single flow rate is about 10%. Furthermore it should be noted that σ_{expt} falls 2.6 standard deviations of σ_{random} from σ_{random} and the probability of this happening if $\sigma_{\text{expt}}=\sigma_{\text{random}}$ is about 1%. This implies that the flow rates are not randomly distributed about a single mean. The former small sample test gives a more pessimistic conclusion than the latter which assumes large sample statistics.

From the foregoing we conclude that there is a regular spacing in the observed flow rates, and from Fig. 3 we see that the value of the spacing is $(0.83\pm0.02)\times10^{-5}$ $\text{cm}^2\,\text{sec}^{-1}$. Harris-Lowe *et al.*[1] and Allen and Armitage[2] have seen regularly spaced flow rates at a given temperature, changing repeatedly (nearly always to lower values) during single runs using the breaker flow method. They found spacings of 0.64×10^{-5} $\text{cm}^2\,\text{sec}^{-1}$ at 0.99°K and 0.5×10^{-5} $\text{cm}^2\,\text{sec}^{-1}$ at 1.19°K, respectively. (No errors were stated.) This seems to be a different but possibly related phenomenon. At present no theory predicts discrete flow rates. The weight of independent evidence in favor of discrete flow rates should stimulate theoretical activity.

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Mobility of a Charged Impurity in a Fermi Liquid*

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The low-temperature mobility of a charged impurity in a polarizable Fermi liquid is calculated. It is shown that the polarization effect is considerably larger than a typical hard-core effect, leading to a smaller low-temperature mobility.

INTRODUCTION

In this paper we present a calculation of the mobility of a charged impurity in a polarizable Fermi liquid. Such a calculation is appropriate for discussing the motion of ions or electrons in low-temperature liquid He^3. A general analysis of this problem would involve two primary considerations: a description of the Fermi liquid without the impurity, and a method of coupling the impurity to the liquid. We will not concern ourselves with the details of these two considerations, but rather start with a simple model which embodies such details in its parameters.

Under the simplest possible assumptions, we regard the Fermi liquid as a gas of noninteracting quasiparticles obeying Fermi statistics and possessing a spectrum

$$\epsilon(k)=\hbar^2k^2/2m. \tag{1}$$

The effect of the interactions between the particles is considered to be included in the effective mass m, which is not the He^3 mass m_3. From the Landau-Fermi liquid theory[1] and specific-heat measurements[2] on He^3, $m=3.08m_3$. Two other quantities will be of interest. For the zero-temperature Fermi gas, one defines a Fermi wave number k_F related to the particle density ρ through

$$\rho=k_F^3/3\pi^2 \tag{2}$$

and a Fermi energy

$$\epsilon_F=\hbar^2k_F^2/2m. \tag{3}$$

With a mass density[2] of 0.081 g/cc, one finds a number density $\rho=1.7\times10^{22}$ particles/cc, $k_F=0.78\times10^8$ cm^{-1}, and $\epsilon_F/\kappa=5m_3/\text{m°K}$, where κ is Boltzmann's constant.

For the impurities, we have the following picture in mind. First of all, their number density $\rho_i=N_i/\Omega$ is to be very small, so that impurities do not interact with each other. The actual impurity may be either an electron or an ion, and hence through its charge will be coupled to the liquid by means of polarizability effects. Several aspects of such mechanisms have been discussed in the literature,[3,4] and the resulting structures resemble bubbles for electrons and "snowballs" for ions. To avoid the difficulties associated with the specific structure of the impurity and its polarization cloud, we approximate the situation as follows. Each impurity will be considered a Boltzmann gas particle of mass M interacting with the Fermi liquid through a repulsive hard-core interaction and an attractive polarization interaction. Hence, part of the interaction between the impurity and the liquid is absorbed in the mass M, and the rest is approximated by a hard-core delta-function pseudopotential and a polarization effect.

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[1] A. A. Abrikisov and I. M. Khalatnikov, Rept. Progr. Phys. **32**, 352 (1959).

[2] J. C. Wheatley, in *Proceedings of the Sussex University Symposium on Quantum Fluids, 1965*, edited by D. F. Brewer (North-Holland Publishing Co., Amsterdam, 1965).

[3] K. R. Atkins, Phys. Rev. **116**, 1339 (1959).

