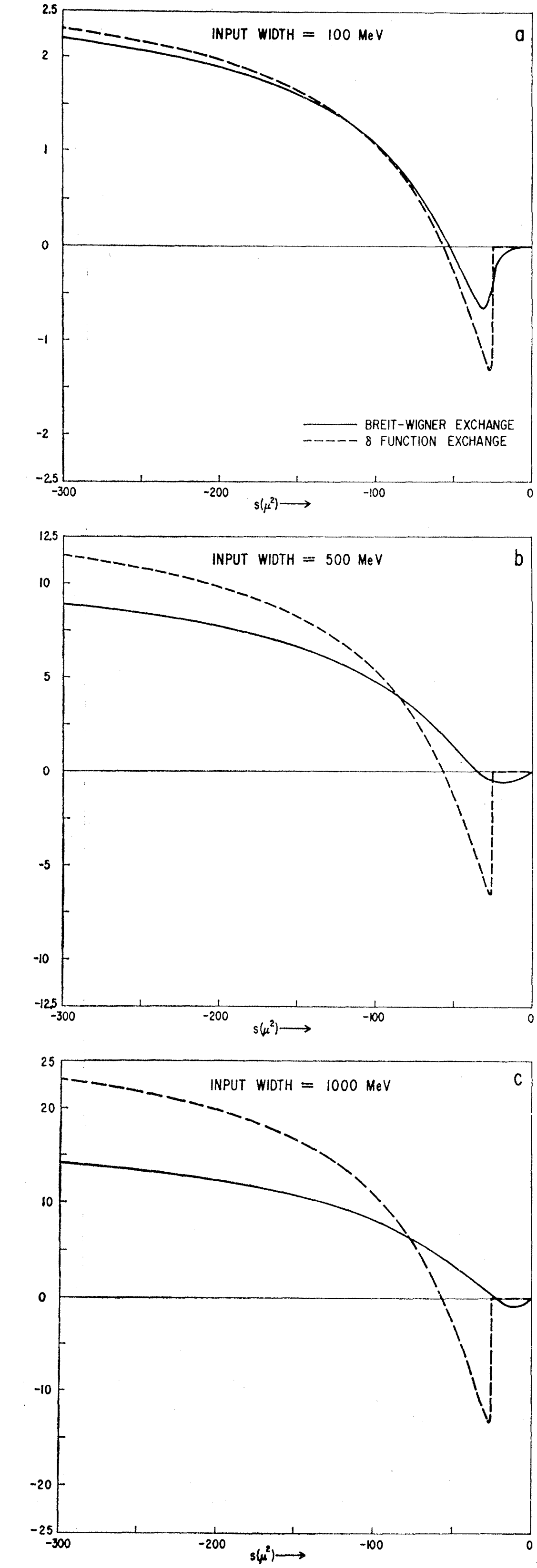

FIG. 2. The discontinuity on the left-hand cut from ρ exchange in the Breit-Wigner and δ-function approximations.

only S and P waves need be retained in the crossed channel for a reasonable description of the low-energy resonances. In view of the negative results of this report, it may be questioned whether this assumption, which is indeed implicit in most bootstrap models, is valid.

In this section, contributions from possible $\sigma(I=0, J=0)$ and $\rho(I=1, J=1)$ resonances are considered by inserting the corresponding imaginary parts of Eq. (2.1) into the sum on the right-hand side of Eq. (2.4). The remainder of the program is to unitarize these forms, by the N/D method, and to require consistency between input and output, if possible.

In Fig. 1, the contributions of the σ and ρ to the right-hand side of Eq. (2.4) are plotted separately (the curves Δ_σ and Δ_ρ), for $m_\sigma^2=8.1\mu^2$ and $m_\rho^2=30\mu^2$ and for several widths. It can be seen that the σ contribution is much smaller than the ρ contribution. In fact, not only does the σ have little effect as input, but it is not produced as a resonance in the $I=0$, $J=0$ wave, even for large σ and ρ input widths.[13] Accordingly, there is no possibility of bootstrapping a σ meson; and in the rest of this paper only the ρ meson will be considered.

