with

$$\begin{aligned}&\text{(a)}\quad \Gamma_\rho{}^a(m_{\pi\pi})\propto\frac{(m_{\pi\pi}{}^2-4m_\pi{}^2)^{3/2}}{m_{\pi\pi}},\\&\text{(b)}\quad \Gamma_\rho{}^b(m_{\pi\pi})\propto\left(\frac{m_{\pi\pi}{}^2-4m_\pi{}^2}{m_{\pi\pi}{}^2}\right)^{3/2}.\end{aligned}\tag{8}$$

Figure 3 shows the phase shift δ_1 as a function of $m_{\pi\pi}$ corresponding to the forms (a) and (b) both having the same ρ width of 150 MeV. The $\pi\pi$ mass distribution

$$N(m_{\pi\pi})=\int\sigma\, d\cos\theta_p d\cos\theta_\pi d\varphi_\pi\,,\tag{9}$$

and the asymmetry $(F-B)/(F+B)$ are very dependent on the values of δ_1. We illustrate this for process (1) in Figs. 4–7.

A number of different sets of $\delta_0{}^I$ have been obtained by a variety of methods.[3,11] We have not attempted to compare all these different sets of $\delta_0{}^I$ with experiment since the number of events at low $m_{\pi\pi}$ is quite limited. For example, it is not clear at all whether $(F-B)/(F+B)$ change sign at low $m_{\pi\pi}$ for processes (1) and (2). However, we hope that Tables I and II will be proven useful in distinguishing between the various $\delta_0{}^I$ (and δ_1) at low $m_{\pi\pi}$ when the data becomes sufficiently accurate.

Finally, there is a nontrivial dependence of the ρ's on the incident energy E_L. Since most of the relevant experiments have been done at $\sim$4 BeV, we have presented our results for this energy. However, an experiment determining the mass plot $N(m_{\pi\pi})$ for the $\pi^0\pi^0$ production process (3) have been done at $E_L\sim 2$ BeV.[12] Thus we give the density matrix elements $(\rho_s m_{\pi\pi})$ at this energy in Fig. (8).

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K_{l5} Form Factors from Partially Conserved Axial-Vector Current and Current Algebra*

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The form factors for the K_{l5} decay ($K\to\pi\pi\pi l\nu_l$) are derived through the use of the algebra of currents and the hypothesis of partially conserved axial-vector current. In obtaining the results, two different methods were used: the single-soft-pion method, in which the momentum of only one pion at a time is set equal to zero, and the multi-soft-pion method, in which all pions in the matrix element are taken off the mass shell simultaneously. The results obtained by the two methods are consistent one with the other; the existence of a pole in the form factors in the limit of two soft pions indicates, however, that the matrix element obtained in the limit of three soft pions is not a valid approximation to the matrix element in the physical region. The $K\pi$ and $\pi\pi$ scattering amplitudes and the transformation properties and matrix elements of the σ field are also discussed, since they are intimately connected with the derivation of the K_{e5} form factors. The rates obtained for the four possible K_{e5} decay modes were found to be $\sim 10^{-3}$–10^{-4} sec^{-1}.

I. INTRODUCTION

In the course of the past several years many significant advances have been made in the theory of weak interactions through the use of the equal-time current commutation relations proposed by Gell-Mann[1] coupled with the concept of a partially conserved axial-vector current (PCAC).[2] The leptonic decay modes of kaons furnish a particularly interesting example of the application of these two hypotheses, since all of the amplitudes for these processes can now be predicted. The K_{l2} amplitude can be given directly in terms of strong-interaction coupling constants by an extension of PCAC and the Goldberger-Treiman[3] relation to the kaon, although experimental errors are too large to draw any definite conclusions about the success of this prediction. Through the work of Callan and Treiman[4] a relation was obtained between the K_{l3} and K_{l2} decay

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modes, while Weinberg's[5] subsequent work related the K_{l4} modes to K_{l3}; both of these relations are surprisingly successful.

It is the purpose of this paper to obtain the form factors for the K_{l5} decay modes through the use of the current algebra and the PCAC hypothesis.[6] While the muonic decay $K_{\mu 5}$ $(K\rightarrow\pi\pi\pi\mu\nu_\mu)$ is energetically impossible, the decay K_{e5} $(K\rightarrow\pi\pi\pi e\nu_e)$ is a possible process with about 80 MeV available; the predicted rates, unfortunately, are too small to be measured. The K_{l5} form factors, however, are of considerable theoretical interest. In addition to a kaon pole which appears in K_{l5} as well as in K_{l4}, there is a pion pole that adds considerably to the complexity of the matrix element; rather surprisingly, there is a singularity in the form factors for K_{l5} at one of the soft-pion limits.

Furthermore, since the calculation is performed both by taking all the pions off the mass shell simultaneously (as in the K_{l4} work of Weinberg[5]) and also by an alternative, simpler procedure involving only single soft pions,[7] some light is thrown on the specific assumptions that are commonly used in work with current commutators. It is notable that the σ fields which arise when several pions are simultaneously taken off the mass shell have matrix elements which are determined by the isotopic transformation properties of these fields.

The $K\pi$ and $\pi\pi$ scattering amplitudes obtained by Weinberg[8] are also discussed briefly in an Appendix since they are intimately connected with the K_{l5} form factors.

II. DEFINITIONS

The spherical basis of $SU(3)$ is used throughout this paper, and a brief discussion of this basis will be found in Appendix A. Since this formalism is not too common in the literature, it will prove useful to bring together the definitions of the otherwise familiar matrix elements and operators that will be used in the subsequent sections of this paper.

Thus the π_{l2}, K_{l2}, K_{l3}, and K_{l4} form factors are defined as[9]

$$\langle 0|\sqrt{2}A_b{}^\mu(0)|\pi_a(p)\rangle = i\eta(a)\delta_{a,\bar{b}}(2p^0)^{-1/2}m_\pi f_\pi p^\mu, \quad (1)$$

$$\langle 0|\sqrt{2}A_j{}^\mu(0)|K_i(k)\rangle = i\eta(i)\delta_{i,\bar{j}}(2k^0)^{-1/2}m_K f_K k^\mu, \quad (2)$$

$$\langle \pi_a(p)|\sqrt{2}V_j{}^\mu(0)|K_i(k)\rangle = 2\sqrt{3}\begin{pmatrix}8 & 8 & 8'\\ i & j & a\end{pmatrix}(4p^0k^0)^{-1/2} \times[f_+(k+p)^\mu+f_-(k-p)^\mu], \quad (3)$$

$$\langle \pi_a(p_1)\pi_b(p_2)|\sqrt{2}A_j{}^\mu(0)|K_i(k)\rangle = -i\sqrt{2}(8p_1{}^0p_2{}^0k^0)^{-1/2}f_+(m_\pi f_\pi)^{-1}[F_1(p_1+p_2)^\mu + F_2(p_1-p_2)^\mu+\tfrac{1}{2}(1+\xi)F_3(k-p_1-p_2)^\mu], \quad (4)$$

where $V_a{}^\mu$ and $A_a{}^\mu$ are the vector and axial-vector currents, respectively, $\xi=f_-/f_+$, and

$$\begin{pmatrix}8 & 8 & 8'\\ i & j & a\end{pmatrix}$$

is the antisymmetric $SU(3)$ Clebsch-Gordan coefficient.[10] The indices $a, b, \cdots, m, n$ are used to indicate $SU(3)$ transformation properties, with $a=(Y,I,I_3)$ and $\bar{a}=(-Y,I,-I_3)$; $\eta(Y,I,I_3)=(-1)^{I_3+\frac{1}{2}Y}$ is a phase factor that occurs repeatedly in calculation.