[4] C. G. Kuper, Phys. Rev. **122**, 1007 (1961).

The zero-order Hamiltonian for the Fermi liquid and impurity structures will be taken as

$$H_0=\sum_{k,\sigma}\frac{\hbar^2k^2}{2m}a_{k,\sigma}{}^\dagger a_{k,\sigma}-\sum_{i=1}^{N}\frac{\hbar^2\nabla_{R_i}{}^2}{2M}. \tag{4}$$

For the Fermi liquid we have used the second-quantization formalism, but the impurities are represented in the ordinary quantum-mechanical description. The interaction Hamiltonian becomes

$$H'=\sum_{i=1}^{N} d^3r\, V(\mathbf{r}-\mathbf{R}_i)\rho(r)^{\text{liquid}}, \tag{5}$$

where $V(\mathbf{r}-\mathbf{R}_i)$ is the potential. The hard-core contribution to the potential is taken as

$$V_{\text{HC}}=\frac{8\pi\hbar^2}{2M^*}a\delta(\mathbf{r}-\mathbf{R}_i), \tag{6}$$

where M^* and a are parameters hav'ng the dimension of a mass and length, respectively. The polarization potential is taken as

$$V_{\text{pol}}=-\tfrac{1}{2}\alpha_0[e/4\pi(\mathbf{r}-\mathbf{R}_i)^2]^2, \quad |\mathbf{r}-\mathbf{R}_i|\geqslant b$$
$$=0 \text{ otherwise}, \tag{7}$$

where α_0 is the static polarizability of the Fermi liquid, and b is a length related to the radius of the impurity structure, which is about the same magnitude as a. The interaction Hamiltonian then becomes

$$H'=\frac{1}{\Omega}\sum_{k,q;\sigma}\sum_{i=1}^{N}a_{k+q,\sigma}{}^\dagger a_{k,\sigma}e^{-i\mathbf{q}\cdot\mathbf{R}_i}V(q), \tag{8}$$

where $V(q)$ is the Fourier transform of the potential

$$V(q)=8\pi\frac{\hbar^2a}{2M^*}-2\pi\alpha_0b^3\left(\frac{e}{4\pi b^2}\right)^2\int_1^\infty\frac{\sin qbx}{qbx}\frac{dx}{x^2}. \tag{9}$$

This defines the model for which we now calculate the impurity mobility.

MOBILITY CALCULATION

In the presence of a uniform constant electric field $\mathbf{E}$ the charged impurity structure will be accelerated, but because of collisions with the quasiparticles of the liquid it will quickly reach a terminal velocity $\mathbf{v}$. The mobility μ is then defined as

$$\mathbf{v}=\mu\mathbf{E}. \tag{10}$$

The above implies an isotropic medium, since μ is taken to be a scalar. Furthermore, it is restricted to weak fields, otherwise $\mathbf{v}$ may not depend linearly on the field. If one multiplies the velocity by the charge density of the impurity structures ρ_ie, assuming each is singly charged, one obtains the current density, or (10) becomes

$$\mathbf{j}=e\rho_i\mu\mathbf{E}=\sigma\mathbf{E}, \tag{11}$$

from which we relate the conductivity σ to the mobility

$$\sigma=e\rho_i\mu. \tag{12}$$

In a quantum-mechanical problem the current density becomes an operator, so that (11) must be interpreted as an expectation value in both a quantum-mechanical and statistical-mechanics sense. Kubo[5] has derived a formalism for discussing linear response problems in quantum-statistical mechanics, and, in particular, Nakano[6] has derived an explicit expression for the conductivity suitable for our purpose. Suppose we have a system consisting of a medium containing charge carriers. If the Hamiltonian is written in the form $H=H_{\text{med}}+H_{\text{ch car}}+H'_{\text{med ch car}}=H_0+H'$, then in a weak constant electric field, Nakano's formula for the conductivity in an isotropic medium takes the form

$$\sigma_{\nu\nu}=\hbar^2\beta^2\langle j_\nu j_\nu\rangle^2\Big/\Big\langle\int_0^\infty d\tau\int_0^\beta d\lambda \times e^{\lambda H_0}[H',j_\nu]e^{-\lambda H_0}[j_\nu,H'(\tau)]\Big\rangle, \tag{13}$$

where $\beta=1/\kappa T$, j_ν is the ν component of the charge-current density operator, and $H(t)'$ is in the Heisenberg representation with respect to H_0. The brackets $\langle\ \rangle$ denote both the quantum-mechanical expectation values and the statistical average with the density operator $\exp(-\beta H_0)$. Equations (12) and (13) are now used to calculate the mobility for our model.