It is a function of this paper to examine the accuracy of the so-called δ-function approximation, in which the imaginary part of Eq. (2.1) is replaced by a δ function. The error is expected to be small only for very narrow resonances. The ρ contribution to the left-hand cut discontinuity is given by

$$\operatorname{Im} f_J(s)=\frac{3\pi\Gamma}{s-4}\left(1-\frac{4}{m^2}\right)\left(1+\frac{2s}{m^2-4}\right) \times P_J\left(1+\frac{2m^2}{s-4}\right)\theta(4-m^2-s)\,, \quad s\leqq 0 \quad (2.6)$$

where m is the mass and Γ is proportional to the width of the ρ meson, as given by Eq. (2.3). In Fig. 2, the ρ contribution to the P wave is shown according to the δ-function approximation, and according to the Breit-Wigner form, for $m^2=30\mu^2$ and widths of (a) 100, (b) 500, and (c) 1000 MeV. For the first case (which is close to the observed width of 120 MeV), the agreement between the two curves is reasonable. In case (b) (the order of magnitude required to produce an output resonance of the correct mass), and in case (c), there is no agreement at all, and one should be very suspicious of any results from a δ-function calculation employing or resulting in such a large width. In particular, it should be noted that the relative importance of the repulsion, in the Breit-Wigner case, is much reduced for large widths. This is not mirrored in the δ-function curves, which are simply proportional to the input width. This is an important factor that explains such specious success as the δ function enjoys; the details will be presented in Sec. 4.

[13] C. F. Kyle, A. W. Martin, and H. P. Pagels, Stanford Report No. ITP229 (unpublished).

The bootstrap condition is of course expressed by the requirement that a resonance be observed in the physical scattering region, with the same mass and width as the input ρ meson. For a narrow resonance (e.g., 120 MeV, the observed width) the equations are

$$\mathrm{Re}D(m^2)=0\,, \tag{2.7}$$

$$\Gamma=-N(m^2)\Big/\left(\rho^2(m^2)\frac{d}{ds}\,\mathrm{Re}D(s)\bigg|_{s=m^2}\right). \tag{2.8}$$

Since it will only prove possible to satisfy Eq. (2.8) for a width of about 500 MeV (and then only in the δ-function case), Eqs. (2.7) and (2.8) should be regarded as "normal bootstrap conditions" rather than real self-consistency requirements for a resonance of width 500 MeV. In any case, one of the results of this report is that the one-channel cutoff ρ bootstrap is in dire straits.

3. N/D METHOD AND DETERMINANTAL APPROXIMATION

In this section, the N/D method will be considered from the point of view of the determinantal approximation, which will be checked against a more accurate method of resolving the equations. The results, in contrast to the generally negative import of the rest of this paper, are encouraging. The determinantal approximation, if used in conjunction with a criterion to be explained below, is capable of high accuracy, even for quite large given inputs.

The N/D equations for the $I=1$, $J=1$ wave are given by

$$f_{11}(s)=N(s)/D(s)\,, \tag{3.1}$$

where

$$N(s)=\frac{s-4}{\pi}\int_{-\infty}^{0}\frac{ds'}{(s'-4)(s'-s)}D(s')\,\mathrm{Im}f_{11}(s') \tag{3.2}$$

and

$$D(s)=1-\frac{s-s_0}{\pi}\int_{4}^{\infty}\frac{ds'}{(s'-s_0)(s'-s)}\rho(s')N(s')\,. \tag{3.3}$$

In Eq. (3.2), the input discontinuity, $\mathrm{Im}f_{11}(s')$, $s'\leqq 0$, is given either by inserting the Breit-Wigner form (2.1) into Eq. (2.4), or by using the δ-function approximation (2.6). In Eq. (3.2), $N(s)$ has been subtracted at the normal threshold, $s=4$, and the P-wave threshold behavior is thereby observed. In Eq. (3.3), $D(s)$ has been normalized at the arbitrary subtraction point $s=s_0$. It can easily be shown that $f_{11}(s)$ is independent of s_0.