Callan and Treiman related the K_{l3} and K_{l2} decays,[4] finding

$$\sqrt{2}(f_+ + f_-)=(m_K f_K)/(m_\pi f_\pi), \quad (5)$$

while, by relating K_{l4} to K_{l3}, Weinberg[5] found the following relations[11]:

$$F_1=\eta(a)\eta(i)\delta_{a,\bar{b}}\delta_{i,\bar{j}}, \quad (6a)$$

$$F_2=6\sum_m \begin{pmatrix}8 & 8 & 8'\\ a & b & m\end{pmatrix}\begin{pmatrix}8 & 8 & 8'\\ i & j & m\end{pmatrix}, \quad (6b)$$

$$F_3=F_1+F_2\frac{k\cdot(p_1-p_2)}{k\cdot(p_1+p_2)}. \quad (6c)$$

Finally, the current commutation relations suggested by the free quark model[1] are stated in spherical tensor form as

$$[V_a{}^0(x),V_b{}^\mu(0)]\delta(x^0) = -\sqrt{3}\sum_c \begin{pmatrix}8 & 8 & 8'\\ a & b & c\end{pmatrix}V_c{}^\mu(x)\delta^4(x), \quad (7a)$$

$$[V_a{}^0(x),A_b{}^\mu(0)]\delta(x^0) = -\sqrt{3}\sum_c \begin{pmatrix}8 & 8 & 8'\\ a & b & c\end{pmatrix}A_c{}^\mu(x)\delta^4(x), \quad (7b)$$

$$[A_a{}^0(x),A_b{}^\mu(0)]\delta(x^0) = -\sqrt{3}\sum_c \begin{pmatrix}8 & 8 & 8'\\ a & b & c\end{pmatrix}V_c{}^\mu(x)\delta^4(x). \quad (7c)$$

III. K_{l5} FORM FACTORS FROM SINGLE-SOFT-PION METHODS

The method of utilizing single-pion reduction in multipion processes is due to Bell[7] and has the advantage of a maximum of physical intuition combined with a minimum of assumptions. The procedure to be

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[9] The vector form factors do not contribute to the K_{l4} amplitude in the soft-pion limit and are presumably suppressed in the physical region by the centrifugal barrier. The same observation holds true for the axial-vector form factors in the case of K_{l5}.
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[11] In Eqs. (6) a sign error in Eq. (24) of Ref. 5 has been corrected.

followed is to construct the amplitude for the process to be considered, taking into consideration all intermediate states that give rise to form factors that, while not constant, are of zeroth order in the pion momenta; this amplitude is then compared with previously known amplitudes by letting the momentum of each of the pions in turn go to zero. Using this method in the case of K_{l4}, it is possible to derive the results obtained by Weinberg[5] without ever considering the commutator

$$[A_a^0(x),\partial_\mu A_b^\mu(0)]\delta(x^0)=\sigma_{ab}(x)\delta^4(x)\,. \tag{8}$$

In the case of K_{l5}, however, the situation is not quite so simple.

The K_{l5} form factors are defined as[9]

$$\begin{aligned}\langle\pi_a(p_1)\pi_b(p_2)\pi_c(p_3)|\sqrt{2}V_j^\mu(0)|K_i(k)\rangle\\ =(16p_1^0p_2^0p_3^0k^0)^{-1/2}[G_1p_1^\mu+G_2p_2^\mu\\ +G_3p_3^\mu+\tfrac{1}{2}(1+\xi)G_4(k-p_1-p_2-p_3)^\mu]\,.\end{aligned} \tag{9}$$

In the case of K_{l4}, it was not possible to assume that the form factors were constant; K-pole diagrams gave rise to terms which, while zeroth order in the pion momenta, still varied considerably in passing to the different soft-pion limits [cf. the last term in Eq. (6c)]. In the case of K_{l5} there are both K-pole and π-pole diagrams that give rise to such terms. These momentum-dependent parts of the form factors are separated by defining

$$G_i=g_i+g_KK_i+g_\pi\Pi_i\,. \tag{10}$$

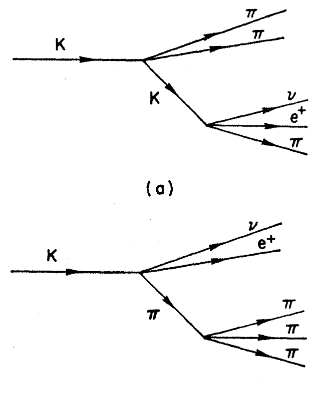

FIG. 1. K-pole and π-pole contributions to the K_{l5} form factors.

Here K_i and Π_i represent the K-pole and π-pole contributions shown in Figs. 1(a) and 1(b), respectively, g_K and g_π being the strengths with which they enter. These momentum-dependent terms being explicitly included in the K_{l5} matrix element, it will be assumed that g_i, g_K, and g_π behave as constants when extrapolating to zero pion momentum.

The K-pole contributions can be written in terms of the K_{l3} form factors and the $K\to K\pi\pi$ amplitude. By crossing symmetry, the $K\to K\pi\pi$ amplitude can be taken from recent work[8] on the low-energy $K\pi$ scattering amplitude (cf. Appendix B):

$$\begin{aligned}\langle\pi_b(p_2)K_j(k_2)|S|\pi_a(p_1)K_i(k_1)\rangle=-3i(2\pi)^4\delta^4(k_1+p_1-k_2-p_2)(16p_1^0p_2^0k_1^0k_2^0)^{-1/2}\\ \times(m_\pi f_\pi)^{-2}\eta(b)\eta(i)\sum_m\begin{pmatrix}8&8&8'\\a&\bar{b}&m\end{pmatrix}\begin{pmatrix}8&8&8'\\i&\bar{j}&\bar{m}\end{pmatrix}(p_1+p_2)\cdot(k_1+k_2)\,.\end{aligned} \tag{11}$$

Using Eqs. (3) and (11) to calculate the contribution of Fig. 1(a), one finds

$$K_1=-12\sqrt{3}\eta(a)\eta(j)f_+(m_\pi f_\pi)^{-2}\sum_{m,n}\begin{pmatrix}8&8&8'\\b&c&m\end{pmatrix}\begin{pmatrix}8&8&8'\\i&\bar{n}&m\end{pmatrix}\begin{pmatrix}8&8&8'\\n&j&a\end{pmatrix}\frac{k\cdot(p_2-p_3)}{k\cdot(p_2+p_3)}\,. \tag{12}$$

K_2 and K_3 are generated from K_1 by the simultaneous cyclic permutation of the indices 1, 2, 3 and a, b, c, while $K_4=K_1+K_2+K_3$.