It is convenient to normalize all wave numbers to k_F and write in second-order perturbation theory

$$\sigma_{\nu\nu}=\rho_ie^2\frac{6\pi^3(\hbar^2k_F{}^2/2M)}{\beta|V(0)k_F{}^3|^2\hbar k_F{}^2\mathfrak{F}}, \tag{14}$$

where ρ_i is the impurity density and $\mathfrak{F}$ is a dimensionless quantity.

$$\mathfrak{F}=\int\frac{d^3k}{4\pi}\int\frac{d^3q}{4\pi}\left|\frac{V(q)}{V(0)}\right|^2q^2f(k)[1-f(\mathbf{k}+\mathbf{q})] \times\int d^3P\, e^{-\beta(\hbar^2k_F{}^2P^2/2M)}\delta\left(\frac{2M\hbar\omega}{\hbar^2k_F{}^2}\right)\Big/\int d^3P e^{-\beta(\hbar^2k_F{}^2/2M)P^2}, \tag{15}$$

and

$$\hbar\omega=\frac{\hbar^2}{2m}[(\mathbf{k}'+\mathbf{q})^2-k'^2]+\frac{\hbar^2}{2M}[(\mathbf{P}-\mathbf{q})^2-P^2], \tag{16}$$

with dimensionless k, q, and $\mathbf{P}$. To discuss the tempera-

[5] R. Kubo, J. Phys. Soc. Japan **12**, 570 (1957).

[6] H. Nakano, Progr. Theoret. Phys. (Kyoto) **17**, 145 (1957).

ture and Fermion density dependence of the mobility or conductivity, one must analyze $\mathfrak{F}$. This is somewhat complicated and is performed in the Appendix.

RESULTS AND DISCUSSION

For low temperatures,

$$\beta_1 = (\hbar^2 k_F^2/2M)\beta \gg 1\,, \tag{17}$$

the final result for the mobility of a charged impurity in a polarizable Fermi liquid is

$$\mu(T\to 0) = \frac{12\pi^3}{\zeta(3)+\zeta(2)}\frac{e\hbar}{M}\frac{\hbar^2 k_F^2}{2M}\frac{\beta_1^2}{(V(0)k_F^3)^2}, \tag{18}$$

where, from (9),

$$V(0)k_F^3 = 8\pi\frac{k_F^2\hbar^2}{2M^*}k_F a - 2\pi\alpha_0\left(\frac{e}{4\pi b^2}\right)^2 k_F^3 b^3\,, \tag{19}$$

and $\zeta(n)$ is the Riemann zeta function[7] of order n. Combining (18) and (19), one obtains

$$\mu(T\to 0) = \frac{3\pi}{32(\zeta(3)+\zeta(2))}\frac{e\hbar^3\beta^2(M^*/mM)^2}{[a-\frac{1}{2}\alpha_0(e/4\pi b^2)^2 M^* b^3/\hbar^2]^2}. \tag{20}$$

Similarly, for high temperatures,

$$\beta_1 \ll 1\,, \tag{21}$$

one finds

$$\mu(T\to\infty) = \frac{9\pi^{3/2}}{2^8\sqrt{2}}\frac{e}{k_F^3 a^2}\left[M^*\left(\frac{m+M}{mM}\right)\right]^{5/2}\left(\frac{\beta}{M^*}\right)^{1/2}. \tag{22}$$

Before discussing the application of these results to liquid He^3, we would like to compare our low-temperature result with a similar calculation by Clark.[8] The approach used by Clark is to start from a Boltzmann transport equation for the impurities. He evaluates the collision integral for a constant differential-scattering cross section σ_{scat} and finds

$$\mu(T\to 0) \approx \frac{1.21 e\hbar^3\beta^2}{(M+m)^2\sigma_{scat}}. \tag{23}$$

