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New Sum Rule for Electromagnetic Mass Differences*

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A new sum rule which is valid for both $|\Delta I|=1$ and $|\Delta I|=2$ mass differences is derived. It gives a reasonable value for the kaon mass difference which cannot be computed by Cottingham's sum rule. It is shown that the new sum rule reduces to that of Cottingham if the latter sum rule exists.

IT is well known that the conventional method of computing electromagnetic mass differences within an isomultiplet using perturbation theory and form-factor cutoff generally leads to incorrect results. In particular, it is difficult to obtain even the right sign for the $|\Delta I|=1$ mass differences by this method. Some time ago, Cottingham[1] derived a sum rule for electromagnetic mass differences under the assumption that the scattering amplitude of spacelike photons off particles in an appropriate isomultiplet satisfies an unsubtracted dispersion relation. Recently, Harari[2] has argued that Cottingham's formula may not be valid for $|\Delta I|=1$ mass differences since the corresponding Compton amplitudes seem to require a subtraction.

In this paper, we derive a new sum rule which is applicable to both $|\Delta I|=1$ and $|\Delta I|=2$ electromagnetic mass differences. This sum rule collapses to the conventional result in case the latter exists. A rough estimate of the kaon mass difference, where the sum rule is saturated with single-particle intermediate states, leads to a qualitively correct answer.

Cottingham started from the equation

$$\Delta m=\frac{i}{8\pi^2}\int d^4q\frac{T(q^2,q_0^2)}{q^2-i\epsilon}, \tag{1}$$

where

$$q^2=\mathbf{q}^2-q_0^2,\quad T(q^2,q_0^2)=T_{\mu\mu}(q^2,q_0^2),$$

and $\epsilon_\mu{}^*\epsilon_\nu T_{\mu\nu}(q^2,q_0^2)$ is the forward Compton-scattering amplitude of a virtual photon with laboratory energy momentum $(q_0,\mathbf{q})$. Assuming an unsubtracted dispersion relation for $T(q^2,q_0^2)$ in q_0^2, when q^2 is held fixed, he obtained the sum rule[3]

$$\Delta m=\frac{-1}{4\pi}\int_0^\infty\frac{dq^2}{q^2}\int_0^\infty dq_0^2[(1+q^2/q_0^2)^{1/2}-1]\times\mathrm{Im}T(q^2,q_0^2). \tag{2}$$

We also start from Eq. (1). Instead of a fixed q^2 dispersion relation, however, we use a fixed $\mathbf{q}^2$ dispersion relation which can be proved rigorously from the Jost-Lehmann-Dyson representation.[4] Bjorken[5] has shown that the leading term of $T(q^2,q_0^2)$ for fixed $\mathbf{q}^2$ is of $O(1/q_0^2)$ at $q_0^2\to\infty$. Our major assumption is that this leading term may be interpreted as a source of the tadpole-type contributions of Coleman and Glashow.[6] The reasonableness of such an assumption has already been discussed in a quark model by Bjorken.[5] After subtracting the leading term from T, we have

$$T(q^2,q_0^2)-\text{leading term}=\frac{1}{\pi q^2}\int_0^\infty dq_0'^2\frac{(\mathbf{q}^2-q_0'^2)\,\mathrm{Im}T(\mathbf{q}^2-q_0'^2,\,q_0'^2)}{q_0'^2-q_0^2}. \tag{3}$$

Here we have assumed the convergence of the integral on the right-hand side. Inserting the right-hand side of Eq. (3) into Eq. (1) and integrating over q_0 and the angular direction of $\mathbf{q}$, we have for non-tadpole-type contributions

$$\Delta m=-\frac{1}{4\pi}\int_0^\infty d\mathbf{q}^2\int_0^\infty\frac{dq_0^2}{q^2}[|\mathbf{q}|/q_0-1-q^2/2\mathbf{q}^2]\times\mathrm{Im}T(\mathbf{q}^2-q_0^2,\,q_0^2). \tag{4}$$