The π-pole contributions can be written in terms of the K_{l3} form factors and the $\pi\pi$ scattering amplitude in a similar manner. Taking the $\pi\pi$ scattering amplitude from recent work in the literature,[8]

$$\begin{aligned}\langle\pi_a(p_1)\pi_b(p_2)|S|\pi_c(p_3)\pi_d(p_4)\rangle=-2i(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)(16p_1^0p_2^0p_3^0p_4^0)^{-1/2}(m_\pi f_\pi)^{-2}\\ \times\{\eta(a)\eta(c)\delta_{a,\bar{b}}\delta_{c,d}[m_\pi^2-(p_1+p_2)^2]+\delta_{a,c}\delta_{b,d}[m_\pi^2-(p_1-p_3)^2]+\delta_{a,d}\delta_{b,c}[m_\pi^2-(p_2-p_3)^2]\}\,,\end{aligned} \tag{13}$$

one can calculate the contribution of the diagram in Fig. 1(b) and find

$$\Pi_i=8\sqrt{3}f_+(m_\pi f_\pi)^{-2}\left\{\eta(b)\begin{pmatrix}8&8&8'\\i&j&a\end{pmatrix}\delta_{b,\bar{c}}[m_\pi^2-(p_2+p_3)^2]+\text{perm.}\right\}[m_\pi^2-(p_1+p_2+p_3)^2]^{-1}\,, \tag{14}$$

where "perm." denotes the other two terms generated by the simultaneous cyclic permutation of the indices 1, 2, 3 and a, b, c.

Finally, the constants g_i, g_K, and g_π are determined by relating K_{l5} to K_{l4} in three separate soft-pion limits. Using the PCAC relation obtained from Eq. (1),

$$\sqrt{2}\partial_\mu A_a^\mu(x)=m_\pi^3f_\pi(\varphi_a(x))^\dagger\,, \tag{15}$$

and the commutator, Eq. (7b), one finds

$$\lim_{p_3\to 0}(2p_3^0)^{1/2}\langle\pi_a(p_1)\pi_b(p_2)\pi_c(p_3)|V_j^\mu(0)|K_i(k)\rangle$$
$$=i(\sqrt{6})(m_\pi f_\pi)^{-1}\eta(c)\sum_m\begin{pmatrix}8 & 8 & 8'\\ \bar{c} & j & m\end{pmatrix}\langle\pi_a(p_1)\pi_b(p_2)|A_m^\mu(0)|K_i(k)\rangle, \quad (16)$$

and two similar equations obtained by the simultaneous cyclic permutation of the indices. Using Eqs. (4), (6), and (9), one obtains a set of simultaneous equations for the constants after equating coefficients of linearly independent terms. The general solution for the form factors is then

$$G_1=g\Bigg\{\begin{pmatrix}8 & 8 & 8'\\ i & j & a\end{pmatrix}\eta(b)\delta_{b,\bar{c}}-2\begin{pmatrix}8 & 8 & 8'\\ i & j & b\end{pmatrix}\eta(c)\delta_{a,\bar{c}}-2\begin{pmatrix}8 & 8 & 8'\\ i & j & c\end{pmatrix}\eta(a)\delta_{b,\bar{a}}$$
$$+\left[\begin{pmatrix}8 & 8 & 8'\\ i & j & b\end{pmatrix}\eta(c)\delta_{a,\bar{c}}-\begin{pmatrix}8 & 8 & 8'\\ i & j & c\end{pmatrix}\eta(b)\delta_{a,\bar{b}}+\tfrac{1}{2}\eta(c)\eta(i)\delta_{i,\bar{j}}\begin{pmatrix}8 & 8 & 8'\\ a & b & \bar{c}\end{pmatrix}\right]\frac{k\cdot(p_2-p_3)}{k\cdot(p_2+p_3)}$$
$$+4\left[\begin{pmatrix}8 & 8 & 8'\\ i & j & a\end{pmatrix}\eta(b)\delta_{b,\bar{c}}\{m_\pi^2-(p_2+p_3)^2\}+\text{perm.}\right][m_\pi^2-(p_1+p_2+p_3)^2]^{-1}\Bigg\}, \quad (17a)$$

$$G_4=g\Bigg\{\begin{pmatrix}8 & 8 & 8'\\ i & j & a\end{pmatrix}\eta(b)\delta_{b,\bar{c}}\left(-1+4\frac{m_\pi^2-(p_2+p_3)^2}{m_\pi^2-(p_1+p_2+p_3)^2}\right)$$
$$+\left[\begin{pmatrix}8 & 8 & 8'\\ i & j & a\end{pmatrix}\eta(b)\delta_{b,\bar{c}}-\begin{pmatrix}8 & 8 & 8'\\ i & j & b\end{pmatrix}\eta(a)\delta_{a,\bar{c}}+\tfrac{1}{2}\eta(c)\eta(i)\delta_{i,\bar{j}}\begin{pmatrix}8 & 8 & 8'\\ a & b & \bar{c}\end{pmatrix}\right]\frac{k\cdot(p_1-p_2)}{k\cdot(p_1+p_2)}\Bigg\}+\text{perm.}, \quad (17b)$$

where $g=-2\sqrt{3}f_+(m_\pi f_\pi)^{-2}$. The quantities G_2 and G_3 are obtained from G_1 by the simultaneous permutation of the indices. Since the structure of these terms is quite obscure, it is convenient to write out the form factors for the four observable K_{e5} decays.

$K^+(k)\to\pi^+(p_1)\pi^-(p_2)\pi^0(p_3)e^+\nu_e$:

$$G_1=3^{-1/2}g\left[1-2\frac{(p_1+p_2)^2-m_\pi^2}{(p_1+p_2+p_3)^2-m_\pi^2}\right], \quad (18a)$$

$$G_2=3^{-1/2}g\left[1-2\frac{(p_1+p_2)^2-m_\pi^2}{(p_1+p_2+p_3)^2-m_\pi^2}+\frac{k\cdot(p_1-p_3)}{k\cdot(p_1+p_3)}\right], \quad (18b)$$

$$G_3=-\tfrac{1}{2}(3)^{-1/2}g\left[1+4\frac{(p_1+p_2)^2-m_\pi^2}{(p_1+p_2+p_3)^2-m_\pi^2}+\frac{k\cdot(p_1-p_2)}{k\cdot(p_1+p_2)}\right], \quad (18c)$$

$$G_4=\tfrac{1}{2}(3)^{-1/2}g\left[1-4\frac{(p_1+p_2)^2-m_\pi^2}{(p_1+p_2+p_3)^2-m_\pi^2}-\frac{k\cdot(p_1-p_2)}{k\cdot(p_1+p_2)}+2\frac{k\cdot(p_1-p_3)}{k\cdot(p_1+p_3)}\right]. \quad (18d)$$

$K^+(k)\to\pi^0(p_1)\pi^0(p_2)\pi^0(p_3)e^+\nu_e$:

$$G_1=G_2=G_3=G_4=-\tfrac{1}{2}(3)^{-1/2}g\left[3-4\frac{(p_1+p_2)^2+(p_1+p_3)^2+(p_2+p_3)^2-3m_\pi^2}{(p_1+p_2+p_3)^2-m_\pi^2}\right]. \quad (19)$$