From Eqs. (3.2) and (3.3), it follows that $N(s)$ satisfies the following integral equation:

$$N(s)=B(s)+\int_{4}^{\infty}ds'K(s,s')N(s')\,, \tag{3.4}$$

where

$$B(s)=\frac{s-4}{\pi}\int_{-\infty}^{0}\frac{ds'}{(s'-4)(s'-s)}\,\mathrm{Im}f_{11}(s') \tag{3.5}$$

and

$$K(s,s')=\frac{s-4}{\pi^2}\frac{\rho(s')}{s'-s_0}\int_{-\infty}^{0}\frac{ds''(s''-s_0)}{(s''-4)(s''-s)(s''-s')}\times\mathrm{Im}f_{11}(s'')\,. \tag{3.6}$$

With either the Breit-Wigner or the δ-function input ρ meson, Eq. (3.4) is not Fredholm; and there has been some interest recently in the nonunique solutions of this equation.[9–11] More usually, however, Eqs. (3.4)–(3.6) are approximated by means of a cutoff, so that Eq. (3.4) becomes Fredholm. This can be done either by cutting off the integrals (3.5) and (3.6),[4,5] or the integral in (3.4).[7] The former possibility wil be adopted here: that is to say, the lower limits $-\infty$ in Eqs. (3.5) and (3.6) will be replaced by $-\Lambda$, and Λ, the cutoff, will be regarded as a parameter.

The modified equation (3.4) can then be solved by a number of standard techniques. The method that has been used consists in transforming the variable s to x, by

$$s=2(x+1)/x\,, \tag{3.7}$$

so that the interval $(4\leqq s<\infty)$ is transformed into $(0<x\leqq 1)$; then the transformed integral equation was replaced by a set of N algebraic equations, corresponding to a set of points x_n, $n=1, 2, \cdots, N$, in the interval $0\leqq x\leqq 1$. Accurate solutions were obtained by matrix inversion (in most cases with $N=8$), although, for

TABLE I. Values of the optimal subtraction point and the corresponding norm $\|K\|$ at 100-MeV input width, for several cutoffs.

Λ (μ^2)	80	120	160	200	400	800	2000	4000
s_0	−32	−8	−200	−280	−280	−360	−600	−800
$\|K\|$	0.0011	0.013	0.027	0.024	0.020	0.020	0.048	0.071

TABLE II. Input and output widths for the δ-function exchange model. The minimum norm $\|K\|$ is included.

Λ (μ^2)	200	400	800	2000	4000
Input width (MeV)	1060 (1400)	375 (410)	220 (230)	135 (140)	110 (130)
Output width (MeV)	172 (270)	780 (820)	850 (840)	840 (840)	800 (850)
$\|K\|$	0.24	0.075	0.043	0.067	0.078

TABLE III. Input and output widths for the Breit-Wigner exchange model.

Λ (μ^2)	200	400	800	2000	4000
Input width (MeV)	545	315	210	140	110
Output width (MeV)	2500	1640	2150	1050	950

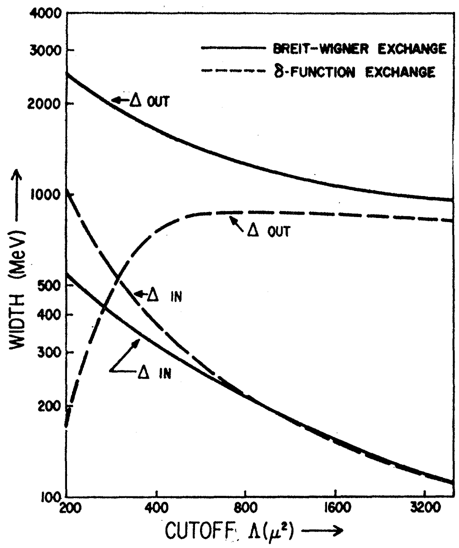

FIG. 3. Input and output ρ widths as functions of the cutoff Λ, with self-consistent ρ mass $m^2=30\mu^2$.

some of the larger cutoffs, $N=16$ was necessary. The accuracy was checked by fitting an interpolating formula to the N points of the solution, and then performing the integration in Eq. (3.4) numerically. This numerical calculation was done not only to investigate the ρ bootstrap but also to check the determinantal approximation which, as will devolve, is capable of much higher accuracy than many of its detractors have suggested. Before the numerical results are presented the determinantal approximation will be discussed.

