Application of the Methods of Quantum Mechanics in the Statistical Theory of Waves in a Fluctuating Medium

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The basic equations for the statistical system of waves in a randomly fluctuating medium are presented in the same form as in quantum mechanics, assuming a Markov process for the temporal change of the medium. A simple model is chosen for the Fokker-Planck equation of the medium which gives rise to a fluctuation of the Gaussian process, and several results are given to illustrate how the methods used in quantum mechanics or quantum field theory can be applied almost without change. Thus, the physical variables are represented by linear operators, and their equations of motion are determined by an equation similar to the Heisenberg equation of motion. The system has two stationary states (corresponding to the vacuum states $\langle 0|$ and $|0\rangle$ in quantum field theory), and the rather standard methods in field theory can be used for the evaluation of the Green's functions. When the temporal changes of the medium are sufficiently large compared with those of the waves, an adiabatic approximation is possible, and the isomorphic transformation (corresponding to the unitary transformation) is employed to lead to the result that the statistical system of the waves for this model is in perfect correspondence with the waves of bosons which interact with each other only through a two-body potential.

1. INTRODUCTION

THE problems of wave propagation in a randomly fluctuating medium have been treated by many authors,[1–10] and there has been great interest in the "renormalization" of the propagation constant and in the evaluation of the correlation function of waves. The procedures hitherto adopted seem to be more or less as follows: The basic physical quantities are the wave function ψ and a fluctuating part of the medium, say q, and for the latter a suitable (space and/or time) correlation function is assumed. The average value of the wave function, $\langle\psi\rangle$ (or, more generally, the Green's function) is evaluated in terms of the correlation function (assuming the multivariate Gaussian distribution of q), and the effective value of the propagation constant or the "renormalized" value is obtained as the solution of an integral equation. Then the fluctuating part of the wave function, $\Delta\psi=\psi-\langle\psi\rangle$, is expressed in terms of $\langle\psi\rangle$ and q,[7–9] and the correlation function of $\Delta\psi$ is obtained either by a direct method using the correlation function of q or by solving a differential equation corresponding to the Bethe-Salpeter equation[11] in quantum field theory.[10,12–14] However, although the correlation function of the waves is a most important statistical quantity, not all statistical information can be obtained in terms of the correlation function; the description of the statistical system cannot be complete unless, for instance, the simultaneous probability density function of q and ψ is known (as a function of time). This situation is analogous to that in quantum mechanics, and the probability-density function in the statistical system corresponds to the probability-amplitude function in quantum mechanics. Hence, it is suggested that, if the equation for the simultaneous probability-density function of q and ψ is found, it corresponds in a certain way to the Schrödinger equation for the probability-amplitude function in quantum mechanics, and hence the "dynamics" of the whole statistical system may be completely determined.

Indeed, when the stochastic change of q is a Markov process, the probability-density function of q satisfies the Fokker-Planck equation, which is formally quite similar to the Schrödinger equation. Further, since the wave function satisfies a deterministic equation when q is given, it can be shown that the whole system of q and ψ is also a Markov process. Hence it follows that the simultaneous probability density function also satisfies the Fokker-Planck equation and its "Hamiltonian" (which is not generally Hermitian) determines the complete "dynamics" of q and ψ.

In this paper, a Markov process is assumed for the temporal change of the medium, and a linear equation

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[1] L. A. Chernov, in *Wave Propagation in a Random Medium* (McGraw-Hill Book Co., New York, 1960).
[2] V. I. Tatarski, in *Wave Propagation in a Turbulent Medium* (McGraw-Hill Book Co., New York, 1961).
[3] J. B. Keller, in *Proceedings of the Symposium on Applied Mathematics* (American Mathematical Society, Providence, R. I., 1964), Vol. 13, p. 145.
[4] R. C. Bourret, Nuovo Cimento **26**, 1 (1962); Can. J. Phys. 43, 619 (1965).
[5] K. Furutsu, J. Res. Natl. Bur. Std. **67D**, 303 (1963).
[6] U. Frisch, Institute d'Astrophysique Report, 1965 (unpublished).
[7] V. I. Tatarski and M. E. Gertsenshtein, Zh. Eksperim. i Teor. Fiz. 44, 676 (1963) [English transl.: Soviet Phys.—JETP **17**, 458 (1963)].
[8] Yu. A. Ryzhov and V. I. Tatarski, Zh. Eksperim. i Teor. Fiz. 48, 656 (1965) [English transl.: Soviet Phys.—JEPT **21**, 433 (1965)].
[9] P. Bassanini, Radio Sci. 2, 429 (1967).
[10] M. J. Beran, IEEE Trans. Antennas Propagation **15**, 66 (1967).
[11] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951); M. Gell-Mann and F. Low, *ibid.* **84**, 350 (1951); J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452 (1951); **37**, 455 (1951).
[12] R. B. Kieburtz, IEEE Trans. Antennas Propagation **15**, 76 (1967).
[13] W. P. Brown, Jr., IEEE Trans. Antennas Propagation **15**, 81 (1967).
[14] R. E. Hufnagel and N. R. Stanley, J. Opt. Soc. Am. **54**, 52 (1964).

for the waves. A simple model is chosen for the Fokker-Planck equation to cause a fluctuation of the Gaussian process. In Secs. 2 and 3, both the wave function and the relevant medium variable are assumed to be functions only of the time, while in Sec. 4, they are treated as functions of the coordinates of three-dimensional space also. In Sec. 3 A, the theory of Green's functions is developed according to Schwinger's method.[15] In Secs. 3 B and 4 E, the isomorphic transformation is used to show that when the temporal changes of the medium are sufficiently fast compared with those of the waves, the "Hamiltonian" of the system is in perfect correspondence with that of a system of boson particles which interact with each other only through a two-body force. This result is similar to the theory of Kraichnan, who assumed a random coupling for each pair of the q's of a system of wave functions.[16]

2. QUANTUM-MECHANICAL TREATMENT OF MARKOV PROCESS

We consider a physical quantity q whose temporal fluctuation is assumed to be a Markov process. Let $\langle q'',t_2|q',t_1\rangle$ be the probability density function defined in such a way that, when q has a definite (real) value q' at the time t_1, the probability that q has a value between q'' and $q''+dq''$ at the later time t_2 is given by $dq''\langle q'',t_2|q',t_1\rangle$. Then, in the case of the (single) Markov process (as is assumed), it holds, for arbitrary t_2 in the range $t_3\geq t_2\geq t_1$, that

$$\langle q''',t_3|q',t_1\rangle=\int \langle q''',t_3|q'',t_2\rangle dq''\langle q'',t_2|q',t_1\rangle, \quad t_3\geq t_2\geq t_1 \tag{2.1}$$

with the conditions

$$\langle q'',t_2|q',t_1\rangle|_{t_2\to t_1}=\delta(q''-q'), \tag{2.2}$$

$$\int dq''\langle q'',t_2|q',t_1\rangle=1. \tag{2.3}$$

Hence, in terms of the notation

$$\langle q''|H|q'\rangle\equiv \lim_{\Delta t\to+0} \Delta t^{-1}\{\langle q'', t+\Delta t|q',t\rangle - \langle q'',t|q',t\rangle\}, \tag{2.4}$$

or, on account of the condition (2.2),

$$\langle q''|H|q'\rangle=-\lim_{\Delta t\to-0} \Delta t^{-1}\{\langle q'',t|q', t+\Delta t\rangle - \langle q'',t|q',t\rangle\}, \tag{2.5}$$

Eq. (2.1) gives the following equations:

$$\partial/\partial t_2\langle q'',t_2|q',t_1\rangle=\int \langle q''|H|q'''\rangle dq'''\langle q''',t_2|q',t_1\rangle,$$

$$-\partial/\partial t_1\langle q'',t_2|q',t_1\rangle=\int \langle q'',t_2|q''',t_1\rangle dq'''\langle q'''|H|q'\rangle, \quad t_2\geq t_1. \tag{2.6}$$

Equation (2.6) is the Fokker-Planck equation, which prescribes the temporal change of the probability-density function in terms of $\langle q''|H|q'\rangle$. The latter may be regarded as a continuous matrix (with respect to q' and q'') operating on the probability-density function, and (2.3) imposes the condition

$$\int dq''\langle q''|H|q'\rangle=0. \tag{2.7}$$

A. Determination of $\langle q''|H|q'\rangle$

We first consider $\langle \partial q/\partial t\rangle|_{q=q'}$, which is the expectation value of the time derivative $\partial q/\partial t$ when q has a definite value q', and is hence given by

$$\langle \partial q/\partial t\rangle|_{q=q'}=\lim_{\Delta t\to+0} \Delta t^{-1}\int dq''(q''-q') \times\langle q'', t+\Delta t|q',t\rangle. \tag{2.8}$$

Since, according to (2.6) and (2.2),

$$\langle q'', t+\Delta t|q',t\rangle =\delta(q'-q'')+\langle q''|H|q'\rangle\Delta t+O(\Delta t^2), \tag{2.9}$$

the right side of (2.8) is also expressed by

$$\int dq''(q''-q')\langle q''|H|q'\rangle = \iint dq''\langle q''|q|q'''\rangle dq'''\langle q'''|H|q'\rangle - \iint dq''\langle q''|H|q'''\rangle dq'''\langle q'''|q|q'\rangle, \tag{2.10}$$

where $\langle q''|q|q'\rangle$ is the diagonal matrix defined by

$$\langle q''|q|q'\rangle=q''\delta(q''-q'), \tag{2.11}$$

and $\delta(q')$ is the ordinary Dirac δ function. Thus, using the usual matrix multiplication convention, (2.8) is found to be expressed by

$$\langle \partial q/\partial t\rangle|_{q=q'}=\int dq''\langle q''|[q,H]|q'\rangle \tag{2.12}$$

in terms of the notation

$$[A,B]=AB-BA. \tag{2.13}$$

In the same way, the expectation values of the higher-

[15] See J. Schwinger (Ref. 11).
[16] R. H. Kraichnan, J. Math. Phys. 2, 124 (1961).

order terms can be expressed in terms of the matrices q and H as follows:

$$\langle\partial^2 q/\partial t\rangle|_{q=q'}\equiv\lim_{\Delta t\to+0}\Delta t^{-1}\int dq''(q''-q')^2\langle q'',t+\Delta t|q',t\rangle$$

$$=\int dq''(q''-q')^2\langle q''|H|q'\rangle$$

$$=\int dq''\langle q''|q^2H-2qHq+Hq^2|q'\rangle$$

$$=\int dq''\langle q''|[q,[q,H]]|q'\rangle, \qquad (2.14)$$

$$\langle\partial^n q/\partial t\rangle|_{q=q'}\equiv\lim_{\Delta t\to+0}\Delta t^{-1}\int dq''(q''-q')^n\langle q'',t+\Delta t|q',t\rangle$$

$$=\int dq''\langle q''|[q,[q,\cdots,[q,H]]\cdots]|q'\rangle. \qquad (2.15)$$

Here, in (2.15), the brackets are $n$fold.