$K^0(k)\to\pi^-(p_1)\pi^-(p_2)\pi^+(p_3)e^+\nu_e$:

$$G_1=6^{-1/2}g\left[1-4\frac{(p_1+p_3)^2+(p_2+p_3)^2-2m_\pi^2}{(p_1+p_2+p_3)^2-m_\pi^2}-\frac{k\cdot(p_2-p_3)}{k\cdot(p_2+p_3)}\right], \quad (20a)$$

$$G_2=6^{-1/2}g\left[1-4\frac{(p_1+p_3)^2+(p_2+p_3)^2-2m_\pi^2}{(p_1+p_2+p_3)^2-m_\pi^2}-\frac{k\cdot(p_1-p_3)}{k\cdot(p_1+p_3)}\right], \quad (20b)$$

$$G_3=4(6)^{-1/2}g\left[1-\frac{(p_1+p_3)^2+(p_2+p_3)^2-2m_\pi^2}{(p_1+p_2+p_3)^2-m_\pi^2}\right], \tag{20c}$$

$$G_4=6^{-1/2}g\left[2-4\frac{(p_1+p_3)^2+(p_2+p_3)^2-2m_\pi^2}{(p_1+p_2+p_3)^2-m_\pi^2}-\frac{k\cdot(p_1-p_3)}{k\cdot(p_1+p_3)}-\frac{k\cdot(p_2-p_3)}{k\cdot(p_2+p_3)}\right]. \tag{20d}$$

$K^0(k)\to\pi^0(p_1)\pi^0(p_2)\pi^-(p_3)e^+\nu_e$:

$$G_1=-(6)^{-1/2}g\left[2-4\frac{(p_1+p_2)^2-m_\pi^2}{(p_1+p_2+p_3)^2-m_\pi^2}+\frac{k\cdot(p_2-p_3)}{k\cdot(p_2+p_3)}\right], \tag{21a}$$

$$G_2=-(6)^{-1/2}g\left[2-4\frac{(p_1+p_2)^2-m_\pi^2}{(p_1+p_2+p_3)^2-m_\pi^2}+\frac{k\cdot(p_1-p_3)}{k\cdot(p_1+p_3)}\right], \tag{21b}$$

$$G_3=6^{-1/2}g\left[1+4\frac{(p_1+p_2)^2-m_\pi^2}{(p_1+p_2+p_3)^2-m_\pi^2}\right], \tag{21c}$$

$$G_4=-(6)^{-1/2}g\left[1-4\frac{(p_1+p_2)^2-m_\pi^2}{(p_1+p_2+p_3)^2-m_\pi^2}+\frac{k\cdot(p_1-p_3)}{k\cdot(p_1+p_3)}+\frac{k\cdot(p_2-p_3)}{k\cdot(p_2+p_3)}\right]. \tag{21d}$$

IV. K_{l5} FORM FACTORS IN THREE-SOFT-PION LIMIT

The K_{l5} form factors can also be calculated in a more traditional manner by going to the limit of all three pions being soft. While this calculation is of considerably greater complexity than the calculation in the previous section, it is of interest since it sheds some light on the matrix elements of the σ field [Eq. (8)]. Reducing in all three pions, one finds

$$\langle\pi_a(p_1)\pi_b(p_2)\pi_c(p_3)|V_j^\mu(0)|K_i(k)\rangle=-i2\sqrt{2}\eta(a)\eta(b)\eta(c)(m_\pi^2-p_1^2)(m_\pi^2-p_2^2)(m_\pi^2-p_3^2)$$
$$\times(8p_1^0p_2^0p_3^0)^{-1/2}(m_\pi^3f_\pi)^{-3}\int d^4xd^4yd^4z\, e^{i(p_1\cdot x+p_2\cdot y+p_3\cdot z)}\langle 0|T(\partial_\alpha A_{\bar a}{}^\alpha(x)\partial_\beta A_{\bar b}{}^\beta(y)\partial_\gamma A_{\bar c}{}^\gamma(z)V_j^\mu(0))|K_i(k)\rangle. \tag{22}$$

To simplify this equation, the identity given in Appendix C is used. In addition to the commutators of Eqs. (7), some assumption concerning the σ field must be made. According to the suggestion of the free quark model or of the Gell-Mann–Lévy σ model,[12] it is here assumed that when a, b are *pion indices* $(0,1,I_3)$, σ transforms as an isotopic scalar:

$$[A_a{}^0(x),\partial_\mu A_b{}^\mu(0)]\delta(x^0)=\eta(a)\delta_{a,\bar b}\sigma(x)\delta^4(x), \tag{23}$$

$$[A_a{}^0(x),\sigma(0)]\delta(x^0)=\partial_\mu A_a{}^\mu(x)\delta^4(x). \tag{24}$$

Equation (24) follows from the fact that σ transforms as an isotopic scalar, as can be seen through the use of the Jacobi identity. After considerable manipulation, one finds

$$\langle\pi_a(p_1)\pi_b(p_2)\pi_c(p_3)|V_j^\mu(0)|K_i(k)\rangle=2\sqrt{2}\eta(a)\eta(b)\eta(c)(8p_1^0p_2^0p_3^0)^{-1/2}$$
$$\times(m_\pi^2-p_1^2)(m_\pi^2-p_2^2)(m_\pi^2-p_3^2)(m_\pi^3f_\pi)^{-3}\sum_{n=1}^{7}N_n{}^\mu, \tag{25}$$

$$N_1{}^\mu=p_{1\alpha}p_{2\beta}p_{3\gamma}\int d^4xd^4yd^4z\, e^{i(p_1\cdot x+p_2\cdot y+p_3\cdot z)}\langle 0|T(A_{\bar a}{}^\alpha(x)A_{\bar b}{}^\beta(y)A_{\bar c}{}^\gamma(z)V_j{}^\mu(0))|K_i(k)\rangle, \tag{26a}$$

$$N_2{}^\mu=-i\sqrt{3}\sum_m\begin{pmatrix}8 & 8 & 8'\\ \bar a & j & m\end{pmatrix}\int d^4xd^4y\, e^{i(p_2\cdot x+p_3\cdot y)}\langle 0|T(\partial_\alpha A_{\bar b}{}^\alpha(x)\partial_\beta A_{\bar c}{}^\beta(y)A_m{}^\mu(0))|K_i(k)\rangle+\text{perm.}, \tag{26b}$$

$$N_3{}^\mu=\tfrac{1}{4}i\eta(a)\delta_{a,\bar b}\int d^4x\, e^{ip_3\cdot x}\langle 0|T(\partial_\alpha A_{\bar c}{}^\alpha(x)V_j{}^\mu(0))|K_i(k)\rangle+\text{perm.}, \tag{26c}$$

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[12] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

$$N_4{}^{\mu}=-\tfrac{3}{4}i\eta(a)\delta_{a,\bar{b}}\int d^4x\, e^{i(p_1+p_2+p_3)\cdot x}\langle 0|T(\partial_\alpha A_{\bar{c}}{}^{\alpha}(x)V_j{}^{\mu}(0))|K_i(k)\rangle+\text{perm.}, \tag{26d}$$

$$N_5{}^{\mu}=\tfrac{1}{4}\sum_m[\eta(a)\delta_{a,\bar{b}}\delta_{c,\bar{m}}-\eta(b)\delta_{b,\bar{c}}\delta_{a,\bar{m}}+\eta(c)\delta_{a,\bar{c}}\delta_{b,\bar{m}}]p_{1\alpha}\int d^4x e^{i(p_1+p_2+p_3)\cdot x}\langle 0|T(A_m{}^{\alpha}(x)V_j{}^{\mu}(0))|K_i(k)\rangle+\text{perm.}, \tag{26e}$$

$$N_6{}^{\mu}=i\tfrac{1}{2}\sqrt{3}\sum_m\begin{pmatrix}8 & 8 & 8'\\ a & b & \bar{m}\end{pmatrix}(p_1-p_2)_\alpha p_{3\beta}\int d^4xd^4y\, e^{i[(p_1+p_2)\cdot x+p_3\cdot y]}\langle 0|T(V_m{}^{\alpha}(x)A_{\bar{c}}{}^{\beta}(y)V_j{}^{\mu}(0))|K_i(k)\rangle+\text{perm.}, \tag{26f}$$

$$N_7{}^{\mu}=\eta(a)\delta_{a,\bar{b}}p_{3\alpha}\int d^4xd^4y\, e^{i[(p_1+p_2)\cdot x+p_3\cdot y]}\langle 0|T(\sigma(x)A_{\bar{c}}{}^{\alpha}(y)V_j{}^{\mu}(0))|K_i(k)\rangle+\text{perm.} \tag{26g}$$