The determinantal approximation consists in keeping only the zeroth-order term in the Liouville-Neumann expansion[14] of the Fredholm equation (3.4): that is, one sets

$$N(s)=B(s)\,, \tag{3.8}$$

and then $D(s)$ is obtained from Eq. (3.3). This approximation will be reasonable only if the series converges quickly. Since it is known that the series certainly converges if the norm of the kernel satisfies

$$\|K\|^2\equiv\int_0^\infty ds\int_0^\infty ds'\,|K(s,s')|^2<1 \tag{3.9}$$

one would expect the determinantal approximation to be excellent if $\|K\|<0.01$, good if $\|K\|<0.1$, and perhaps indicative of trends for $\|K\|<0.5$. However, for values greater than unity, there is no reason to expect the determinantal approximation to have any validity.

Whereas the exact solution $f_{11}(s)$ does not depend on the subtraction point s_0, the norm of the kernel (3.6) does depend on s_0, as does the determinantal solution itself. Clearly, in order that the exact solution be approximated as closely as possible, the subtraction point should be chosen in such a way as to minimize the norm of the kernel. In fact, this optimum choice of s_0 often reduces the norm by a substantial factor from its value at $s_0=0$, and the usefulness of the determinantal approximation is accordingly enhanced.[15]

In Table I, the optimal s_0 values are given, together with the corresponding norms, for a nominal input width of 100 MeV in the δ-function approximation. The norms quoted in the table correspond in fact to the symmetrized form of the kernel (3.6), which may be shown to possess a smaller norm than the unsymmetrized one.[15]

The final row of Table I indicates that the determinantal approximation should be very good, even for large cutoffs. However, it is in fact necessary to use input widths four or five times larger than the nominal 100 MeV employed in Table I. For some cutoffs still larger widths are necessary; but even in the least favorable cases, $\|K\|$ is appreciably less than unity.

4. NUMERICAL RESULTS

The method of searching for a bootstrap solution consisted in fixing the input mass at the correct value of $m^2=30\mu^2$ and then varying the input width until the output mass was also equal to this value. The input and output widths could thereby be compared for a range of cutoff values. These results are displayed in the δ-function case in Table II. The numbers in parentheses are the results from the matrix inversion, and they agree reasonably well with the determinantal approximates. In the last row of Table II, the minimum norm of the kernel is given for the actual input width in the second row.

In Table III, corresponding results are given for the

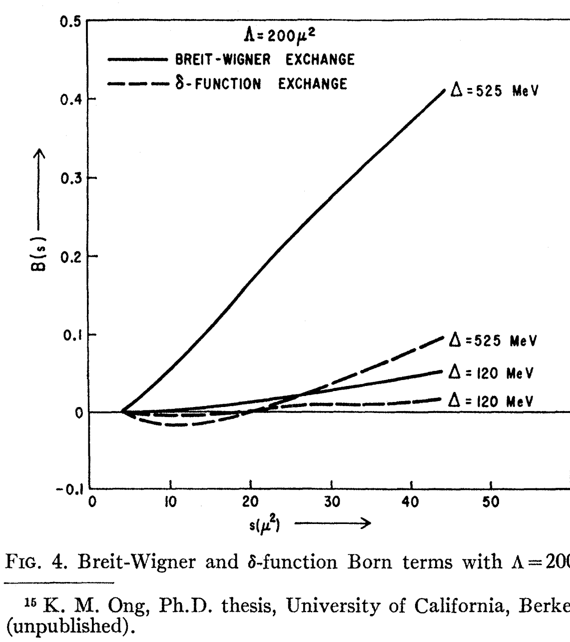

FIG. 4. Breit-Wigner and δ-function Born terms with $\Lambda=200\mu^2$.

[14] See, for example, F. G. Tricomi, *Integral Equations* (Interscience Publishers, Inc., New York, 1957).

[15] K. M. Ong, Ph.D. thesis, University of California, Berkeley (unpublished).

Breit-Wigner input meson, this time in the determinantal approximation only.