Now, we introduce the Hermitian matrix p defined by

$$\langle q''|p|q'\rangle\equiv-i(\partial/\partial q'')\delta(q''-q') \qquad (2.16)$$

with the commutation relation

$$[q,p]=i, \qquad (2.17)$$

and also the constant vector $\langle 0|q'\rangle=1$, which can be regarded as the eigenvector of p with vanishing eigenvalue:

$$\langle 0|q'\rangle=1, \quad \int\langle 0|q''\rangle dq''\langle q''|p|q'\rangle=0, \qquad (2.18a)$$

or

$$\langle 0|p=0. \qquad (2.18b)$$

Then (2.15) can be expressed symbolically by

$$\langle\partial^n q/\partial t\rangle|_{q=q'}=\langle 0|[q,[q,\cdots,[q,H]]\cdots]|q'\rangle, \quad n=1,\,2,\,\cdots \qquad (2.19)$$

and (2.7) and (2.3) by

$$\langle 0|H|q'\rangle=0, \quad \langle 0|q',t_1\rangle=1 \qquad (2.20)$$

in terms of the notation

$$\int\langle 0|q''\rangle dq''\langle q''|A|q'\rangle=\langle 0|A|q'\rangle. \qquad (2.21)$$

Equation (2.19) with the condition (2.20) is sufficient to determine H uniquely in terms of the expectation values $\langle\partial^n q/\partial t\rangle$, $n=1, 2, \cdots$; indeed, by the help of (2.18b) and the successive use of the commutation relation (2.17), the general solution of (2.19) for the boundary condition (2.20) is readily proven to be

$$H=\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}p^n\langle\partial^n q/\partial t\rangle. \qquad (2.22)$$

For example, in the case in which

$$\langle\partial q/\partial t\rangle=-\nu q, \quad \langle\partial^2 q/\partial t\rangle=m, \quad \langle\partial^n q/\partial t\rangle=0, \quad n\geq 3 \qquad (2.23)$$

ν and m being positive constants, (2.22) becomes

$$H=-\tfrac{1}{2}mp^2+i\nu pq, \qquad (2.24)$$

and hence, from (2.6),

$$[\partial/\partial t_2-\tfrac{1}{2}m(\partial/\partial q'')^2-\nu(\partial/\partial q'')q'']\langle q'',t_2|q',t_1\rangle=0. \qquad (2.25)$$

The solution of (2.25) for the boundary condition (2.2) is found to be [see (2.71)]

$$\langle q'',t_2|q',t_1\rangle=(\nu/\pi m)^{1/2}\{1-e^{-2\nu(t_2-t_1)}\}^{-1/2}\times\exp[-m^{-1}\nu\{q''-e^{-\nu(t_2-t_1)}q'\}^2/\{1-e^{-2\nu(t_2-t_1)}\}], \quad t_2>t_1 \qquad (2.26)$$

and tends to a stationary distribution, say $\langle q''|0\rangle$, as $t_2-t_1\to+\infty$:

$$\langle q''|0\rangle\equiv\langle q'',t_2|q',t_1\rangle|_{t_2-t_1\to\infty}=(\nu/\pi m)^{1/2}\exp[-m^{-1}\nu q''^2]. \qquad (2.27)$$

B. Time-Dependent Matrix Representation of Physical Variables

As we have seen in the preceding section, the Fokker-Planck equation (2.6) evidently corresponds to the Schrödinger equation in quantum mechanics and $\langle q'',t_2|q',t_1\rangle$ to the probability-amplitude function or the transformation function. The physical quantity q is represented by the diagonal matrix (2.11), and the other operator p, which is quite naturally introduced in the Hamiltonian H, is represented by the matrix (2.16), which has the commutation relation (2.17), and hence is "canonically conjugate" to q.

So far the matrices q and p are independent of time, and hence they are those of the Schrödinger representation. On the other hand, there is the Heisenberg representation in which the physical variables are represented by time-dependent matrices, and their equations of motion often result in clear insights and powerful methods for the problems.

For any function F of q and p which is represented by $\langle q''|F|q'\rangle$ in the Schrödinger representation, we define the time-dependent matrix $F(t)$ by the "mixed" representation

$$\langle q_2',t_2|F(t)|q_1',t_1\rangle=\int\int\langle q_2',t_2|q',t\rangle dq'\langle q'|F|q''\rangle\times dq''\langle q'',t|q_1',t_1\rangle. \qquad (2.28)$$

Hence, for instance, (2.6) is expressed by

$$\begin{aligned} \partial/\partial t_2\langle q'',t_2|q',t_1\rangle&=\langle q'',t_2|H(t_2)|q',t_1\rangle, \\ -\partial/\partial t_1\langle q'',t_2|q',t_1\rangle&=\langle q'',t_2|H(t_1)|q',t_1\rangle. \end{aligned} \tag{2.29}$$

Putting $F=q$ and $t=t_2$ or t_1 in (2.28), we find that

$$\begin{aligned} \langle q_2',t_2|q(t_2)|q_1',t_1\rangle&=q_2'\langle q_2',t_2|q_1',t_1\rangle, \\ \langle q_2',t_2|q(t_1)|q_1',t_1\rangle&=\langle q_2',t_2|q_1',t_1\rangle q_1'. \end{aligned} \tag{2.30}$$

Hence, the mixed representation (2.28) can be interpreted as that in which $q(t_2)$ and $q(t_1)$ are diagonal on the left (column) and right (row) sides, respectively.

Operating $\partial/\partial t$ on both sides of (2.28) and using (2.29), we readily obtain

$$\begin{aligned} (\partial/\partial t)\langle q_2',t_2|F(t)|q_1',t_1\rangle=\langle q_2',t_2|& \\ -H(t)F(t)+F(t)H(t)&|q_1',t_1\rangle \end{aligned} \tag{2.31}$$

or, omitting $\langle q_2',t_2|$ and $|q_1',t_1\rangle$ on both sides,

$$(\partial/\partial t)F(t)=[F(t),H(t)], \tag{2.32}$$

which corresponds to the Heisenberg equation of motion in quantum mechanics.

In the special case of $F=H$, (2.32) yields $\partial H/\partial t=0$ and hence H is a constant of the motion, provided that H is not an explicit function of time.

In the case of the example (2.23), H is given by (2.24) or, adding some extra terms for later convenience, by

$$H=H_q\equiv-\tfrac{1}{2}mp^2+i\nu pq+jq+kp. \tag{2.33}$$

Here j and k are ordinary numbers and have no physical meaning at present; they are to vanish in the final results. Using (2.33) and the commutation relation (2.17), the equations of motion (2.32) for q and p become

$$\begin{aligned} \partial q/\partial t&=-imp-\nu q+ik, \\ \partial p/\partial t&=\nu p-ij, \end{aligned} \tag{2.34}$$

which, for arbitrary times t_1 and t_2, give the solutions

$$\begin{aligned} q(t)&=e^{-\nu(t-t_1)}q_1+i\int_{t_2}^{t}dt'e^{-\nu(t-t')}\{k(t')-mp(t')\}, \\ p(t)&=e^{-\nu(t_2-t)}p_2+i\int_{t}^{t_2}dt'e^{-\nu(t'-t)}j(t') \end{aligned} \tag{2.35}$$

in terms of the notation $q_1=q(t_1)$ and $p_2=p(t_2)$. Hence, using (2.17),

$$[q(t_1),p(t_2)]=ie^{-\nu(t_1-t_2)}. \tag{2.36}$$

It may be noted that $q(t)$ given by (2.35) is not always Hermitian, even though it is so at some particular time. However, this fact is not contradictory with the original definition of q given by the diagonal matrix (2.11)

In the case of $j=k=0$, (2.35) means, for $t_2\geq t\geq t_1$, that

$$\begin{aligned} \langle q_2',t_2|q(t)|q_1',t_1\rangle=&e^{-\nu(t-t_1)}\langle q_2',t_2|q_1|q_1',t_1\rangle \\ &-im\int_{t_1}^{t}dt'e^{-\nu(t-t')}\langle q_2',t_2|p(t')|q_1',t_1\rangle, \end{aligned} \tag{2.37}$$

$$\langle q_1',t_2|p(t)|q_1',t_1\rangle=e^{-\nu(t_2-t)}\langle q_2',t_2|p_2|q_1',t_1\rangle. \tag{2.38}$$

Here the right side of (2.38) is significant for all values of t as long as $t_2\geq t_1$, and it gives an explicit expression for the matrix elements of $p(t)$. Hence it follows that the right side of (2.37) is also significant for all values of t.[17] Now, since $\langle q_2',t_2|q_1|q_1',t_1\rangle$ and $\langle q_2',t_2|p(t)|q_1',t_1\rangle$ tend to Hermitian matrices as $t_2\to t_1$, it follows from (2.37) that $\langle q_1'',t_1|q(t)|q_1',t_1\rangle$ is not Hermitian for $t\neq t_1$. However, this fact simply means that there is no such representation in which $q(t_1)$ and $q(t_1)$ $(t\neq t_1)$ are simultaneously Hermitian.

The abbreviation $\langle q_2'|q_1'\rangle$ will be used hereinafter for $\langle q_2',t_2|q_1',t_1\rangle$, and $\langle q_2'|F(t)|q_1'\rangle$ for $\langle q_2',t_2|F(t)|q_1',t_1\rangle$.