Retaining only the terms *in the form factors* that are of zeroth order in the pion momenta in the limit $p_1, p_2, p_3\to 0$, it is apparent that most of these terms can be handled fairly easily. $N_1{}^{\mu}$ contributes form factors that are first or higher order in the pion momenta. $N_2{}^{\mu}$ is related to the K_{l4} matrix elements. $N_3{}^{\mu}$ and $N_4{}^{\mu}$ are related to the K_{l3} matrix elements. $N_5{}^{\mu}$ may be obtained from a consideration of the K_{l3} matrix element[13]:

$$\int d^4x\, e^{ip\cdot x}(2k^0)^{1/2}\langle 0|T(A_{\bar{a}}{}^{\nu}(x)V_j{}^{\mu}(0))|K_i(k)\rangle=\sqrt{3}m_\pi f_\pi f_+(1-\xi)\eta(a)\begin{pmatrix}8 & 8 & 8'\\ i & j & a\end{pmatrix}g^{\mu\nu}+O(p). \tag{27}$$

$N_6{}^{\mu}$ has zeroth-order terms that arise from a kaon intermediate state; using Eq. (27) and the fact that the isotopic spin current of the kaon is conserved, one finds

$$\int d^4xd^4y\, e^{i(p\cdot x+p'\cdot y)}(2k^0)^{1/2}\langle 0|T(V_m{}^{\alpha}(x)A_{\bar{c}}{}^{\beta}(y)V_j{}^{\mu}(0))|K_i(k)\rangle =3im_\pi f_\pi f_+(1-\xi)\sum_n\begin{pmatrix}8 & 8 & 8'\\ i & m & n\end{pmatrix}\begin{pmatrix}8 & 8 & 8'\\ n & j & c\end{pmatrix}\frac{g^{\mu\beta}k^\alpha}{(k-p)^2-m_K{}^2}. \tag{28}$$

$N_7{}^{\mu}$, finally, is a matrix element of the σ field and will be neglected. [There is no singularity arising from a kaon intermediate state since the right-hand side of Eq. (B2) vanishes.]

When all of this has been substituted into Eq. (25), considerable manipulation reveals that the same results are obtained that are found in Eqs. (17), with the exception that $p_1{}^2=p_2{}^2=p_3{}^3=p_1\cdot p_2=p_2\cdot p_3=p_3\cdot p_1=0$; i.e., $[m_\pi{}^2-(p_i+p_j)^2][m_\pi{}^2-(p_1+p_2+p_3)^2]^{-1}=1$.

V. DISCUSSION OF MATRIX ELEMENT

One of the more interesting features of the K_{l5} form factors is that they can have a pole in the limit of two soft pions [e.g., $p_1=p_2=0$ in Eqs. (18)], and for this reason the matrix element obtained in the limit of all three pions soft is not a valid approximation to the matrix element in the physical region. This singularity arises from a matrix element of the σ field: If one goes to the limit $p_1=p_2=0$ in the manner of the K_{l4} calculation of Weinberg,[5] one arrives at an expression whose only singular term is of the form

$$\int d^4x\, e^{i(p_1+p_2)\cdot x}\langle\pi_c(p_3)|T(\sigma(x)V_j{}^{\mu}(0))|K_i(k)\rangle. \tag{29}$$

This is singular by virtue of a pion intermediate state and the nonvanishing of the right-hand side of Eq. (B3).

Of central importance to the calculation of the K_{l5} form factors is the transformation property of the σ field; this enters the single-soft-pion calculation through the $\pi\pi$ scattering amplitude and the three-soft-pion calculation through Eqs. (23) and (24). If it is assumed that σ transforms as an operator with $I=2$, then a different (though similar) set of form factors is obtained for K_{l5}.[14]

Finally, it is somewhat surprising that the matrix elements of the σ field are determined by the specification of the field's transformation properties, at least within the context of the parametrization of the $K\pi$ and $\pi\pi$ scattering amplitudes. It is particularly important that not all of these matrix elements can be neglected—e.g., Eq. (B3), and especially Eq. (29), which is singular.

VI. NUMERICAL RESULTS

The only decays that are energetically possible for the K_{l5} decay modes are the K_{e5} modes, and these have about 80-MeV phase space available. Since the mass of the electron is negligible over almost all of phase space, it will consistently be assumed that $m_e=0$. The total decay rate for these modes can then be

[13] S. Adler and Y. Dothan, Phys. Rev. **151**, 1267 (1966).

[14] P. McNamee, Ph.D. thesis, Stanford University, Stanford, Calif., 1967 (unpublished).

written as

$$\Gamma=\frac{G^2\sin^2\theta_V}{4k^0(2\pi)^5 n!}\int\frac{d^3p_1}{2p_1^0}\frac{d^3p_2}{2p_2^0}\frac{d^3p_3}{2p_3^0}[G_1p_1^\mu+G_2p_2^\mu+G_3p_3^\mu]^*\times[G_1p_1^\nu+G_2p_2^\nu+G_3p_3^\nu]I_{\mu\nu}(k-p_1-p_2-p_3)\,,\quad(30)$$

where $1/n!$ is a statistical factor which occurs if there are n identical pions in the final state. The matrix element of the lepton current is contained in $I^{\mu\nu}$, which is defined by

$$I^{\mu\nu}(K)=\sum_{\text{spin}}\int d^3p_e d^3p_\nu\;(2\pi)^{-6}\delta^4(K-p_e-p_\nu)\times\langle e^+,\nu_e|j^\mu(0)|0\rangle^*\langle e^+,\nu_e|j^\nu(0)|0\rangle.\quad(31)$$