Our approach through Nakano's formula for the conductivity seems simpler and the pertinent expressions can be evaluated exactly. To compare (20) with (23), we take the parameter M^* as the reduced mass of the impurity and He^3 quasiparticle

$$M^* = mM/M+m\,. \tag{24}$$

Then our differential-scattering cross section in Born approximation becomes

$$\sigma_{scat} = \left|\frac{M^*}{2\pi\hbar^2}V(0)\right|^2 = 4\left[a - \tfrac{1}{2}\alpha_0\left(\frac{e}{4\pi b^2}\right)^2\frac{M^* b^3}{\hbar^2}\right]^2, \tag{25}$$

so that (20) with (24) and (25) becomes

$$\mu(T\to 0) = \frac{0.41 e\hbar^3\beta^2}{\sigma_{scat}(M+m)^2}. \tag{26}$$

We believe that part of the numerical discrepancy between our result (26) and that of Ref. 8 is that (23) should be a factor of 2 smaller, since the collision integral [Ref. 8, Eq. (2)] should contain a spin factor of 2. The rest is presumably due to approximating the Fermi functions in the derivation of Ref. 8. The main difference between this calculation and that of Ref. 8 is that our scattering cross section includes the polarizability effect, and as we now show for He^3, this dominates over the hard-core effect.

For example, Atkins's[3] model for a positive ion in He^4 results in a "snowball" of about 7 Å radius and a mass of about 100 amu. In our calculation for the mobility we then take $a=b=7$ Å, $M=100$ amu, and M^* the effective mass (24). The static polarizability α_0 of He^3 is roughly the same as that of He^4, and from the index of refraction[1] $n_{He^4}=1.027=(1+\rho_{He^4}\alpha_0)^{1/2}$, one finds $\rho_{He^3}\alpha_0=0.04$, where ρ is the particle density. With these numbers one finds

$$\left[a-\tfrac{1}{2}\alpha_0\left(\frac{e}{4\pi b^2}\right)^2\frac{M^* b^3}{\hbar^2}\right]^2 = [7-4.05\times 10^2]^2,$$

i.e., the polarizability effect is considerably larger. The low-temperature mobility of a positive ion in He^3 then becomes

$$\mu_+ \approx 1.7\times 10^{-4}/(100/T°\mathrm{K})^2\ \mathrm{cm^2/V\ sec}, \quad T\ll 1°\mathrm{K} \tag{27}$$

since low temperature means $\beta\epsilon_F \gg 1$. A similar result can be derived for μ_-, depending of course on the model used for the negative charged impurity.

At this point, a few remarks about the validity of our results (20) and (22) should be mentioned. The high-temperature limit Eq. (22) is included only for completeness; in reality it would correspond to a dilute Boltzmann gas instead of a Fermi liquid. To estimate the temperature range for which the Fermi-liquid result Eq. (20) is valid, we make the following observations. Our calculation is based on independent Fermi quasiparticles scattering off the impurity structures. This implies a mean free path for the fermions larger than the radius of the impurity. For the mean free path, we take the Fermi velocity v_f multiplied by the collision time τ_c, which is inversely proportional to the square of the temperature. With the appropriate values[2] for He^3, we find $1/T^2 > a$, where a is the radius of the impurity in

[7] *Handbook of Mathematical Functions*, edited by Milton Abramowitz and Irene Stegun (Dover Publications, Inc., New York, 1965).

[8] R. C. Clark, Proc. Phys. Soc. (London) **82**, 785 (1963).

angstroms and T is the temperature in °K. For $a \approx 10$ Å this requires $T<0.3$°K.

Another restriction on the temperature arises from the requirement that the weak coupling limit be valid. That is, the kinetic energy of the impurity must be larger than some measure of the interaction energy between an impurity and the excitations of the liquid. For the kinetic energy of the impurity we take κT and for the interaction energy, the maximum matrix element[9] multiplied by the effective quasiparticle density $\rho_{\mathrm{He}^3}(\kappa T/\epsilon_F)^2$. With the numerical estimates in the preceding paragraph, this criterion

$$|V(0)|\rho_{\mathrm{He}^3}\left(\frac{\kappa T}{\epsilon_F}\right)^2 < \kappa T$$

requires a temperature $T<10$ m°K.