C. Stationary States

The probability-density function of q may tend to a time-independent function, say $\langle q'|0\rangle$, as time elapses [refer to (2.27)]. Then, by (2.29),

$$\langle q_2'|H|0\rangle=0 \quad\text{or}\quad H|0\rangle=0. \tag{2.39}$$

Hence, when H is given by (2.24), (2.39) yields

$$[-\tfrac{1}{2}mp+i\nu q]|0\rangle=0, \tag{2.40}$$

and (2.27) is obtained as the solution.

Another time-independent density function is $\langle 0|q'\rangle$, defined by (2.18) and, as is evident from (2.20), satisfying

$$\langle 0|H=0. \tag{2.41}$$

However, (2.39) and (2.41) are not true for H_q of (2.33), on account of the additional terms jq and kp, and hence also $\langle 0|0\rangle\neq 1$ for $j\neq 0$. In this general case, we shall define $\langle 0|$ by

$$\langle 0|p(t_2)=0, \quad t_2\to+\infty \tag{2.42a}$$

instead of (2.18), and $|0\rangle$ by

$$q(t_1)|0\rangle=0, \quad t_1\to-\infty. \tag{2.42b}$$

D. Expectation Values and Correlation Functions

We now suppose that the values of q are known to be q_4' and q_1' at the times t_4 and t_1 $(t_4>t_1)$, respectively, and we ask for the expectation value $E[q_3q_2]$ of the product q_3q_2 when the times t_3 and t_2 are involved between t_4

[17] It is noticed that, since $\langle q_1'',t_1|q',t\rangle$ generally does not exist for $t_1<t$, $\langle q_1'',t_1|q(t)|q_1',t_1\rangle$ cannot be constructed directly by (2.28) in this range.

and t_1. Then, when $t_3 \geq t_2$,

$$E[q_3q_2] = \int\int dq_3'dq_2'\langle q_4'|q_3'\rangle q_3'\langle q_3'|q_2'\rangle q_2'\langle q_2'|q_1'\rangle$$
$$/\langle q_4'|q_1'\rangle = \langle q_4'|q_3q_2|q_1'\rangle/\langle q_4'|q_1'\rangle,$$
$$t_4 > t_3 \geq t_2 > t_1 \quad (2.43)$$

while, when $t_3 < t_2$, the order of q_3q_2 on the right side is exchanged. Hence, in terms of the notation

$$T[A(t_1)B(t_2)] = A(t_1)B(t_2), \quad t_1 > t_2$$
$$= B(t_2)A(t_1), \quad t_1 < t_2 \quad (2.44)$$

(2.43) is expressed by

$$E[q_3q_2] = \langle q_4'|T[q_3q_2]|q_1'\rangle/\langle q_4'|q_1'\rangle. \quad (2.45)$$

More generally, if F is any functional of q involved between the times t_4 and t_1, the expectation value of F for the same condition as in (2.45) is given by

$$E[F] = \langle q_4'|T[F]|q_1'\rangle/\langle q_4'|q_1'\rangle. \quad (2.46)$$

On the other hand, when the medium is already in the stationary state at the time t_1 and the expectation value is required for all possible values of q_4', then (2.46) is replaced by

$$E[F] = \langle T[F]\rangle \equiv \langle 0|T[F]|0\rangle/\langle 0|0\rangle. \quad (2.47)$$

For instance, the correlation function $g(t_2,t_1)$ in the stationary state ($j=k=0$) is given by

$$g(t_2,t_1) = \langle T[q_2q_1]\rangle = \langle 0|q_2q_1|0\rangle, \quad t_2 \geq t_1 \quad (2.48)$$

which becomes, by using (2.40),

$$m(2i\nu)^{-1}\langle 0|q_2p_1|0\rangle = m(2i\nu)^{-1}\langle 0|[q_2,p_1]|0\rangle \quad (2.49)$$

on account of (2.18). Hence, using (2.36) for the commutator, we find that

$$g(t_2,t_1) = m(2\nu)^{-1}e^{-\nu|t_2-t_1|}, \quad t_2 \gtrless t_1. \quad (2.50)$$

The correlation function in the general case is given by

$$g^{(n)}(t_n,\cdots,t_2,t_1) = \langle T[q_n\cdots q_2q_1]\rangle, \quad n=1,2,\cdots \quad (2.51)$$

and can be evaluated by the successive use of the method used for (2.48). However, there is the more simple method, as follows.

We suppose an infinitesimal variation $\delta H(t)$ of the Hamiltonian $H(t)$ in (2.29). Then the resultant variation of the probability density function, $\delta\langle q_2''|q_1'\rangle$, is given by

$$\delta\langle q_2''|q_1'\rangle = \langle q_2''|\int_{t_1}^{t_2}\delta H(t)dt|q_1'\rangle. \quad (2.52)$$

Hence, for any variable $F=F(t)$ ($t_2 \geq t \geq t_1$),

$$\delta\langle q_2''|F|q_1'\rangle$$
$$= \delta\left[\int\int\langle q_2''|q'\rangle dq'\langle q'|F|q''\rangle dq''\langle q''|q_1'\rangle\right]$$
$$= \langle q_2''|\int_{t_1}^{t_2}dt'T[F\delta H(t')]|q_1'\rangle, \quad (2.53)$$

where $q=q(t)$. The expression (2.53) is true also in the more general case where F is a time-ordered functional $F=T[F]$ of q and/or p involved between t_2 and t_1.

Now, when δH is caused by the variations of j and k in (2.33),

$$\delta H = q\delta j + p\delta k. \quad (2.54)$$

Hence, from (2.53),

$$\{\delta/\delta j(t_1)\}\langle 0|0\rangle = \langle 0|q(t_1)|0\rangle,$$
$$\{\delta/\delta j(t_1)\}\{\delta/\delta j(t_2)\}\langle 0|0\rangle = \langle 0|T[q(t_1)q(t_2)]|0\rangle, \quad (2.55)$$
etc.

Thus, according to (2.51),

$$g^{(n)}(t_n,\cdots,t_2,t_1) = \{\delta/\delta j(t_n)\}\cdots$$
$$\{\delta/\delta j(t_2)\}\{\delta/\delta j(t_1)\}\langle 0|0\rangle|_{j=k=0}, \quad (2.56)$$

and hence $\langle 0|0\rangle$ is found to be the generating function for the correlation functions.

E. Evaluation of $\langle 0|0\rangle$

Using the notation (2.47), we have from (2.55)

$$\{\delta/\delta j(t)\}\langle 0|0\rangle/\langle 0/0\rangle = \langle q(t)\rangle. \quad (2.57)$$

Hence, in the case of the preceding example, we can use (2.35) for $q(t)$, and hence

$$\langle q(t)\rangle = i\int_{t_1}^{t}dt'e^{-\nu(t-t')}\{k(t')-m\langle p(t')\rangle\},$$
$$\langle p(t)\rangle = i\int_{t}^{t_2}dt'e^{-\nu(t'-t)}j(t'), \quad (2.58)$$

where $\langle q(t)\rangle$ is assumed to vanish at $t=t_1$ and $\langle p(t)\rangle$ at $t=t_2$; these boundary conditions correspond to those of (2.42) and are valid for $t_1=-\infty$ and $t_2=+\infty$. Thus, we find from (2.58) that

$$\langle q(t)\rangle = i\int_{t_1}^{t}dt'e^{-\nu(t-t')}k(t') + m(2\nu)^{-1}\left[\int_{t_1}^{t_2}dt'e^{-\nu|t-t'|}j(t')\right.$$
$$\left.-e^{-\nu(t-t')}\int_{t_1}^{t_2}dt'e^{-\nu(t'-t_1)}j(t')\right]. \quad (2.59)$$

Now Eq. (5.57) with (2.59) gives the solution

$$\langle 0|0\rangle=\exp\Big[i\int_{t_1}^{t_2}dt\int_{t_1}^{t}dt' j(t)e^{-\nu(t-t')}k(t')$$
$$+m(4\nu)^{-1}\Big\{\int\int_{t_1}^{t_2}dtdt' j(t)e^{-\nu|t-t'|}j(t')$$
$$-\Big(\int_{t_1}^{t_2}dt' e^{-\nu(t'-t_1)}j(t')\Big)^2\Big\}\Big], \quad (2.60)$$

where the integration constant is chosen so that $\langle 0|0\rangle=1$ for $j=k=0$.

The results (2.60) is obtained on the condition (2.42) for finite values of t_2 and t_1. If $t_1=-\infty$, $t_2=+\infty$, and $k=0$, (2.60) becomes

$$\langle 0|0\rangle=\exp\Big[m(4\nu)^{-1}\int\int_{-\infty}^{\infty}dtdt' j(t)e^{-\nu|t-t'|}j(t')\Big]. \quad (2.61)$$

Hence, (2.56) gives

$$g^{(2n+1)}(t_{2n+1},\cdots,t_2,t_1)=0,$$
$$g^{(2n)}(t_{2n},\cdots,t_2,t_1)=\sum g(t_{2n},t_{2n-1})\cdots g(t_4,t_3)g(t_2,t_1), \quad (2.62)$$

where the summation $\sum$ is over all possible combinations of pairs of $t_{2n},\cdots, t_2, t_1$; and $g(t_2,t_1)$ is the same as given by (2.50). Thus the fluctuation of q is found to be also a perfect Gaussian process.