This may be integrated by covariant methods[15] to give

$$I^{\mu\nu}(K)=(48\pi^5)^{-1}(K^\mu K^\nu-g^{\mu\nu}K^2)\,.\quad(32)$$

After the lepton variables have been summed, the K_{e5} decay rate depends on an integral of the form[16]

$$\rho=\int\frac{d^3p_1}{2p_1^0}\frac{d^3p_2}{2p_2^0}\frac{d^3p_3}{2p_3^0}F\,,\quad(33)$$

where F is a Lorentz-invariant function. It will be convenient to define a new set of variables:

$$t=p_1+p_2\,,\quad(34a)$$

$$r=p_1-p_2\,,\quad(34b)$$

$$\sigma=k-p_1-p_2\,,\quad(34c)$$

$$\lambda=k-p_1-p_2+p_3\,,\quad(34d)$$

and to choose the six independent Lorentz scalars of which F is a function to be t^2, σ^2, λ^2, $t\cdot p_3$, $k\cdot r$, and $r\cdot p_3$. After a simple substitution,

$$\rho=\int\frac{d^3p_3}{2p_3^0}d^4t d^4p_1\;\delta(p_1^2-m_1^2)\delta([t-p_1]^2-m_2^2)F\,,\quad(35)$$

it is possible to perform the integration over $\mathbf{p}_1$ in the frame $\mathbf{t}=\mathbf{0}$, $\hat{k}=\hat{z}$; since this frame is a center-of-mass frame, there are no kinematic constraints on the polar and azimuthal angles of $\mathbf{p}_1$. The integration is greatly simplified by assuming that F is independent of $r\cdot p_3$:

$$\rho=\tfrac{1}{2}\pi\int\frac{d(k\cdot r)}{\Lambda(k^2,t^2,\sigma^2)}d^4t\frac{d^3p_3}{2p_3^0}F\,,\quad(36)$$

where

$$\Lambda(x,y,z)=[x^2+y^2+z^2-2xy-2xz-2yz]^{1/2}.\quad(37)$$

It is then possible to perform the integration over $\mathbf{p}_3$ in the frame $\boldsymbol{\sigma}=\mathbf{0}$, $\hat{t}=\hat{z}$;

$$\rho=\tfrac{1}{2}\pi^2\int\frac{d\lambda^2 d(t\cdot p_3)d(k\cdot r)}{[\Lambda(k^2,t^2,\sigma^2)]^2}d^4t\;F\,,\quad(38)$$

and finally to perform the integration over t in the frame $\mathbf{k}=\mathbf{0}$;

$$\rho=\frac{\pi^3}{4k^2}\int\frac{dt^2d\sigma^2d\lambda^2d(t\cdot p_3)d(k\cdot r)}{\Lambda(k^2,t^2,\sigma^2)}F.\quad(39)$$

In the matrix elements of the four possible K_{e5} decay modes [Eqs. (18)–(21)], $G_1=G_2$ except for terms containing the factor $k\cdot(p_i-p_j)/k\cdot(p_i+p_j)$ which, in the region available for the decay, is $\lesssim\frac{1}{8}$. Since these are the only terms that involve $r\cdot p_3$ [cf. Eq. (36)], they will be neglected. The decay rate for K_{e5} may therefore be written, in the rest frame of the kaon, as

$$\Gamma=\frac{G^2\sin^2\theta_V}{192(2\pi)^7 n! m_K^3}\int\frac{dt^2d\sigma^2d\lambda^2d(t\cdot p_3)d(k\cdot r)}{\Lambda(m_K^2,t^2,\sigma^2)}\times[G_1t^\mu+G_3p_3^\mu][G_1t^\nu+G_3p_3^\nu]\times[(k-t-p_3)_\mu(k-t-p_3)_\nu-g_{\mu\nu}(k-t-p_3)^2],\quad(40)$$

where n is the number of identical pions in the final state. The limits on the integrals are

$$k\cdot r_\pm=(1/2t^2)[(m_K^2+t^2-\sigma^2)(m_1^2-m_2^2)\pm\Lambda(m_K^2,t^2,\sigma^2)\Lambda(t^2,m_1^2,m_2^2)],\quad(41a)$$

$$t\cdot p_{3\pm}=(1/4\sigma^2)[(m_K^2-t^2-\sigma^2)(\lambda^2-\sigma^2-m_3^2)\pm\Lambda(m_K^2,t^2,\sigma^2)\Lambda(\sigma^2,\lambda^2,m_3^2)],\quad(41b)$$

$$[(\sigma^2)^{1/2}+m_3]^2\le\lambda^2\le2(\sigma^2+m_3^2)\,,\quad(41c)$$

$$m_3^2\le\sigma^2\le[m_K-(t^2)^{1/2}]^2\,,\quad(41d)$$

$$(m_1+m_2)^2\le t^2\le(m_K-m_3)^2.\quad(41e)$$

The parameters f_π and f_+ are evaluated from the experimental rates for the $\pi_{\mu2}$ and K_{e3} decays. It will be noted that since f_π and f_+ are inversely proportional to $\cos\theta_A$ and $\sin\theta_V$, respectively, the only dependence of the K_{e5} decay rate on the Cabbibo angles is a weak dependence on θ_A: $\Gamma(K_{e5})\sim\cos^4\theta_A$. The numerical values of the experimentally determined constants[17] were taken to be $G^2=1.3510\times10^{-22}$ MeV^{-4}, $\sin^2\theta_V=0.044$, $\sin^2\theta_A=0.07$, $\Gamma(\pi^+\to\mu^+\nu_\mu)=3.92\times10^7$ sec^{-1}, and $\Gamma(K^+\to\pi^0e^+\nu_e)=3.61\times10^6$ sec^{-1}. These give the values $|f_\pi|=0.967$ and $|f_+|=0.747$.

It was possible to perform the integration over $k\cdot r$ and $t\cdot p_3$ in Eq. (40) explicitly; because at the complexity of the integrand, integration over the variables t^2, σ^2, and λ^2 was done numerically. Electromagnetic mass differences were neglected and all constants have the

[15] J. D. Jackson, *Weak Interactions in Elementary Particle Physics and Field Theory, Brandeis Summer Institute, 1962* (W. A. Benjamin, Inc., New York, 1963), Vol. 1, p. 263.

[16] W. Williamson, Jr., Am. J. Phys. **33**, 987 (1965).

[17] N. Brene, L. Veje, M. Roos, and C. Cronström, Phys. Rev. **149**, 1288 (1966); A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **37**, 633 (1965); G. H. Trilling, in Proceedings of the International Conference on Weak Interactions, Argonne National Laboratory Report No. ANL 7130, 1965 (unpublished).

values quoted above. The rates obtained are

$$\Gamma(K^+ \to \pi^+\pi^-\pi^0 e^+\nu_e) = 2.8\times 10^{-4}\ \text{sec}^{-1},$$
$$\Gamma(K^+ \to \pi^0\pi^0\pi^0 e^+\nu_e) = 2.7\times 10^{-4}\ \text{sec}^{-1},$$
$$\Gamma(K^0 \to \pi^-\pi^-\pi^+ e^+\nu_e) = 8.1\times 10^{-4}\ \text{sec}^{-1},$$
$$\Gamma(K^0 \to \pi^0\pi^0\pi^- e^+\nu_e) = 2.8\times 10^{-4}\ \text{sec}^{-1}.$$

These rates are substantially higher than previous estimates indicate,[18] but they are far too small for the process to be seen in the near future.