Changing variables from $\mathbf{q}^2$ to $q^2=\mathbf{q}^2-q_0^2$ and interchanging the orders of integration, we obtain our basic formula

$$\Delta m=-\frac{1}{4\pi}\int_0^\infty\frac{dq^2}{q^2}\int_0^\infty dq_0^2[(1+q^2/q_0^2)^{1/2}-1-\tfrac{1}{2}q^2/(q^2+q_0^2)]\,\mathrm{Im}T(q^2,q_0^2)$$
$$-\frac{1}{4\pi}\int_{-\infty}^0\frac{dq^2}{q^2}\int_{-q^2}^\infty dq_0^2[(1+q^2/q_0^2)^{1/2}-1-\tfrac{1}{2}q^2/(q^2+q_0^2)]\,\mathrm{Im}T(q^2,q_0^2). \tag{5}$$

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[1] W. N. Cottingham, Ann. Phys. (N. Y.) **25**, 424 (1963). We note that Cottingham's sum rule is not in general equivalent to the conventional graphical formula. Although for spin-0 mass differences the contributions from the elastic part (that is, the contributions from the single-particle intermediate state composed of the external isomultiplet itself) of Cottingham's sum rule is the same as those of the graphical formula, it is not the case for spin-½ mass differences. This was shown by Y. Taguchi and K. Yamamoto, Progr. Theoret. Phys. (Kyoto) **38**, 1152 (1967).

[2] H. Harari, Phys. Rev. Letters **17**, 1303 (1966).

[3] The sum rule (2), while it looks similar to Cottingham's original one obtained in Ref. 1, is not in fact identical to it. It was derived by W. N. Cottingham and J. Gibb, Phys. Rev. Letters **18**, 883 (1967). [Insert their Eq. (9) into their Eq. (2).]

[4] R. Jost and H. Lehmann, Nuovo Cimento **5**, 1598 (1957); F. J. Dyson, Phys. Rev. **110**, 1460 (1958).

[5] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

[6] S. Coleman and S. L. Glashow, Phys. Rev. **134**, B671 (1964).

Here $\mathrm{Im}T(q^2,q_0^2)$ is given by

$$\mathrm{Im}T(q^2,q_0^2)=\pi(2\pi)^4\sum[\langle n|j_\mu(0)|\kappa\rangle\langle\kappa|j_\mu(0)|n\rangle]_{\mathrm{av}}\times[\delta^4(p+q-k)-\delta^4(p-q-k)],\quad(6)$$

where $\sum$ is the sum over the complete set of intermediate states $|\kappa\rangle$, k is the four-momentum of the state $|\kappa\rangle$, $|n\rangle$ is the state vector for the one-particle state under consideration with the four-momentum $p=(m,0)$ and "av" means average over all possible spin states.

The sum rule (5) is different from (2) in two respects. First, there is an additional term $-\frac{1}{2}q^2/(q^2+q_0^2)$ in each of the two square brackets of the integrand in (5). By virtue of this term, (5) converges even for $|\Delta I|=1$ mass differences. Second, unlike the Cottingham formula, (5) has contributions also from timelike (massive) photons. This is a disadvantage as it is hard to obtain experimental information on the corresponding total cross sections. We feel however, that in practice, where the integrals are saturated with a few low-lying intermediate states, this disadvantage is a minor one and that it is much more important to obtain convergent integral representation for Δm. We shall show later that (5) reduces to (2) if the right-hand side of (2) converges. To compute the kaon and nucleon mass differences using (5), we assume that the integral (5) is saturated by the contributions of low-lying levels.

In this paper we shall estimate only the elastic contribution. The elastic contributions to the sum rule (5) minus that to the conventional graphical formula is given by

$$-\frac{3\alpha}{8\pi m_k}\int_0^\infty dq^2(F_+^2-F_0^2)=-1.30\text{ MeV}.\quad(7)$$

for the kaon mass difference[7] $m(K^+)-m(K^0)$, and by

$$-\frac{3\alpha}{8\pi m_N}\int_0^\infty dq^2\left[-\frac{q^2}{q^2+4m_N^2}\{(F_{p1}+\mu_pF_{P2})^2-(F_{n1}+\mu_nF_{n2})^2\}+F_{p1}^2-F_{n1}^2\right]\quad(8)$$