F. Evaluation of the Characteristic Function

As in quantum mechanics, the probability density function $\langle q_2'|q_1'\rangle$ used so far can be interpreted as the transformation function between the eigenvectors $|q_1'\rangle$ of q_1 and $\langle q_2'|$ of q_2, and this fact is explicitly shown in (2.30). In the same way, if $\langle p_2'|$ is the eigenvector of p_2 having the eigenvalue p_2', we can obtain the transformation function $\langle p_2'|q_1'\rangle$ between $\langle p_2'|$ and $|q_1'\rangle$ as

$$\langle p_2'|q_1'\rangle=\int\langle p_2'|q_2'\rangle dq_2'\langle q_2'|q_1'\rangle, \quad (2.63)$$

which is usually called the "characteristic function." Here

$$\langle p_2'|p_2|q_2'\rangle=p_2'\langle p_2'|q_2'\rangle, \quad (2.64)$$

which, using (2.16), gives a solution

$$\langle p_2'|q_2'\rangle=\exp[-ip_2'q_2'], \quad (2.65a)$$

and the integration constant is chosen so that it agrees with $\langle 0|q_2'\rangle$ defined by (2.18) for $p_2'=0$. The corresponding $\langle q_2'|p_2'\rangle$ will be defined by

$$\langle q_2'|p_2'\rangle=(2\pi)^{-1}\exp[iq_2'p_2'], \quad (2.65b)$$

with the relation

$$\int\langle q_2''|p_2'\rangle dp_2'\langle p_2'|q_2'\rangle=\delta(q_2''-q_2'). \quad (2.66)$$

Hence, for instance, using (2.11), we find that

$$\langle p_2''|q_2|p_2'\rangle=(i\partial/\partial p_2'')\delta(p_2''-p_2'). \quad (2.67)$$

Now, expanding $\langle p_2'|q_1'\rangle$ in a power series of p_2',

$$\langle p_2'|q_1'\rangle=\sum_{n=0}^{\infty}(n!)^{-1}p_2'^n(\partial/\partial p_2'')^n\langle p_2''|q_1'\rangle|_{p_2''=0}, \quad (2.68)$$

which can be expressed, on referring to (2.67) and (2.55), as

$$\sum_{n=0}^{\infty}(n!)^{-1}(-ip_2')^n\langle 0|q_2^n|q_1'\rangle$$
$$=\sum_{n=0}^{\infty}(n!)^{-1}\{-ip_2'\delta/\delta j(t_2)\}^n\langle 0|q_1'\rangle.$$

Hence, we find that

$$\langle p_2'|q_1'\rangle=\langle 0|q_1'\rangle|_{j(t)\to j(t)-ip_2'\delta(t-t_2)}. \quad (2.69)$$

In the same way, we can express $\langle 0|q_1'\rangle$ in terms of $\langle 0|0\rangle$ and thus have

$$\langle p_2'|q_1'\rangle=\langle 0|0\rangle|_{j(t)\to j(t)-ip_2'\delta(t-t_2),\ k(t)\to k(t)-iq_1'\delta(t-t_1)}. \quad (2.70)$$

Now, using the result (2.60) for $\langle 0|0\rangle$ and putting $j=k=0$ in (2.70), we obtain

$$\langle p_2'|q_1'\rangle=\exp[-ip_2'q_1'e^{-\nu(t_2-t_1)}$$
$$-m(4\nu)^{-1}\{1-e^{-2\nu(t_2-t_1)}\}p_2'^2], \quad (2.71)$$

which gives $\langle q_2'|q_1'\rangle$ of (2.26) after the transformation by (2.65).

Thus we have seen that all the statistical characteristics of the system can be derived from $\langle 0|0\rangle=\langle p_2'|q_1'\rangle|_{p_2'=q_1'=0}$.

3. EXTENSION TO THE TOTAL SYSTEM OF WAVE FUNCTION AND MEDIUM

In the preceding sections, we have seen that any system following a Markov process has a mathematical basis similar to that of quantum mechanics, and several results were illustrated by a simple example. In the following, we shall consider the total system of waves and fluctuating media, and the latter will be assumed to be a Markov process. Thus, the (real) wave function $\psi=\{\psi_\alpha\}$, $\alpha=1, 2, \cdots$ is assumed to satisfy a linear equation of the following form:

$$(\partial/\partial t)\psi_\alpha=(a_{\alpha\beta}+b_{\alpha\beta}q)\psi_\beta+\eta_\alpha, \quad (3.1a)$$

or, symbolically,

$$(\partial/\partial t)\psi=(a+bq)\psi+\eta. \tag{3.1b}$$

Here $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are constant matrices with respect to the Greek subscripts, η_α is the external source of wave, and q is a fluctuating quantity of the Markov process. The repeated subscripts are to be summed.

The probability-density function $\langle q'',\psi'',t_2|q',\psi',t_1\rangle$ for this system will be defined so that, when q and ψ have definite values q' and ψ' at the time t_1, the probability that they have values between q'', ψ'' and $q''+dq''$, $\psi''+d\psi''$ at the later time t_2 is given by $dq''d\psi''\langle q'',\psi'',t_2|q',\psi',t_1\rangle$. Here, it is not difficult to show that, when q is a Markov process as assumed, the total system of q and ψ is also a Markov process, because of the deterministic wave equation (3.1) for ψ when q is given. Thus, using the notation

$$\langle q_2'',\psi_2''|q_1',\psi_1'\rangle \tag{3.2}$$

instead of $\langle q_2'',\psi_2'',t_2|q_1',\psi_1',t_1\rangle$, we have, for $t_3\geq t_2\geq t_1$,

$$\langle q_3',\psi_3'|q_1',\psi_1'\rangle=\int\int\langle q_3',\psi_3'|q_2',\psi_2'\rangle \times dq_2'd\psi_2'\langle q_2',\psi_2'|q_1',\psi_1'\rangle,\quad d\psi'=\textstyle\prod_\alpha d\psi_\alpha'. \tag{3.3}$$

Now we can develop the theory in the same way as in the preceding sections. In particular, no change is necessary for the medium part and the corresponding Hamiltonian remains the same. For the wave part, we have from (3.1)

$$\begin{aligned}\langle\partial\psi/\partial t\rangle|_{\psi=\psi',\ q=q'}&=(a+bq')\psi'+\eta,\\ \langle\partial^n\psi/\partial t\rangle|_{\psi=\psi',\ q=q'}&=0,\quad n\geq 2\end{aligned} \tag{3.4}$$

in terms of the notation of (2.8) and (2.15), since, when $\psi(t)'$ and $q(t)'$ are given, $\psi(t+\Delta t)'$ is deterministic to the first order of Δt. Hence, according to (2.22), the Hamiltonian of the wave part, say H_ψ, is found to be given by

$$H_\psi=-i\psi_\alpha^\dagger\{(a_{\alpha\beta}+b_{\alpha\beta}q)\psi_\beta+\eta_\alpha\}-i\eta_\alpha^\dagger\psi_\alpha. \tag{3.5}$$

Here, $\psi_\alpha^\dagger$ is an operator having matrix elements similar to (2.16), and satisfies the commutation relation

$$\begin{aligned}&[\psi_\alpha,\psi_\beta^\dagger]=i\delta_{\alpha\beta},\\ &[\psi_\alpha,\psi_\beta]=[\psi_\alpha^\dagger,\psi_\beta^\dagger]=0;\end{aligned} \tag{3.6}$$

and all the wave variables ψ and $\psi^\dagger$ are to commute with the medium variables q and p. Also, the additional term $-i\eta_\alpha^\dagger\psi_\alpha$ has no physical meaning, and $\eta_\alpha^\dagger$ is an ordinary number to vanish in the final results.

The total Hamiltonian H is given by

$$H=H_\psi+H_q, \tag{3.7}$$

and the equations of motion for q, p, ψ, and $\psi^\dagger$ are obtained according to (2.32). Using H_ψ and H_q given by (3.5) and (2.33) with the replacement $j\rightarrow j_e$, the equations for q and ψ are found to be exactly the same as the one given by (2.34) and the original wave equation (3.1), respectively. On the other hand, the equations for p and $\psi^\dagger$ become

$$\begin{aligned}\partial p/\partial t&=\nu p-ij,\\ -\partial\psi^\dagger/\partial t&=\psi^\dagger(a+bq)+\eta^\dagger,\\ j&=-i\psi^\dagger b\psi+j_e.\end{aligned} \tag{3.8}$$

Here, it is remarked that the equation for p includes a term depending on the wave variables through j. Hence it formally follows that the media affect the waves, and vice versa. Of course, this mutual interaction is of mathematical products and has no physical meaning; it simply means that the matrix elements of the medium variables, e.g., $\langle q_2',\psi_2'|q(t)|q_1',\psi_1'\rangle$, cannot be free from the values of ψ_2' and ψ_1' or from the knowledge of the values of the wave function at the times t_2 and t_1. As a result, it generally follows that the wave variables and the medium variables do not commute at different times, although they do at the same time.

Many times, the representation corresponding to (2.63), i.e., $\langle p_2',\psi_2^{\dagger\prime}|q_1',\psi_1'\rangle$, is more convenient than the original probability-density function (3.2), and we shall define the states $|0\rangle$ and $\langle 0|$, for $t_2\rightarrow+\infty$ and $t_1\rightarrow-\infty$, by

$$\langle 0|q',\psi'\rangle=\langle p_2',\psi_2^{\dagger\prime}|q',\psi'\rangle|_{\psi_2^{\dagger\prime}=p_2'=0}, \tag{3.9a}$$

$$\langle q',\psi'|0\rangle=\langle q',\psi'|q_1',\psi_1'\rangle|_{\psi_1'=q_1'=0}, \tag{3.9b}$$

where $\psi'=\psi(t)'$ and $q'=q(t)'$. Hence, for $t_2\rightarrow+\infty$ and $t_1\rightarrow-\infty$, they satisfy the equations corresponding to (2.42):

$$\langle 0|\psi^\dagger(t_2)=\langle 0|p(t_2)=0,\quad t_2\rightarrow+\infty \tag{3.10a}$$

$$\psi(t_1)|0\rangle=q(t_1)|0\rangle=0,\quad t_1\rightarrow-\infty. \tag{3.10b}$$

When $\eta^\dagger=j_e=0$, Eq. (3.10a) holds for the whole range of t_2 in view of (3.8). Thus

$$\langle 0|q',\psi'\rangle=1,\quad \eta^\dagger=j_e=0. \tag{3.11}$$

A. Expectation Values and Green's Functions

The expectation value of any physical quantity can be treated in the same way as in the preceding sections and, for instance, when F is any functional of $\psi(t)$ and $q(t)$ $(+\infty>t>-\infty)$, its expectation value for the boundary conditions $\psi(-\infty)'=q(-\infty)'=0$ and for all possible values of $\psi(+\infty)'$ and $q(+\infty)'$ is given by the same formula as (2.47).