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APPENDIX A

It is convenient to use the spherical-tensor basis for $SU(3)$ calculations since all matrix elements are then expressed immediately in terms of physical eigenstates rather than in cartesian components. The price that is paid is, of course, the complexity of the "angular momentum" algebra and the question of normalizations and phases. For present purposes a few details are sufficient.[14]

The relation of a general octet spherical-tensor operator $T_{Y,I,I_3}{}^8$ to the same operator in cartesian form T_i is given as

$$T_{0,0,0}{}^8 = T_8, \qquad \text{(A1)}$$
$$T_{0,1,0}{}^8 = T_3, \qquad \text{(A2)}$$
$$T_{0,1,\pm1}{}^8 = \mp\tfrac{1}{2}\sqrt{2}(T_1 \pm iT_2), \qquad \text{(A3)}$$
$$T_{\pm1,\frac{1}{2},\pm\frac{1}{2}}{}^8 = \mp\tfrac{1}{2}\sqrt{2}(T_4 \pm iT_5), \qquad \text{(A4)}$$
$$T_{\pm1,\frac{1}{2},\mp\frac{1}{2}}{}^8 = -\tfrac{1}{2}\sqrt{2}(T_6 \pm iT_7). \qquad \text{(A5)}$$

The importance of the spherical-tensor operators lies in the Wigner-Eckart theorem, which is stated for the group $SU(3)$ as[10]

$$\langle N_3,c|T_b{}^{N_2}|N_1,a\rangle = \sum_\gamma \begin{pmatrix} N_1 & N_2 & N_{3\gamma} \\ a & b & c \end{pmatrix} \langle N_3\|T^{N_2}\|N_1\rangle_\gamma, \qquad \text{(A6)}$$

where $a, b, \cdots, m, n$ are used to denote $SU(3)$ transformation properties, with $a = (Y,I,I_3)$ and $\bar{a} = (-Y,I,-I_3)$.

In the assignment of $SU(3)$ transformation properties to physical eigenstates, there are several phases that must be determined by convention. For the octet of pseudoscalar mesons, this is most conveniently done by defining the free-field second quantized operator

$$\varphi_m(x) = (2\pi)^{-3/2}\int d^3p \times[f_p(x)a_m(p) + \eta(m)f_p{}^*(x)a^\dagger{}_{\bar{m}}(p)], \qquad \text{(A7)}$$

where $\eta(Y,I,I_3) = (-1)^{I_3+\frac{1}{2}Y}$. These conventions are consistent with those usually used with $SU(2)$ in the more common case of isotopic spin.[19] Two points are worthy of notice: (1) The phase $\eta(m)$ must be introduced when crossing symmetry is used; (2) the more useful convention is that the spherical tensor operator $T_m{}^8$ is $(\varphi_m)^\dagger = \eta(m)\varphi_{\bar{m}}$ [cf. Eq. (15)] since $(\varphi_m)^\dagger$ creates a meson of $SU(3)$ index m.

If the weak Hamiltonian is defined to be

$$\mathcal{H} = 2^{-1/2}Gj_\mu J^\mu + \text{H.c.}$$

with the usual definition of the lepton current $[j^\mu = \bar{\psi}_\nu\gamma^\mu(1-\gamma^5)\psi_e]$, then the hadron current for both strangeness-changing and strangness-nonchanging interactions is

$$J^\mu = \sqrt{2}(\cos\theta_V\, V_{0,1,-1}{}^\mu - \cos\theta_A\, A_{0,1,-1}{}^\mu) + \sqrt{2}(\sin\theta_V\, V_{-1,\frac{1}{2},-\frac{1}{2}}{}^\mu - \sin\theta_A\, A_{-1,\frac{1}{2},-\frac{1}{2}}{}^\mu), \qquad \text{(A8)}$$

where $V_a{}^\mu$ and $A_a{}^\mu$ are the vector and axial-vector currents, respectively.

Finally, several useful identities can be derived through the use of the 6μ recoupling coefficients for the Clebsch-Gordan coefficients of $SU(3)$[20]; of particular interest is the general identity

$$\eta(a)\sum_m \begin{pmatrix} 8 & 8 & 8' \\ a & c & m \end{pmatrix}\begin{pmatrix} 8 & 8 & 8' \\ b & d & \bar{m} \end{pmatrix} - \eta(b)\sum_m \begin{pmatrix} 8 & 8 & 8' \\ b & c & m \end{pmatrix}\begin{pmatrix} 8 & 8 & 8' \\ a & d & \bar{m} \end{pmatrix} = \eta(d)\sum_m \begin{pmatrix} 8 & 8 & 8' \\ a & b & m \end{pmatrix}\begin{pmatrix} 8 & 8 & 8' \\ c & d & \bar{m} \end{pmatrix}. \qquad \text{(A9)}$$

By inserting explicit values of Clebsch-Gordan coefficients, one can also obtain the following special cases which are useful in calculations with pions and kaons. For a, b pion indices $(0,1,I_3)$ and c, d kaon indices $(\pm1,\frac{1}{2},I_3)$

$$\sum_m \begin{pmatrix} 8 & 8 & 8' \\ a & c & m \end{pmatrix}\begin{pmatrix} 8 & 8 & 8' \\ b & d & \bar{m} \end{pmatrix} = -(1/12)\delta_{a,\bar{b}}\delta_{c,\bar{d}} + \tfrac{1}{2}\eta(a)\eta(d)\sum_m \begin{pmatrix} 8 & 8 & 8' \\ a & b & m \end{pmatrix}\begin{pmatrix} 8 & 8 & 8' \\ c & d & \bar{m} \end{pmatrix}, \qquad \text{(A10)}$$

and for a, b, c, d all pion indices $(0,1,I_3)$

$$\sum_m \begin{pmatrix} 8 & 8 & 8' \\ a & c & m \end{pmatrix}\begin{pmatrix} 8 & 8 & 8' \\ b & d & \bar{m} \end{pmatrix} = \tfrac{1}{3}(\delta_{a,\bar{d}}\delta_{b,\bar{c}} - \delta_{a,\bar{b}}\delta_{c,\bar{d}}). \qquad \text{(A11)}$$

[18] V. A. Kolkunov and I. V. Lyagin, Zh. Eksperim. i Teor. Fiz. **45**, 2009 (1963) [English transl.: Soviet Physics—JETP **18**, 1379 (1964)]. [*Note added in proof.* A. Gaffur, Nuovo Cimento **29**, 302 (1963); G. W. Intemann and I. R. Lapidus, *ibid.* **52A**, 432 (1967), obtain higher rates through the use of pole models.]

[19] P. A. Carruthers and F. P. Krisch, Ann. Phys. (N. Y.) **33**, 1 (1964).

[20] M. Krammer, in *Weak Interactions and Higher Symmetries*, edited by P. Urban (Springer-Verlag, Vienna, 1964), p. 183; M. Resnikoff, J. Math. Phys. **8**, 79 (1967); S. Fubini, G. Segré, and J. D. Walecka, Ann. Phys. (N. Y.) **39**, 381 (1966).

APPENDIX B

The $K\pi$ scattering amplitude can be derived either by the single-soft-pion methods used in Sec. III or by the Adler self-consistency argument[21] used by Weinberg[8]; in either case, the scattering amplitude is determined with no reference to assumptions concerning the σ field. The derivation of the $\pi\pi$ scattering amplitude is, however, more ambiguous. Starting from the parametrization

$$\begin{aligned}\langle\pi_a(p_1)\pi_b(p_2)|S|\pi_c(p_3)\pi_d(p_4)\rangle=&(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)(16p_1^0p_2^0p_3^0p_4^0)^{-1/2}[\eta(a)\eta(c)\delta_{a,\bar{b}}\delta_{c,\bar{d}}\\&\times\{A+B[(p_1-p_3)^2+(p_2-p_3)^2]+C(p_1+p_2)^2\}+\delta_{a,c}\delta_{b,d}\{A+B[(p_1+p_2)^2+(p_2-p_3)^2]+C(p_1-p_3)^2\}\\&+\delta_{a,d}\delta_{b,c}\{A+B[(p_1+p_2)^2+(p_1-p_3)^2]+C(p_2-p_3)^2\}],\end{aligned}\tag{B1}$$

one finds either through single-soft-pion techniques or through Weinberg's use of the Adler self-consistency argument that $A+m_\pi^2(2B+C)=0$ and $B-C=-2i(m_\pi f_\pi)^{-2}$. To proceed any further, one must add some additional piece of information. If it is assumed that the σ field transforms as an isotopic singlet, then the solution is $A=-2if_\pi^{-2}$, $B=0$, and $C=2i(m_\pi f_\pi)^{-2}$; if, on the other hand, it is assumed that σ transforms as an operator with $I=2$ [$I=1$ is ruled out since Eq. (8) is symmetric in a,b], then the solution is $A=8i(5f_\pi^2)^{-1}$, $B=-6i(5m_\pi^2f_\pi^2)^{-1}$, and $C=4i(5m_\pi^2f_\pi^2)^{-1}$. The assumption that σ transforms as an isotopic scalar is commonly accepted, and it is that solution that is adopted in Sec. III.