The mobility of positive and negative ions in liquid He^3 have been measured,[9] but the temperature range of 0.3–3°K is not low enough to apply our low-temperature result. The orders of magnitude observed are $3\text{–}8\times10^{-2}$ cm^2/V sec. On the other hand, our high-temperature limit (28) does not apply either. At higher temperatures, the noninteracting quasiparticle picture for the liquid must break down, and one should include viscosity effects, or a transition to Stokes's-law–type flow. The experiment mentioned above seems to show a temperature dependence of the mobility which is closer to the Stokes-law explanation

$$\mu = e/6\pi\eta a, \tag{28}$$

where η is the viscosity of the liquid of the liquid and a the radius of the impurity.

In conclusion, we summarize the main results of this calculation. The low-temperature mobility of a charged impurity in a polarizable Fermi liquid (He^3) is determined by the polarization effect, which gives a mobility a few orders of magnitude less than typical hard-core effects.

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APPENDIX

In this Appendix we perform the integrations in Eq. (15)

$$\mathfrak{F}(\beta)=\int\frac{d^3k}{4\pi}\int\frac{d^3q}{4\pi}\left|\frac{V(q)}{V(0)}\right|^2 q^2 f(k)[1-f(\mathbf{k}+\mathbf{q})]$$
$$\times\frac{4}{\sqrt{\pi}}\left(\frac{\hbar^2k_F{}^2\beta}{2M}\right)^{-3/2}\int\frac{d^3P}{4\pi}e^{-\beta(\hbar^2k_F{}^2P^2/2M)}\delta\left(\frac{2M\hbar\omega}{\hbar^2k_F{}^2}\right) \tag{A1}$$

for the limiting cases of low temperature

$$\beta_1=(\hbar^2k_F{}^2/2M)\beta\gg1, \tag{A2}$$

and high temperature

$$\beta_1\ll1. \tag{A3}$$

Consider the energy-conservation delta function (16) which becomes

$$\delta(\hbar\omega 2M/\hbar^2k_F{}^2)=\delta[(M/m)(2\mathbf{k}\cdot\mathbf{q}+q^2)-2\mathbf{P}\cdot\mathbf{q}+q^2]. \tag{A4}$$

Perform the $d\theta_{Pq}$ integration with $\mathbf{q}$ as the polar axis. This eliminates the $\delta(\omega)$, and since $|\cos\theta_{Pq}|\leqslant1$, it gives a lower limit for the dP integration

$$P_{\min}=\left|\frac{q^2+(M/m)(2\mathbf{k}\cdot\mathbf{q}+q^2)}{2q}\right|. \tag{A5}$$

Performing the dP integration then gives

$$\mathfrak{F}=\tfrac{1}{2}\left(\frac{\beta_2}{\pi}\right)^{1/2}\int\frac{d^3k}{4\pi}\int\frac{d^3q}{4\pi}\left|\frac{V(q)}{V(0)}\right|^2 qf(k)$$
$$\times[1-f(\mathbf{k}+\mathbf{q})]e^{-\beta_2P_{\min}{}^2}, \tag{A6}$$

where

$$\beta_2=(\hbar^2k_F{}^2/2M)\beta=(m/M)\beta_1. \tag{A7}$$

It is convenient to include the factor $\sqrt{\beta_1}$ with the dimensionless variables $\mathbf{q}$ and $\mathbf{k}$, i.e.,

$$k=\mathfrak{k}/\sqrt{\beta_1}, \tag{A8}$$

and

$$q=\mathfrak{q}/\sqrt{\beta_1}, \tag{A9}$$

so that

$$\mathfrak{F}=\tfrac{1}{2}\left(\frac{\beta_2}{\pi}\right)^{1/2}\beta_1{}^{-7/2}\int\frac{d^3\mathfrak{k}}{4\pi}\int\frac{d^3\mathfrak{q}}{4\pi}\left|\frac{V(\mathfrak{q}/\sqrt{\beta_1})}{V(0)}\right|^2$$
$$\times f(\mathfrak{k})[1-f(\mathfrak{k}+\mathfrak{q})]$$
$$\times\exp\left(-\frac{m}{4M}\left[\frac{\mathfrak{q}^2+(M/m)(2\mathfrak{k}\cdot\mathfrak{q}+\mathfrak{q}^2)}{\mathfrak{q}}\right]^2\right). \tag{A10}$$

(i) Low temperature: Since $\beta_1\to\infty$, the matrix element satisfies

$$\left.\frac{V(\mathfrak{q}/\sqrt{\beta_1})}{V(0)}\right|_{\beta_1\to\infty}\to1. \tag{A11}$$