The Green's functions of the wave and the medium are defined by

$$\begin{aligned}G_{\alpha\beta}(t_1,t_2)&=\{\delta/\delta\eta_\beta(t_2)\}\langle\psi_\alpha(t_1)\rangle|_{\eta=\eta^\dagger=0},\\ G_{\alpha\beta,\gamma\delta}(t_1,t_2;\,t_3,t_4)&=\{\delta/\delta\eta_\gamma(t_3)\}\{\delta/\delta\eta_\delta(t_4)\}\\ &\quad\times\langle T[\psi_\alpha(t_1)\psi_\beta(t_2)]\rangle|_{\eta=\eta^\dagger=0},\\ D(t_1,t_2)&=\{\delta/\delta j_e(t_2)\}\langle q(t_1)\rangle|_{\eta=\eta^\dagger=0},\quad\text{etc.}\end{aligned} \tag{3.12}$$

in terms of the notation (2.47). Here the external sources η, $\eta^\dagger$, and k are to vanish after the differentiation, while j_e will be assumed to take on some fixed value for a given moment. Hence, it follows that the expectation value of any functional of the wave function can be expressed in terms of the Green's functions, provided that the wave is excited only through the external source η and the power-series expansion of the expectation value with respect to η is possible.

On the other hand, for infinitesimal variations $\delta\eta$, $\delta\eta^\dagger$, δk, and δj_e, $\langle 0|T[F]|0\rangle$ undergoes the variation $\delta\langle 0|T[F]|0\rangle$ given by

$$\delta\langle 0|T[F]|0\rangle = \langle 0|\int_{-\infty}^{\infty} dt T[F\delta H(t)]|0\rangle, \quad (3.13)$$

with

$$\delta H = -i(\delta\eta_\alpha{}^\dagger\psi_\alpha + \psi_\alpha{}^\dagger\delta\eta_\alpha) + q\delta j_e + p\delta k, \quad (3.14)$$

as in (2.53). Hence we have

$$\delta\langle T[F]\rangle = \int_{-\infty}^{\infty} dt\{\langle T[F\delta H(t)]\rangle - \langle T[F]\rangle\langle\delta H(t)\rangle\}. \quad (3.15)$$

Thus, from the definition (3.12), we find that, since $\langle\psi\rangle|_{\eta=0}=0$,

$$G_{\alpha\beta}(t_1,t_2) = -i\langle T[\psi_\alpha(t_1)\psi_\beta{}^\dagger(t_2)]\rangle,$$
$$G_{\alpha\beta,\gamma\delta}(t_1,t_2;\, t_3,t_4) = -\langle T[\psi_\alpha(t_1)\psi_\beta(t_2)\psi_\gamma{}^\dagger(t_3)\psi_\delta{}^\dagger(t_4)]\rangle, \text{ etc.} \quad (3.16)$$

On the other hand, since $\langle j\rangle = j_e$ for $\eta^\dagger=0$, we have, on using (3.8) and (2.59),

$$\langle q(t)\rangle = m(2\nu)^{-1}\int_{-\infty}^{\infty} dt' e^{-\nu|t-t'|} j_e(t'). \quad (3.17)$$

Hence, according to (3.12), $D(t_1,t_2)$ coincides with the correlation function $g(t_1,t_2)$ of (2.50).

Schwinger's Green's-function theory[15] can be developed in the same way as in quantum electrodynamics: From the wave equation (3.1),

$$(\partial/\partial t - a)\langle\psi\rangle - b\langle q\psi\rangle = \eta, \quad (3.18)$$

where, using (3.15),

$$\langle q\psi\rangle = (\delta/\delta j_e)\langle\psi\rangle + \langle q\rangle\langle\psi\rangle. \quad (3.19)$$

Hence, according to the definition of (3.12), the single Green's function is found to satisfy

$$[\partial/\partial t - a - b\{\langle q(t)\rangle + \delta/\delta j_e(t)\}]G(t,t') = \delta(t-t'). \quad (3.20)$$

Here the Greek subscripts are suppressed and $\langle q(t)\rangle$ is given by (3.17).

The "mass" operator $\Delta a(t,t')$ and the "vertex" operator $B(t',t,t'')$ are defined by

$$b\{\delta/\delta j_e(t_1)\}G(t_1,t_2) = \int dt \Delta a(t_1,t)G(t,t_2),$$
$$\delta G(t_1,t_2)/\delta\langle q(t)\rangle = \iint dt'dt'' G(t_1,t')B(t',t,t'')G(t'',t_2), \quad (3.21)$$

and it is straightforward to show that

$$\Delta a(t_1,t_2) = \iint dt'dt'' bG(t_1,t')B(t',t'',t_2)g(t'',t_1),$$
$$B(t_1,t,t_2) = b\delta(t_1-t)\delta(t_1-t_2) + \{\delta/\delta\langle q(t)\rangle\}\Delta a(t_1,t_2), \quad (3.22)$$

with $g(t,t')$ given by (2.50). Hence these operators can be evaluated by successive approximation, using the system of Eqs. (3.21) and (3.22).

Also, for the double Green's function in (3.12), we can employ the same methods as in quantum field theory and obtain the results as follows:

$$[\mathcal{H}(t_1)\mathcal{H}(t_2) - I_{12}(t_1,t_2)]G(t_1,t_2;\, t_3,t_4) = \delta(t_1-t_3)\delta(t_2-t_4) + \delta(t_1-t_4)\delta(t_2-t_3). \quad (3.23)$$

Here, the Greek subscripts are included in the coordinates t_i ($i=1, 2, 3, 4$);

$$\mathcal{H}(t_1) = \partial/\partial t_1 - a_1 - \Delta a_1, \quad \mathcal{H}(t_2) = \partial/\partial t_2 - a_2 - \Delta a_2 \quad (3.24)$$

are operators only with respect to the coordinates t_1 and t_2, respectively; and the interaction operator $I_{12}(t_1,t_2)$ is characterized by

$$I_{12}(t_1,t_2)G(t_1,t_2;\, t_3,t_4) = b_1 B_2 g(t_1,t_2)G(t_1 t_2;\, t_3,t_4) + b_1\int dt_1' G(t_1,t_1')\{\delta/\delta j_e(t_1')\} \times\{I_{12}(t_1',t_2)G(t_1',t_2;\, t_3,t_4)\}. \quad (3.25)$$

B. Isomorphic Transformation

When the temporal change of the medium is sufficiently fast compared with that of the waves, so that the latter do not appreciably change within the time interval ν^{-1}, then an adiabatic approximation may be used for the medium variables, in the sense that the medium is always in a stationary state for the "excitation" by the waves.

In this case, it is convenient to introduce the isomorphic operators $\mathbf{q}$, $\mathbf{p}$, $\boldsymbol{\psi}$, and $\boldsymbol{\psi}^\dagger$ defined by

$$\mathbf{q} = UqU^{-1}, \quad \mathbf{p} = UpU^{-1}, \quad \boldsymbol{\psi} = U\psi U^{-1}, \quad \boldsymbol{\psi}^\dagger = U\psi^\dagger U^{-1}, \quad (3.26)$$

with

$$U = \exp[-m(4\nu)^{-1}(p+i\nu^{-1}j)^2]\exp[-\nu^{-1}jq], \quad (3.27)$$

and express the physical operators in terms of them.[18] Here, the new operators have the same algebraic relations as those for the old operators, and hence their commutation relations do not change. Thus, since $\mathbf{U} = UUU^{-1} = U$ ($\mathbf{U}$ being the same function of $\mathbf{q}$, $\mathbf{p}$, and

[18] In the following, boldface letters will be used for all isomorphic operators obtained with U.

j as U is of q, p, and j), we find that

$$q=\boldsymbol{U}^{-1}\boldsymbol{q}\boldsymbol{U}=\boldsymbol{q}-im(2\nu)^{-1}\boldsymbol{p}+m\nu^{-2}\boldsymbol{j}\,,$$
$$p=\boldsymbol{p}+i\nu^{-1}\boldsymbol{j}\,,\quad j=\boldsymbol{j}\,, \tag{3.28}$$
$$\psi=W\boldsymbol{\psi}\,,\qquad \psi^{\dagger}=\boldsymbol{\psi}^{\dagger}W^{-1}\,,$$

with

$$W=\exp[-m(4\nu)^{-1}\{2\boldsymbol{p}+2i\nu^{-1}\boldsymbol{j})(i\nu^{-1}b)-\nu^{-2}b^2\}]\times\exp[-\nu^{-1}b\boldsymbol{q}]. \tag{3.29}$$

Here, in the derivation of (3.28), the following lemmas are used:

$$\boldsymbol{\psi}\boldsymbol{j}^n=(\boldsymbol{j}+b)^n\boldsymbol{\psi}\,,\qquad \boldsymbol{j}^n\boldsymbol{\psi}^{\dagger}=\boldsymbol{\psi}^{\dagger}(\boldsymbol{j}+b)^n\,,\quad n=1,2,\cdots$$
$$\boldsymbol{\psi}f(\boldsymbol{j})=f(\boldsymbol{j}+b)\boldsymbol{\psi}\,,\quad f(\boldsymbol{j})\boldsymbol{\psi}^{\dagger}=\boldsymbol{\psi}^{\dagger}f(\boldsymbol{j}+b)\,. \tag{3.30}$$

In the same way, for the Hamiltonian H of (3.7), we have, using (3.5) and (2.33),

$$H=-i\boldsymbol{\psi}^{\dagger}(\boldsymbol{a}\boldsymbol{\psi}+\boldsymbol{\eta})+m(2\nu^2)^{-1}\boldsymbol{j}^2+i\nu\boldsymbol{p}\boldsymbol{q}\,. \tag{3.31}$$

Here, we assumed $j_e=\eta^{\dagger}=k=0$, and

$$\boldsymbol{\eta}=W^{-1}\eta\,,$$
$$\boldsymbol{a}=W^{-1}aW=a+\nu^{-1}[b,a]\boldsymbol{q}+m(4\nu)^{-1}\times\{2i\nu^{-1}[b,a](\boldsymbol{p}+2i\nu^{-1}\boldsymbol{j})-\nu^{-2}[b^2,a]\}+\cdots. \tag{3.32}$$

Hence, using the expression (3.31) for H, the equations of motion for the new operators are found to be

$$\partial\boldsymbol{q}/\partial t=[\boldsymbol{q},H]=-\nu\boldsymbol{q}-i\boldsymbol{\psi}^{\dagger}[\boldsymbol{q},\boldsymbol{a}]\boldsymbol{\psi}-i\boldsymbol{\psi}^{\dagger}[\boldsymbol{q},\boldsymbol{\eta}]\,,$$
$$\partial\boldsymbol{p}/\partial t=\nu\boldsymbol{p}-i\boldsymbol{\psi}^{\dagger}[\boldsymbol{p},\boldsymbol{a}]\boldsymbol{\psi}-i\boldsymbol{\psi}^{\dagger}[\boldsymbol{p},\boldsymbol{\eta}]. \tag{3.33}$$