It is important to note that the $K\pi$ and $\pi\pi$ scattering amplitudes determine certain matrix elements of the σ field:

$$\lim_{k_1\to k_2}(4k_1^0k_2^0)^{1/2}\langle K_j(k_2)|\sigma(0)|K_i(k_1)\rangle=0\,,\tag{B2}$$

$$\lim_{k_1\to k_2}(4k_1^0k_2^0)^{1/2}\langle\pi_b(k_2)|\sigma(0)|\pi_a(k_1)\rangle=-im_\pi^2\delta_{a,b}.\tag{B3}$$

The former of these relations is independent of the transformation properties of σ, but the latter is based on the assumption that σ transforms as an isotopic scalar; both relations are valid up to terms of the order of $(k_1-k_2)^2$.

APPENDIX C

$$\begin{aligned}T(\partial X\partial Y\partial ZV)=&\partial_x^\alpha\partial_y^\beta\partial_z^\gamma T(X_\alpha Y_\beta Z_\gamma V)-\tfrac14(3\partial_y^\alpha-\partial_x^\alpha-\partial_z^\alpha)\delta_{xy}\delta_{yz}T([Z_0,[X_0,Y_\alpha]]V)-\tfrac14(3\partial_z^\alpha-\partial_x^\alpha-\partial_y^\alpha)\delta_{xy}\delta_{yz}T\\&\times([Y_0,[X_0,Z_\alpha]]V)-\tfrac14\delta_{xy}\delta_{yz}T((\partial_x^\alpha+\partial_y^\alpha+\partial_z^\alpha)[Z_0,[X_0,Y_\alpha]]V)-\tfrac14\delta_{xy}\delta_{yz}T((\partial_x^\alpha+\partial_y^\alpha+\partial_z^\alpha)[Y_0,[X_0,Z_\alpha]]V)\\&-\tfrac12\delta_{xy}\delta_{yz}T([[X_0,Y_0],\partial Z]V)-\tfrac12\delta_{xy}\delta_{yz}T([[X_0,Z_0],\partial Y]V)-\tfrac12\delta_{xy}\delta_{yz}T([Y_0,[Z_0,\partial X]]V)\\&-\tfrac12\delta_{xy}\delta_{yz}T([Z_0,[Y_0,\partial X]]V)-\delta_xT([X_0,V]\partial Y\partial Z)-\delta_yT([Y_0,V]\partial Z\partial X)-\delta_zT([Z_0,V]\partial X\partial Y)-\tfrac12\delta_{xy}T((\partial_x^\alpha+\partial_y^\alpha)\\&\times[X_0,Y_\alpha]\partial ZV)-\tfrac12\delta_{yz}T((\partial_y^\alpha+\partial_z^\alpha)[Y_0,Z_\alpha]\partial XV)-\tfrac12\delta_{zx}T((\partial_z^\alpha+\partial_x^\alpha)[Z_0,X_\alpha]\partial YV)-\tfrac12\delta_{xy}\delta_zT((\partial_x^\alpha+\partial_y^\alpha)\\&\times[X_0,Y_\alpha][Z_0,V])-\tfrac12\delta_{yz}\delta_xT((\partial_y^\alpha+\partial_z^\alpha)[Y_0,Z_\alpha][X_0,V])-\tfrac12\delta_{zx}\delta_yT((\partial_z^\alpha+\partial_x^\alpha)[Z_0,X_\alpha][Y_0,V])\\&-\partial_z^\alpha\delta_{xy}T([Y_0,\partial X]Z_\alpha V)-\partial_x^\alpha\delta_{yz}T([Z_0,\partial Y]X_\alpha V)-\partial_y^\alpha\delta_{zx}T([X_0,\partial Z]Y_\alpha V)+\delta_{xy}\delta_{yz}T([Y_0,[X_0,\partial Z]]V)\\&+\delta_{xy}\delta_{yz}T([Z_0,[Y_0,\partial X]]V)+\delta_{xy}\delta_{yz}T([X_0,[Z_0,\partial Y]]V)-\tfrac12(\partial_y^\alpha-\partial_x^\alpha)\partial_z^\beta\delta_{xy}T([X_0,Y_\alpha]Z_\beta V)-\tfrac12(\partial_z^\alpha-\partial_y^\alpha)\\&\times\partial_x^\beta\delta_{yz}T([Y_0,Z_\alpha]X_\beta V)-\tfrac12(\partial_x^\alpha-\partial_z^\alpha)\partial_y^\beta\delta_{zx}T([Z_0,X_\alpha]Y_\beta V)+\tfrac12(\partial_y^\alpha-\partial_x^\alpha)\delta_{xy}\delta_{yz}T([Z_0,[X_0,Y_\alpha]]V)\\&+\tfrac12(\partial_z^\alpha-\partial_y^\alpha)\delta_{xy}\delta_{yz}T([X_0,[Y_0,Z_\alpha]]V)+\tfrac12(\partial_x^\alpha-\partial_z^\alpha)\delta_{xy}\delta_{yz}T([Y_0,[Z_0,X_\alpha]]V)-\tfrac12\delta_x\delta_yT([X_0,[Y_0,V]]\partial Z)\\&-\tfrac12\delta_y\delta_zT([Y_0,[Z_0,V]]\partial X)-\tfrac12\delta_z\delta_xT([Z_0,[X_0,V]]\partial Y)-\tfrac12\delta_x\delta_yT([Y_0,[X_0,V]]\partial Z)-\tfrac12\delta_y\delta_zT([Z_0,[Y_0,V]]\partial X)\\&-\tfrac12\delta_z\delta_xT([X_0,[Z_0,V]]\partial Y)-\tfrac14\delta_x\delta_y\delta_z\{[X_0,[Y_0,[Z_0,V]]]+[X_0,[Z_0,[Y_0,V]]]\\&+[Y_0,[Z_0,[X_0,V]]]+[Z_0,[Y_0,[X_0,V]]]\}\,.\end{aligned}$$

In the interest of compactness the following abbreviations have been used: $X^\alpha=A^\alpha(x)$, $Y^\beta=A^\beta(y)$, $Z^\gamma=A^\gamma(z)$, $V=V^\mu(0)$, $\partial_x^\alpha=\partial/\partial x_\alpha$, $\delta_x=\delta(x^0)$, and $\delta_{xy}=\delta(x^0-y^0)$.

[21] S. Adler, Phys. Rev. **137**, B1022 (1965); **139**, B1638 (1965).