for the nucleon mass difference $m(p)-m(n)$, where F and F_i are the usual form factors. Although the approximation where only the elastic contribution is retained may be too crude[8] to explain the observed mass differences, Eq. (7) gives[9]

$$\begin{aligned}m(K^+)-m(K^0)&=2.2\text{ MeV (conventional formula)}\\&\quad-4.7\text{ MeV (tadpole contribution)}\\&\quad-1.3\text{ MeV [Eq. (7)]}=-3.8\text{ MeV}.\end{aligned}\quad(9)$$

The experimental value is -4.05 ± 0.12 MeV. For nucleons, the correction given by (8) is rather small. For pions, although the two sum rules (2) and (5) coincide when no approximation are made, they are not equivalent in the elastic approximation. It is therefore not clear which of the two sum rules is suitable for approximations. Our opinion on this point is that the sum rule (5) should be used only for those cases where the Cottingham sum rule does not converge. A more detailed analysis will be given elsewhere.

One might think that there may be some convergence difficulty associated with the singularity at $q^2+q_0^2=0$ in the new term of the sum rule (5). If this is indeed the case, we may use

$$T(q^2,q_0^2)-\text{leading term}=\frac{1}{\pi(q^2+A^2)}\times\int_0^\infty dq'^2\frac{(\mathbf{q}^2-q_0'^2+A^2)\,\mathrm{Im}T(\mathbf{q}^2-q_0'^2,\,q_0'^2)}{q_0'^2-q_0^2}\quad(10)$$

instead of Eq. (3), where A is an arbitrary real parameter introduced to avoid the singularity. Then (5) is changed into

$$\begin{aligned}\Delta m=&-\frac{1}{4\pi}\int_0^\infty\frac{dq^2}{q^2}\int_0^\infty dq_0^2\left[(1+q^2/q_0^2)^{1/2}-1-\frac{q^2}{q^2+q_0^2+A^2+(q^2+q_0^2+A^2)^{1/2}(q^2+q_0^2)^{1/2}}\right]\mathrm{Im}T\\&-\frac{1}{4\pi}\int_{-\infty}^0\frac{dq^2}{q_2}\int_{-q^2}^\infty dq_0^2\left[(1+q^2/q_0^2)^{1/2}-1-\frac{q^2}{q^2+q_0^2+A^2+(q^2+q_0^2+A^2)^{1/2}(q^2+q_0^2)^{1/2}}\right]\mathrm{Im}T.\end{aligned}\quad(11)$$

Finally we show that (11)[10] reduces to (2) if the tadpole-type contribution is zero and the latter converges. For this purpose, we make use of the Jost-Lehmann-Dyson representation[11]:

$$\begin{aligned}\mathrm{Im}T=&|\mathbf{q}|^{-1}\int_0^m dx\int_c^\infty du\,\rho(u,x)[\theta(-u-q^2+2x|\mathbf{q}|)-\theta(-u-q^2-2x|\mathbf{q}|)]=|\mathbf{q}|^{-1}\int_0^m dx\int_c^\infty dv\,\sigma(v,x)\\&\times[\delta(v+q^2-2x|\mathbf{q}|)-\delta(v+q^2+2x|\mathbf{q}|)],\end{aligned}\quad(12)$$

with

$$c=2m[m-(m^2-x^2)^{1/2}]$$

and

$$\sigma(v,x)=\int_c^v du\,\rho(u,x).$$

Substituting Eq. (12) into Eq. (11) and integrating over

[7] For kaon form factors, we have used Eq. (15) of R. H. Socolow, Phys. Rev. **137**, B1221 (1965).

[8] The contribution of the vector-meson intermediate state is small. See R. H. Socolow, Ref. 7.

[9] Eq. (9) does not include the contribution from the vector meson.

[10] We use the sum rule (11) to avoid the unnecessary complication coming from the singularity at $q^2+q_0^2=0$.