The Fermi distribution is of the form

$$f(\mathfrak{k})=1/e^{\mathfrak{k}^2-\beta_1}+1, \tag{A12}$$

so that for low T only $\mathfrak{k}$ and $|\mathfrak{k}+\mathfrak{q}|$ near β_1 contribute in the integration in (A10). With the change of variables

$$t=\mathfrak{k}^2 \tag{A13}$$

[9] P. de Magistris, I. Modena, and F. Scaramuzzi, in *Proceedings of the Ninth International Conference on Low-Temperature Physics, Columbus, Ohio*, edited by J. A. Daunt *et al.* (Plenum Press, Inc., New York, 1965), p. 349.

and

$$u=|\mathfrak{k}+\mathfrak{q}|^2=|\mathfrak{p}|^2, \tag{A14}$$

one obtains, after integrating over $\theta_{\mathfrak{kp}}$,

$$\mathfrak{F}(\beta_1\to\infty)=\tfrac{1}{4}\frac{M}{m}\beta_1^{-3}\int_0^\infty dt\int_0^t du$$

$$\times[\tfrac{1}{2}(t-u)+1]\left(\frac{1}{e^{t-\beta_1}+1}\right)\left(\frac{e^{u-\beta_1}}{e^{u-\beta_1}+1}\right). \tag{A15}$$

After shifting both u and t by β_1, the double integral becomes

$$\int_{-\beta_1}^\infty dt\int_{-\beta_1}^t du[\tfrac{1}{2}(t-u)+1]\left(\frac{1}{e^t+1}\right)\left(\frac{e^u}{e^u+1}\right)$$

$$\xrightarrow[\beta_1\to\infty]{}\zeta(3)+\zeta(2), \tag{A16}$$

where $\zeta(n)$ is the Riemann zeta function[7] of order n. The last step in (A16) follows easily after the change of variables $t-u=x$ and $e^t=y$, so that in the $\beta_1\to\infty$ limit (A16) becomes

$$\int_0^\infty dy\int_0^\infty dx(\tfrac{1}{2}x+1)\frac{1}{(y+e^x)(y+1)}=\zeta(3)+\zeta(2). \tag{A17}$$

The final result for the low-temperature integral is

$$\mathfrak{F}(\beta_1\to\infty)=[\zeta(3)+\zeta(2)](M\beta_1^{-3}/4m). \tag{A18}$$

(ii) High temperature: Since $\beta_1\to 0$, the matrix element (A11) becomes

$$\left|\frac{V(\infty)}{V(0)}\right|^2\to\left[\frac{8\pi(\hbar^2a/2M^*)}{V(0)}\right]^2 \tag{A19}$$

for our example (9). The Fermi distributions become Boltzmann factors[10]

$$f(\mathfrak{k})\to\frac{8}{3\sqrt{\pi}}\beta_1^{3/2}\exp(-\mathfrak{k}^2), \tag{A20}$$

so that (A10) becomes

$$\mathfrak{F}=\left(\frac{V(\infty)}{V(0)}\right)^2\frac{4}{3\pi}\frac{\sqrt{\beta_2}}{\beta_1^2}\int_0^\infty\frac{d^3\mathfrak{k}}{4\pi}\int_0^\infty\frac{d^3\mathfrak{q}}{4\pi}e^{-\mathfrak{k}^2}\mathfrak{q}$$

$$\times\exp\left\{-\frac{m}{4M}\left[\frac{\mathfrak{q}^2+(M/m)(2\mathfrak{k}\cdot\mathfrak{q}+\mathfrak{q}^2)}{\mathfrak{q}}\right]^2\right\}. \tag{A21}$$

With the change of variables

$$\mathbf{x}=\mathfrak{k}+\tfrac{1}{2}\mathfrak{q} \tag{A22}$$

but

$$q=\mathfrak{q}. \tag{A23}$$

The integrations are easily performed, giving

$$\mathfrak{F}(\beta_1\to 0)=\frac{8}{3\sqrt{\pi}}\beta_1^{-3/2}\left(\frac{V(\infty)}{V(0)}\right)^2 m/M(1+m/M)^{5/2}.$$

[10] K. Huang, *Statistical Mechanics* (John Wiley & Sons, Inc., New York, 1963).