For the wave operator $\boldsymbol{\psi}$, we notice that the second term on the right side of (3.31) can be expressed also by

$$-m(2\nu^2)^{-1}\{\boldsymbol{\psi}^{\dagger}b(\boldsymbol{\psi}^{\dagger}b\boldsymbol{\psi})\boldsymbol{\psi}\}-i\boldsymbol{\psi}^{\dagger}m(2\nu^2)^{-1}b^2\boldsymbol{\psi}. \tag{3.34}$$

Hence,

$$\partial\boldsymbol{\psi}/\partial t=(\boldsymbol{a}+\Delta a+m\nu^{-2}b\boldsymbol{j})\boldsymbol{\psi}+\boldsymbol{\eta}-i\boldsymbol{\psi}^{\dagger}[\boldsymbol{\psi},\boldsymbol{a}]\boldsymbol{\psi}-i\boldsymbol{\psi}^{\dagger}[\boldsymbol{\psi},\boldsymbol{\eta}]\,, \tag{3.35}$$

$$\partial\boldsymbol{\psi}^{\dagger}/\partial t=-\boldsymbol{\psi}^{\dagger}(\boldsymbol{a}+\Delta a+m\nu^{-2}b\boldsymbol{j})-i\boldsymbol{\psi}^{\dagger}[\boldsymbol{\psi}^{\dagger},\boldsymbol{a}]\boldsymbol{\psi}-i\boldsymbol{\psi}^{\dagger}[\boldsymbol{\psi}^{\dagger},\boldsymbol{\eta}]\,, \tag{3.36}$$

with

$$\Delta a=m(2\nu^2)^{-1}b^2. \tag{3.37}$$

Here, referring to (3.32), Δa can be regarded as a correction to a and, indeed, agrees with that given by (3.22) to first order. Also we notice that the term $m\nu^{-2}b\boldsymbol{j}$ takes the same form as the potential of the field of boson particles, which interact only through a two-body force with the "charge" $\boldsymbol{j}$. On the other hand, the last two terms in (3.35) are of the higher-order "interaction," as may be seen from the expansion of $\boldsymbol{a}$ on the right side of (3.32); it is of the order of the magnitude of $a\nu^{-1}$, which becomes very small when the temporal change of the medium is sufficiently large compared with that of the wave function, as is assumed. In particular, they vanish exactly when $[b,a]=0$, or when $\boldsymbol{j}$ is a constant of the motion. The situation is the same also for the last two terms in (3.33).

On the other hand, according to (3.31), the neglect of these higher-order terms is equivalent to the approximation $\boldsymbol{a}\simeq a$ and $\boldsymbol{\eta}\simeq\eta$, and H is then divided into the two independent parts, for the wave and for the medium.

The original probability-density function is given by $\langle q_2',\psi_2'|q_1',\psi_1'\rangle$ and, as has already been noticed, it can be regarded as the representation of the eigenvector $|q_1',\psi_1'\rangle$ (of the operators q_1 and ψ_1 for the eigenvalues q_1' and ψ_1') by the eigenvector $\langle q_2',\psi_2'|$ (of the operators q_2 and ψ_2 for the eigenvalues q_2' and ψ_2'). On the other hand, the solution of the Fokker-Planck equation (2.6) for the Hamiltonian (3.31) may be expressed by $\langle\boldsymbol{q}_2',\boldsymbol{\psi}_2'|\boldsymbol{q}_1',\boldsymbol{\psi}_1'\rangle$ in terms of the new operators in the same way as for the original operators. Here, using (3.26),

$$qU^{-1}|\boldsymbol{q}'\rangle=U^{-1}\boldsymbol{q}|\boldsymbol{q}'\rangle=\boldsymbol{q}'U^{-1}|\boldsymbol{q}'\rangle\,, \tag{3.38}$$

and hence it follows that $U^{-1}|\boldsymbol{q}'\rangle$ is the eigenvector of q of the eigenvalue $\boldsymbol{q}'$, or

$$U^{-1}|\boldsymbol{q}'\rangle=|q'\rangle|_{q'=\boldsymbol{q}'}. \tag{3.39}$$

In the same way,

$$\langle\boldsymbol{q}'|U=\langle q'||_{q'=\boldsymbol{q}'}. \tag{3.40}$$

Thus,

$$\langle q_2',\psi_2'|q_1',\psi_1'\rangle=\langle\boldsymbol{q}_2',\boldsymbol{\psi}_2'|U_2U_1^{-1}|\boldsymbol{q}_1',\boldsymbol{\psi}_1'\rangle\,, \tag{3.41}$$

where $\boldsymbol{q}_2'=q_2'$, $\boldsymbol{\psi}_2'=\psi_2'$, $\boldsymbol{q}_1'=q_1'$, and $\boldsymbol{\psi}_1'=\psi_1'$.

The two stationary states $|0\rangle$ and $\langle 0|$ were originally defined by (2.40) and (2.18b) with respect to the medium and, using (3.28), the equations for these states are found to be expressed exactly by

$$\boldsymbol{q}|0\rangle=\langle 0|\boldsymbol{p}=0\,,$$
$$\boldsymbol{\psi}|0\rangle=\langle 0|\boldsymbol{\psi}^{\dagger}=0\,, \tag{3.42}$$

and, of course,

$$H|0\rangle=\langle 0|H=0 \tag{3.43}$$

for vanishing external sources $\eta=\eta^{\dagger}=k=j_e=0$. Hence, these stationary states are found to be the eigenvectors of the new operators, and the evaluation of expectation values of physical quantities may become easier in view of (2.47).

In the case of the approximation $\boldsymbol{a}\simeq a$ and $\boldsymbol{\eta}\simeq\eta$, the Hamiltonian is divided into the two independent parts for the wave and for the medium, as has already been noted, and hence we may put

$$\langle\boldsymbol{p}_2',\boldsymbol{\psi}_2^{\dagger\prime}|\boldsymbol{q}_1',\boldsymbol{\psi}_1'\rangle=\langle\boldsymbol{p}_2'|\boldsymbol{q}_1'\rangle\langle\boldsymbol{\psi}_2^{\dagger\prime}|\boldsymbol{\psi}_1'\rangle. \tag{3.44}$$

Here, the solution of (3.33) for this approximation becomes

$$\boldsymbol{q}_2=e^{-\nu t_{21}}\boldsymbol{q}_1\,,\quad \boldsymbol{p}_2=e^{\nu t_{21}}\boldsymbol{p}_1\,,\quad t_{21}=t_2-t_1\,, \tag{3.45}$$

and hence

$$i\partial/\partial\boldsymbol{p}_2'\langle\boldsymbol{p}_2'|\boldsymbol{q}_1'\rangle=\langle\boldsymbol{p}_2'|\boldsymbol{q}_2|\boldsymbol{q}_1'\rangle=\langle\boldsymbol{p}_2'|e^{-\nu t_{21}}\boldsymbol{q}_1|\boldsymbol{q}_1'\rangle=e^{-\nu t_{21}}\boldsymbol{q}_1'\langle\boldsymbol{p}_2'|\boldsymbol{q}_1'\rangle\,, \tag{3.46}$$

which gives a solution

$$\langle\boldsymbol{p}_2'|\boldsymbol{q}_1'\rangle=\exp[-i\boldsymbol{p}_2'e^{-\nu t_{21}}\boldsymbol{q}_1']. \tag{3.47}$$

The Hamiltonian of the Fokker-Planck equation for $\langle\psi_2^{\dagger\prime}|\psi_1'\rangle$ is given by (3.31) with the omission of the medium part $i\nu pq$.

In order to obtain the probability-density function, we need to evaluate $\langle q_2',\psi_2'|U_2$ and $U_1^{-1}|q_1',\psi_1'\rangle$, in view of (3.41). Now, using (3.27) and the lemma (3.30), we find that

$$\begin{aligned}\langle q'|U|p'\rangle &= \langle q'|\exp[-\nu^{-1}jq]\exp[-m(4\nu)^{-1}(p+2i\nu^{-1}j)^2]|p'\rangle\\ &=(2\pi)^{-1}\exp[iq'(p'+i\nu^{-1}j)-m(4\nu)^{-1}(p'+2i\nu^{-1}j)^2],\end{aligned}\tag{3.48}$$

where the definition (2.65b) is taken into account.

In the same way,

$$\langle p'|U^{-1}|q'\rangle=\exp[-iq'(p'+i\nu^{-1}j)+m(4\nu)^{-1}(p'+2i\nu^{-1}j)^2].\tag{3.49}$$

Hence, in the case of the approximation (3.44), the probability-density function is given by

$$\begin{aligned}\langle q_2',\psi_2'|q_1',\psi_1'\rangle=&(2\pi)^{-1}\langle\psi_2'|\int_{-\infty}^{\infty}dp_2'\\ &\times\exp[iq_2'(p_2'+i\nu^{-1}j_2)-m(4\nu)^{-1}(p_2'+2i\nu^{-1}j_2)^2]\\ &\times\exp[-iq_1'(p_1'+i\nu^{-1}j_1)\\ &\qquad+m(4\nu)^{-1}(p_1'+2i\nu^{-1}j_1)^2]|\psi_1'\rangle,\end{aligned}$$

with $p_1'=e^{-\nu t_{21}}p_2'$, $\psi_2'=\psi_2'$, and $\psi_1'=\psi_1'$. Here, for $\nu t_{21}\gg1$, $p_1'\sim0$, and hence

$$\begin{aligned}\langle q_2',\psi_2'|q_1',\psi_1'\rangle=&(\nu/\pi m)^{1/2}e^{-(\nu/m)q_2'^2}\\ &\times\langle\psi_2'|e^{\nu^{-1}j_2q_2'}e^{\nu^{-1}j_1q_1'-m\nu^{-3}j_1^2}|\psi_1'\rangle,\\ &\qquad\nu t_{21}\gg1.\end{aligned}\tag{3.50}$$