[11] The final expression of Eq. (12) holds when the source of the tadpole-type contribution [the term of $O(1/q_0^2)$ in T] is zero.

q_0^2 and q^2, we get

$$\int_0^\infty \frac{dq^2}{q^2}\int_0^\infty dq_0^2 \frac{q^2\,\mathrm{Im}T}{q^2+q_0^2+A^2+(q^2+q_0^2+A^2)^{1/2}(q^2+q_0^2)^{1/2}}$$

$$=-\int_{-\infty}^0 \frac{dq^2}{q^2}\int_{-q^2}^\infty dq_0^2 \times\frac{q^2\,\mathrm{Im}T}{q^2+q_0^2+A^2+(q^2+q_0^2+A^2)^{1/2}(q^2+q_0^2)^{1/2}}$$

$$=\int_0^m dx\int_c^\infty dv \frac{4x\sigma(v,x)}{v+(v^2+4x^2A^2)^{1/2}} \quad (13)$$

and

$$\int_{-\infty}^0 \frac{dq^2}{q^2}\int_{-q^2}^\infty dq_0^2[(1+q^2/q_0^2)^{1/2}-1]\,\mathrm{Im}T=0\,, \quad (14)$$

which proves our claim.

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Comments on $\pi\pi$ Phase Shifts as Determined from the Peripheral Model*

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The determination of the S-wave $\pi\pi$ phase shifts δ_0^I at low energy from the analysis of $\pi N\rightarrow(2\pi)N$ is examined critically from the standpoint of the one-pion-exchange model with absorptive corrections. It is found that: (1) The value of δ_0^I depends strongly on the P-wave phase shifts, which cannot be unambiguously determined, at $m_{\pi\pi}<600$ MeV, by using a Breit-Wigner formula. (2) The ratio of the production density matrix elements ρ (with the $\pi\pi$ elastic scattering amplitudes factored out) depends strongly on $m_{\pi\pi}$ for $m_{\pi\pi}<600$ MeV. (3) The $(F-B)/(F+B)$ asymmetry shows a sizeable dependence on the momentum transfer t to the nucleon. It is concluded that more accurate data at low $m_{\pi\pi}$ are required in order to determine δ_0^I for $m_{\pi\pi}<600$ MeV. Tables of the $\rho(m_{\pi\pi},t)$ calculated from the absorption model for an incident-pion laboratory kinetic energy of 4 BeV are included. These could be directly applied to the data to obtain the low-energy $\pi\pi$ phase shifts.

The determination of the S-wave $\pi\pi$ phase shifts $\delta_0^I(m_{\pi\pi})$ at low energy ($m_{\pi\pi}\lesssim 600$ MeV) is of considerable importance because of the following factors: (1) They enter into a variety of processes; in all of them either the theoretical understanding of the dynamics is somewhat shaky or more experimental data is needed, thus no unambiguous values of δ_0^I have been obtained from these experiments.[1] (2) From the theoretical standpoint there have been a number of predictions made by the utilization of current-algebra techniques together with low-energy theorems.[2] These predictions depend critically on the smallness of the $\pi\pi$ S-wave scattering lengths.

The production processes

$$\pi^-+p\rightarrow\pi^-+\pi^++n \quad (1)$$

$$\rightarrow\pi^-+\pi^0+p \quad (2)$$

$$\rightarrow\pi^0+\pi^0+n \quad (3)$$

have been widely studied, using the (experimentally observed) peripheral nature of the interaction, to determine the $\pi\pi$ S- and P-wave amplitudes A_0 and A_1, mainly for $m_{\pi\pi}$ in the region of the ρ resonance.[3] The purpose of this article is to make a critical analysis of the possibility of using (1)–(3) to determine the $\pi\pi$ phase shifts at low $m_{\pi\pi}$. We use the one-pion-exchange

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[1] See, for example, P. Singer, Finnish Summer School, 1966 (to be published).
[2] See, for example, R. Dashen, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967), p. 51.
[3] W. D. Walker *et al.*, Phys. Rev. Letters **18**, 630 (1967), this paper contains references to earlier work; E. Malamud and P. E. Schlein, *ibid.* **19**, 1056 (1967).