On comparing Tables II and III, one notices that there is fair agreement between the δ-function and Breit-Wigner cases for cutoffs of $800\mu^2$ or more, but that there is no agreement at all for smaller cutoffs.

The input and output curves are plotted in Fig. 3 for both cases. It can be seen that, for the δ-function input, there is a self-consistent point, because of the fact that the output curve descends very steeply for small cutoffs. The self-consistent width is 525 MeV and corresponds to a cutoff $\Lambda=250\mu^2$. For the Breit-Wigner ρ meson, there is no self-consistent solution, and this lack can be traced to the failure of the output width curve to turn down for small cutoffs. The reason for this disparity between the δ-function and Breit-Wigner cases must be examined.

It has already been seen that the left-hand cut discontinuities do not resemble one another too closely in the two cases for widths much in excess of 100 MeV (see Fig. 2). In particular, the long-range repulsive part is much less pronounced for the Breit-Wigner than for the δ-function case. For a small cutoff, the repulsion succeeds in overwhelming the attraction at very low energies for the δ-function but not for the Breit-Wigner inputs. The result is that the δ-function Born term $B(s)$ is negative for small s and positive for large s. For a suitably small cutoff, the zero of $N(s)=B(s)$ is close to the resonance position. Hence the output width is drastically decreased [cf. Eq. (2.8)]. This explains the small cutoff behavior of the δ-function output curve in Fig. 3. The comparative weakness of the repulsion for the Breit-Wigner input explains the absence of a zero in this case. These considerations are illustrated in Fig. 4, where the Born terms for a small cutoff $\Lambda=200\mu^2$ are plotted for both the δ-function and Breit-Wigner cases, for widths of 120 and 525 MeV. Figure 5 shows the self-consistent N and D functions, for the δ-function input, corresponding to a cutoff of $\Lambda=250\mu^2$ and a width of 525 MeV.

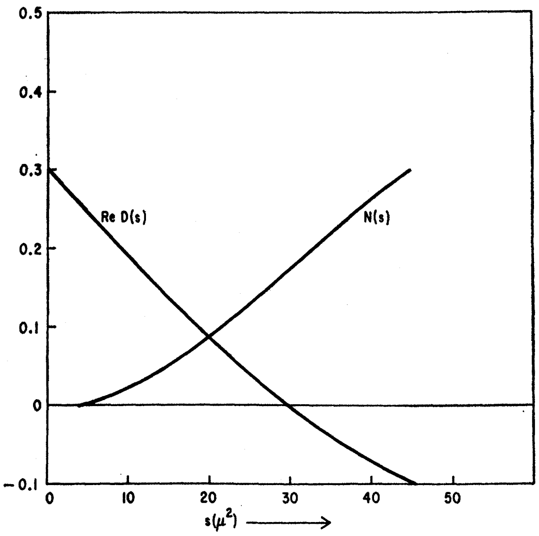

FIG. 5. Self-consistent N and D functions for the δ-functoin ρ bootstrap.

5. CONCLUSION

The general verdict of this report is that the qualitative success of the δ-function bootstrap for the ρ meson is very suspect because there is no self-consistent solution for the Breit-Wigner exchange. The reason that the output-width curve decreases for small cutoff in the δ-function case, and so intersects the input-width curve (see Fig. 3), is that there is still a considerable proportion of repulsive forces, even for a width of about 500 MeV (see Fig. 2b); and so, for a small cutoff, there is considerable cancellation in the low-energy Born term. So much is this so that in fact for a cutoff of $200\mu^2$, the δ-function N functions actually have zeros, which greatly decreases the output width. This cancellation is lacking in the Breit-Wigner exchange at 500 MeV (see Fig. 4).

If the ρ bootstrap is to be saved, there seem to be two possibilities. First, a many-channel calculation, perhaps with some very-high-mass external particles, may be inescapable. Secondly, there is perhaps the hope still that by modifying the short-range part of the ρ exchange (instead of simply cutting it off) one could preserve the beneficial cancellations in the low-energy Born term but enhance the effective binding strength of the exchanged ρ, so that a width of 100–200 MeV would suffice to produce an output meson. In this range of widths, the δ-function and Breit-Wigner forms are comparable, and would both give substantial cancellations.