On the other hand, using the lemma (3.30),

$$\psi e^{\alpha j}|\psi'\rangle=e^{\alpha(j+b)}\psi|\psi'\rangle=e^{\alpha b}\psi'e^{\alpha j}|\psi'\rangle,$$

and hence $e^{\alpha j}|\psi'\rangle$ is found to be the eigenvector of ψ with the eigenvalue $e^{\alpha b}\psi'$. In this way, we find that

$$\begin{aligned}e^{\alpha j}|\psi'\rangle&=|e^{\alpha b}\psi'\rangle, & \langle\psi^{\dagger\prime}|e^{\alpha j}&=\langle\psi^{\dagger\prime}e^{\alpha b}|,\\ e^{\alpha j}|\psi^{\dagger\prime}\rangle&=|\psi^{\dagger\prime}e^{-\alpha b}\rangle, & \langle\psi'|e^{\alpha j}&=\langle e^{-\alpha b}\psi'|.\end{aligned}\tag{3.51}$$

Thus, using the integral representation

$$e^{-m\nu^{-3}j^2}=(\nu/\pi m)^{1/2}\int_{-\infty}^{\infty}d\lambda e^{i2\nu^{-1}j\lambda-(\nu/m)\lambda^2},\tag{3.52}$$

(3.50) can be expressed by

$$\begin{aligned}&\langle q_2',\psi_2'|q_1',\psi_1'\rangle\\ &\quad=(\nu/\pi m)\int_{-\infty}^{\infty}d\lambda e^{-(\nu/m)(q_2'^2+\lambda^2)}\langle\psi_2''|\psi_1''\rangle,\end{aligned}\tag{3.53}$$

where

$$\psi_2''=e^{-\nu^{-1}bq_2'}\psi_2',\quad\psi_1''=e^{\nu^{-1}b(q_1'+i2\lambda)}\psi_1',\quad\nu t_{21}\gg1.\tag{3.54}$$

In the integral (3.53), the effective range of λ is $|\lambda|\lesssim(m/\nu)^{1/2}$, and hence, when the condition

$$|2b(m/\nu^3)^{1/2}|\ll1\tag{3.55}$$

holds, (3.53) becomes

$$\langle q_2',\psi_2'|q_1',\psi_1'\rangle\simeq(\nu/\pi m)^{1/2}e^{-(\nu/m)q_2'^2}\langle\psi_2''|\psi_1''\rangle,\tag{3.56}$$

where ψ_2'' and ψ_1'' are given by (3.54) with $\lambda=0$. Here, since the right side of (3.56) is appreciable only for $|q_2'|\lesssim(m/\nu)^{1/2}$, we can put $\psi_2''\simeq\psi_2'$, and further, if (3.56) is averaged over the stationary distribution of q_1', we can put $\psi_1''\simeq\psi_1'$.

4. CASE OF WAVES IN THREE-DIMENSIONAL SPACE

The waves and the media we have considered so far in the preceding sections are functions only of the time. However, it is quite straightforward to extend the methods to the ordinary waves and media in three-dimensional space, and we shall illustrate the results by an example corresponding to (2.23).

The medium variable q is now a function of space coordinates x, say $q(x)$, and changes stochastically with the time; the temporal change is assumed to be a Markov process and also to be spatially homogeneous. The probability-density function $\langle q'',t_2|q',t_1\rangle$ is defined in the same way as in Sec. 2, except that q'' and q' now represent values of $q(x)$ at all points in space.

Now we assume, using a notation similar to (2.15), that

$$\begin{aligned}\langle\partial q(x)/\partial t\rangle&=\mathcal{K}q(x),\\ \langle\partial q(x_1)\partial q(x_2)/\partial t\rangle&=m(x_1-x_2),\\ \langle\partial q(x_1)\partial q(x_2)\cdots\partial q(x_n)/\partial t\rangle&=0,\quad n\geq3\end{aligned}\tag{4.1}$$

for arbitrary points x_i ($i=1, 2, \cdots$). Here, $\mathcal{K}$ is an operator operating on $q(x)$, and, for instance,

$$\mathcal{K}=-\nu+\alpha\nabla^2,\tag{4.2}$$

where ν and α are positive constants and ∇^2 is the Laplacian. Then, the Hamiltonian H_q for the medium which corresponds to (2.24) is found to be given by

$$\begin{aligned}H_q=&-\tfrac{1}{2}\int\int(dx)(dx')p(x)m(x-x')p(x')\\ &-i\int(dx)p(x)\mathcal{K}q(x)+\int(dx)j(x,t)q(x).\end{aligned}\tag{4.3}$$

Here, (dx) stands for the space element, and $q(x)$ and $p(x)$ are now operators having the matrix representations

$$\begin{aligned}\langle q''|q(x)|q'\rangle&=q(x)''\delta(q''-q'),\\ \langle q''|p(x)|q'\rangle&=-i\{\delta/\delta q(x)''\}\delta(q''-q'),\end{aligned}\tag{4.4}$$

and satisfy the commutation relations

$$[q(x),p(x')]=i\delta(x-x'),\quad [q(x),q(x')]=[p(x),p(x')]=0, \tag{4.5}$$

$\delta(x)$ being the ordinary Dirac δ function in three-dimensional space. Also, $j(x,t)$ is an external source introduced for later convenience, and is to vanish in the final results.

Indeed, on referring to (2.14), we have, for instance,

$$\begin{aligned}\langle\partial q(x_1)\partial q(x_2)/\partial t\rangle&=\lim_{\Delta t\to 0}\Delta t^{-1}\int(dq'')\{q(x_1)''-q(x_1)'\}\\&\quad\times\{q(x_2)''-q(x_2)'\}\langle q'',t+\Delta t|q',t\rangle\\&=\int(dq'')\langle q'',t|[q(x_1),[q(x_2),H_q]]|q',t\rangle\\&=m(x_1-x_2),\end{aligned} \tag{4.6}$$

where (dq') represents the product of the elements $dq(x)'$ at all points in space.

A. Stationary States

For the stationary state $|0\rangle$, we obtain, from $H_q|0\rangle=0$,

$$\left[(\tfrac{1}{2}i)\int(dx')m(x-x')p(x')-\mathcal{H}q(x)\right]|0\rangle=0,\quad j=0,$$

or

$$\left[(\tfrac{1}{2}i)\int(dx')m_1(x-x')p(x')+q(x)\right]|0\rangle=0. \tag{4.7}$$

Here

$$m_n(x)=\int(dx')w(x-x')m_{n-1}(x'),\quad n=1,2,\cdots \quad m_0(x)=m(x), \tag{4.8}$$

and $w(x)$ is the solution of

$$\mathcal{H}w(x)=-\delta(x) \tag{4.9}$$

or, in the case of (4.2),

$$w(x)=(4\pi\alpha|x|)^{-1}e^{-k|x|},\quad k=(\nu/\alpha)^{1/2}. \tag{4.10}$$

Since the fluctuation is assumed to be spatially homogeneous, we have $w(x)=w(-x)$ and $m_n(x)=m_n(-x)$.

The characteristic function $\langle p'|0\rangle$ is thus found to be

$$\langle p'|0\rangle=\exp\left[-\tfrac{1}{4}\int\int(dx)(dx')p(x)'m_1(x-x')p(x')'\right], \tag{4.11}$$

which corresponds to (2.71) for $t_2-t_1\to+\infty$.

Another stationary state is defined in the same way as in (2.18), and satisfies

$$\langle 0|p=\langle 0|H_q=0. \tag{4.12}$$

B. Equations of Motion and Solutions

The time-dependent representations of $q(x)$ and $p(x)$ are defined by (2.28), and will be denoted by $q(x,t)$ and $p(x,t)$. Also, $\langle q_2'|q_1'\rangle$ will denote $\langle q_2',t_2|q_1',t_1\rangle$, as in the preceding sections.

The equations of motion of these operators are given by (2.32) with $H=H_q$ of (4.3) and, on using the commutation relations of (4.5), are found to be

$$\begin{aligned}(\partial/\partial t)q(x,t)&=-i\int(dx')m(x-x')p(x',t)+\mathcal{H}q(x,t),\\(\partial/\partial t)p(x,t)&=-p(x,t)\overleftarrow{\mathcal{H}}-ij(x,t).\end{aligned} \tag{4.13}$$

The formal solutions of (4.13) can be obtained in terms of the function $S(x,t)$ defined by

$$(\partial/\partial t)S(x,t)=\mathcal{H}S(x,t),\quad t\geq 0,\qquad S(x,t)|_{t\to+0}=\delta(x), \tag{4.14}$$

and are expressed by

$$\begin{aligned}q(x_2,t_2)&=\int(dx_1)S(x_2-x_1,t_2-t_1)q(x_1,t_1)-i\int_{t_1}^{t_2}dt\\&\quad\times\int\int(dx)(dx')S(x_2-x,t_2-t)m(x-x')p(x',t),\quad t_2\geq t_1\end{aligned} \tag{4.15}$$

$$\begin{aligned}p(x_2,t_2)&=\int(dx_3)p(x_3,t_3)S(x_3-x_2,t_3-t_2)-i\int_{t_3}^{t_2}dt'\\&\quad\times\int(dx')j(x',t')S(x'-x_2,t'-t_2),\quad t_3\geq t_2.\end{aligned} \tag{4.15'}$$

Here, when $\mathcal{H}$ is given by (4.2),

$$S(x,t)=(k/2)^3(\pi\nu t)^{-3/2}\exp[-k^2x^2/(4\nu t)-\nu t],\quad \alpha=\nu/k^2,\quad t>0. \tag{4.16}$$

Hence, we find that

$$\begin{aligned}&[q(x_2,t_2),p(x_1,t_1)]=iS(x_2-x_1,t_2-t_1),\\&[p(x_2,t_2),p(x_1,t_1)]=0,\\&[q(x_2,t_2),q(x_1,t_1)]\\&\quad=-\int_{t_1}^{t_2}dt\int\int(dx)(dx')S(x_2-x,t_2-t)m(x-x')\\&\qquad\times S(x_1-x',t_1-t).\end{aligned} \tag{4.17}$$

C. Expectation Values and Correlation Functions

If F is a functional of q and p as functions of time between t_2 and t_1, the expectation value of F is given by the same equation as (2.46) or (2.47), and hence the correlation function of q in the stationary state, say

$g(x,t)$, is given by

$$g(x_2-x_1,\ t_2-t_1)=\langle T[q(x_2,t_2)q(x_1,t_1)]\rangle. \quad (4.18)$$

Here, using (4.7), (4.12), and the commutation relations (4.17), we find that

$$g(x_2-x_1,\ t_2-t_1)$$
$$=\tfrac{1}{2}\int (dx')S(x_2-x',\ |t_2-t_1|)m_1(x'-x_1), \quad (4.19)$$

which tends to $\frac{1}{2}m_1(x_2-x_1)$ as $t_2\rightarrow t_1$.

In the special case where $m(x)=m\delta(x)$, (4.19) yields

$$g(x,t)=\tfrac{1}{2}m\int_t^\infty dt'S(x,t') \quad (4.20)$$

by use of the relation

$$w(x)=\int_0^\infty dt\, S(x,t).$$

More generally, the generating function for the moments of q of any order can be obtained in the same way as in Sec. 2 E, and is found to be

$$\langle 0|0\rangle=\exp\left[\tfrac{1}{2}\int\int_{-\infty}^{\infty}(dx)dt\int\int_{-\infty}^{\infty}(dx')dt \times j(x,t)g(x-x',\ t-t')j(x',t')\right]. \quad (4.21)$$

The expectation values are obtained from (4.21) by

$$E[q(x_1,t_1)q(x_2,y_2)\cdots q(x_n,t_n)]=\{\delta/\delta j(x_1,t_1)\} \times\{\delta/\delta j(x_2,t_2)\}\cdots\{\delta/\delta j(x_n,t_n)\}\langle 0|0\rangle|_{j=0}. \quad (4.22)$$

Thus, we find that the process is also perfectly Gaussian.

D. Wave-Medium System

We assume a linear wave equation of the same form as (3.1):

$$\partial\psi/\partial t=(a+bq)\psi+\eta. \quad (4.23)$$

Here a is generally a function of the spatial differential operators $\nabla_i=\partial/\partial x_i$ $(i=1,2,3)$ and hence, for instance,

$$a=a_i\nabla_i+a_0. \quad (4.24)$$

Also, ψ may consist of several components ψ_α $(\alpha=1,2,\cdots)$, and then a and b are also (real) matrices $a_{\alpha\beta}$ and $b_{\alpha\beta}$ operating on ψ_β.

The Hamiltonian H_ψ for the wave can be formulated as in Sec. 2 A and, on introducing an external source $\eta^\dagger$, it is given by

$$H_\psi=-i\int (dx)\{\psi^\dagger[(a+bq)\psi+\eta]+\eta^\dagger\psi\}, \quad (4.25)$$

where $\psi^\dagger$ is the operator satisfying the commutation relation

$$[\psi_\alpha(x),\psi_\beta{}^\dagger(x')]=i\delta(x-x')\delta_{\alpha\beta}. \quad (4.26)$$

For the total system of the waves and the medium, the Hamiltonian $H=H_q+H_\psi$, and we find the same equations of motion for q and ψ as in (4.13) and (4.23), respectively, while for p and $\psi^\dagger$, with the replacement $j\rightarrow j_e$, we find

$$\partial p/\partial t=-p\mathcal{H}-ij,$$
$$-\partial\psi^\dagger/\partial t=\psi^\dagger(a+bq)+\eta^\dagger, \quad (4.27)$$
$$j=-i\psi^\dagger b\psi+j_e.$$

Also j is a constant of the motion when $\eta=\eta^\dagger=j_e=0$ and $[a,b]=0$, and anti-Hermitian when $\mathrm{Tr}[b]=0$ (in the representation in which ψ and $\psi^\dagger$ are Hermitian).

E. Adiabatic Approximation

When the temporal change of the waves is sufficiently small compared with that of the medium, we may employ the adiabatic approximation as in Sec. 3 B. Thus, referring to the definition (4.8) for $m_n(x)$, we introduce U defined by

$$U=\exp\left[-\tfrac{1}{4}\int\int (dx')(dx'')\{p(x')m_1(x'-x'')p(x'') +2ip(x')m_2(x'-x'')j(x'')-j(x')m_3(x'-x'')j(x'')\}\right] \times\exp\left[-\int\int (dx')(dx'')j(x')w(x'-x'')q(x'')\right]. \quad (4.28)$$

Then, in terms of the notation $\boldsymbol{F}=UFU^{-1}$, we find that

$$q(x)=\boldsymbol{U}^{-1}\boldsymbol{q}(x)\boldsymbol{U}=\boldsymbol{q}(x)-(\tfrac{1}{2}i)\int (dx')m_1(x-x')\boldsymbol{p}(x') +\int (dx')m_2(x-x')\boldsymbol{j}(x'),$$

$$p(x)=\boldsymbol{p}(x)+i\int (dx')w(x-x')\boldsymbol{j}(x'),\quad j(x)=\boldsymbol{j}(x),$$

$$\psi(x)=W(x)\boldsymbol{\psi}(x),\quad \psi^\dagger(x)=\boldsymbol{\psi}^\dagger(x)W^{-1}(x), \quad (4.29)$$

$$W(x)=\exp\left[b\int (dx')\{(-\tfrac{1}{2}i)m_2(x-x')\boldsymbol{p}(x') +m_3(x-x')\boldsymbol{j}(x')\}+(\tfrac{1}{4})m_3(0)b^2\right] \times\exp\left[-b\int (dx')w(x-x')\boldsymbol{q}(x')\right],$$

and, for $\eta^\dagger=0$,

$$H=-i\int(dx)[\boldsymbol{\psi}^\dagger(x)\{\boldsymbol{a}(x)\boldsymbol{\psi}(x)+\boldsymbol{\eta}(x)\}+\boldsymbol{p}(x)\mathcal{K}\boldsymbol{q}(x)]$$
$$+\tfrac{1}{2}\int\int(dx')(dx'')\boldsymbol{j}(x')m_2(x'-x'')\boldsymbol{j}(x''). \quad (4.30)$$

Here

$$\boldsymbol{a}(x)=W^{-1}(x)aW(x)$$
$$=a+[b,a]\int(dx')[w(x-x')\boldsymbol{q}(x')$$
$$+(\tfrac{1}{2}i)m_2(x-x')\boldsymbol{p}(x')-m_3(x-x')\boldsymbol{j}(x')]$$
$$-(\tfrac{1}{4})m_3(0)[b^2,a]+\cdots,$$
$$\boldsymbol{\eta}(x)=W^{-1}(x)\eta(x), \quad (4.31)$$

and the last term in (4.30) can also be expressed by

$$-\tfrac{1}{2}\int\int(dx')(dx'')m_2(x'-x'')$$
$$\times\{\boldsymbol{\psi}^\dagger(x')b(\boldsymbol{\psi}^\dagger(x'')b\boldsymbol{\psi}(x''))\boldsymbol{\psi}(x')\}$$
$$-i\int(dx)\boldsymbol{\psi}^\dagger(x)\Delta a\boldsymbol{\psi}(x), \quad (4.32)$$

with

$$\Delta a=\tfrac{1}{2}m_2(0)b^2. \quad (4.33)$$

The equations of motion for the boldface operators are immediately obtained by the use of (4.30) and, for instance,

$$(\partial/\partial t)\boldsymbol{\psi}(x)=(\mathbf{a}(x)+\Delta a+bV(x))\boldsymbol{\psi}(x)+\boldsymbol{\eta}(x)$$
$$-i\int(dx')\boldsymbol{\psi}^\dagger(x')\{[\boldsymbol{\psi}(x),\boldsymbol{a}(x')]\boldsymbol{\psi}(x')$$
$$+[\boldsymbol{\psi}(x),\boldsymbol{\eta}(x')]\}, \quad (4.34)$$
$$(\partial/\partial t)\boldsymbol{q}(x)=\mathcal{K}\boldsymbol{q}(x)-i\int(dx')\boldsymbol{\psi}^\dagger(x')$$
$$\times\{[\boldsymbol{q}(x),\boldsymbol{a}(x')]\boldsymbol{\psi}(x')+[\boldsymbol{q}(x),\boldsymbol{\eta}(x')]\},$$

where

$$V(x)=\int(dx')m_2(x-x')\boldsymbol{j}(x'), \quad (4.35)$$

and the time coordinate t is suppressed for all the variables.

As is discussed in Sec. 3, the integral terms in (4.34) are higher-order terms, and can be neglected when the temporal change of the medium is very large compared with that of the waves. Hence, in this case, H is divided into two independent parts for the medium and the wave, and the latter interact only through the two-body "potential" $m_2(x_1-x_2)$ with the "charge" $\boldsymbol{j}(x)$.

When $\mathcal{K}$ is given by (4.2) and

$$m(x)=me^{-|x|/l}, \quad (4.36)$$

$m_2(x)$ is found to be

$$m_2(x)=mv^{-2}\{(kl)^{-2}-1\}^{-2}[e^{-|x|/l}+(kl)^{-1}e^{-k|x|}$$
$$-4\{1-(kl)^2\}^{-1}(l/|x|)\{e^{-k|x|}-e^{-|x|/l}\}]. \quad (4.37)$$

Hence, when l is sufficiently large so that $kl\gg1$,

$$m_2(x)\simeq mv^{-2}e^{-|x|/l}, \quad kl\gg1 \quad (4.38)$$

while, by replacing $m\to m(8\pi l^3)^{-1}$ and $l\to0$, we can obtain the result in the case of $m(x)=m\delta(x)$:

$$m_2(x)=(8\pi)^{-1}k^3mv^{-2}e^{-k|x|}. \quad (4.39)$$

The original probability-density function and the equations for the stationary states $|0\rangle$ and $\langle0|$ are expressed by the same equations as (3.42). Also, when the condition corresponding to (3.55) holds, i.e.,

$$|2bm_3^{1/2}|\ll1, \quad (4.40)$$

then $\langle q_2',\psi_2'|q_1',\psi_1'\rangle$ is given by $\langle \boldsymbol{q}_2',\boldsymbol{\psi}_2'|\boldsymbol{q}_1',\boldsymbol{\psi}_1'\rangle$ for $\boldsymbol{q}_2=q_2'$, $\boldsymbol{\psi}_2'=\psi_2'$, $\boldsymbol{q}_1'=q_1'$, and $\boldsymbol{\psi}_1'=\psi_1'$ times the stationary distribution function of the medium, insofar as q_1' is not much greater than the rms value of q.