(d) The magnitude of this dip is sensitive to the parameter $\rho_{\pi N}$ (the ratio of the real to the imaginary part of the πN scattering amplitude), which has only been roughly determined by high-energy π-N scattering experiments. Accurate measurements of cross section in this region would therefore provide an independent check on this point.

(e) Present experiments seem to indicate that various parameters of πN and NN interactions will approach a constant limit as energy increases. As a consequence, in this model the $d\sigma/dt$ of particle-nucleus scattering would also approach a limiting form at very high energies. Since the various parameters change only slightly from 8 BeV/c on, we expect that the present calculation would also fit future experiments performed at higher energies reasonably well.

I would like to thank Professor C. N. Yang for suggesting this investigation and for numerous enlightening discussions on this and other related problems and many valuable comments on the manuscript.

PHYSICAL REVIEW VOLUME 168, NUMBER 5 25 APRIL 1968

Feynman-Diagram Models of Regge and Multi-Regge Couplings*

R. BLANKENBECLER AND R. L. SUGAR
Department of Physics, University of California, Santa Barbara, California
(Received 11 December 1967)

Some Feynman-diagram models are presented which lead to a natural framework for discussions of the kinematic properties of Regge poles. The models do not require an infinite number of recurrences of a trajectory. The basic property of the Feynman prescriptions that is useful here is the fact that the analyticity properties of the amplitude are preserved at all stages of the calculation. The coupling of particles with integer spin to Regge trajectories is discussed in detail. These models allow one to consider various aspects of ghost-killing mechanisms and to clarify the kinematic properties, especially the kinematic singularities, of Regge residues. The extension of the discussion to the coupling of two Regge trajectories to a physical particle is carried out and applied to the multiple-Regge model of production amplitudes. Some experimental consequences of these models are briefly explored for particle production.

I. INTRODUCTION

ONE of the major problems in Regge pole theory is that one really does not know how to couple Regge poles to particles of unequal mass, to particles with spin, or to other Regge poles. It is well known that the usual prescription, carried over from potential scattering, of making a partial wave decomposition of the scattering amplitude and then replacing J everywhere by α, leads to Regge pole contributions which violate our basic notions about analyticity. This difficulty has led to the introduction of daughter and conspirator Regge trajectories.[1]

In this paper we shall study the coupling of Regge poles to other particles by considering some simple, but relativistically invariant, models. The model we will discuss will be a sum of selected Feynman diagrams, a not uncommon laboratory for testing theoretical ideas in dispersion theory. An example of the type of model we have in mind is the one originally discussed by Van Hove and Durand.[2] They pointed out that the exchange of a Regge pole of spin α can be simulated by taking the sum of the diagrams arising from the exchange of particles of spin 0, 1, 2, $\cdots$. (Throughout this paper we shall consistently ignore signature. It can be added at the end in a trivial fashion.) The advantage of using Feynman-diagram models as a guide to the correct coupling of Regge poles to particles of unequal mass or to particles with spin is that the Feynman prescriptions always lead to scattering amplitudes with "good" analyticity properties. As a result, these models will automatically have daughter or conspirator trajectories. One can use them to write down a simple expression for the contribution of a Regge pole which will automatically include daughter or conspirator poles and which can be studied for all values of energy and momentum transfer. One is not necessarily restricted to the asymptotic region.

In Secs. II–V we review the one-particle exchange model of Regge poles and extend it to the case in which the external particles have spin. We also give a simple model for the case of four-particle coupling. This model is important in illustrating the point that an infinite number of Regge recurrences are not required by the perturbation approach. In Secs. VI–VIII we apply our results to the multiperipheral Regge model.[3] In subsequent papers we shall extend the Feynman-diagram models to the case of spinor particles and use our results

* Work supported by the National Science Foundation.
[1] D. Z. Freedman and J. M. Wang, Phys. Rev. Letters **17**, 569 (1966); Phys. Rev. **153**, 1596 (1967); **160**, 1560 (1967).
[2] L. Van Hove, Phys. Letters **24B**, 183 (1967); Loyal Durand, III, Phys. Rev. **154**, 1537 (1967).
[3] N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. Letters **19**, 614 (1967). See also F. Zachariasen and G. Zweig, Phys. Rev. **160**, 1322 (1967); **160**, 1326 (1967); Chan Hong-Mo, K. Jajantie, and G. Ranft, Nuovo Cimento **49**, 157 (1967).

to study conspiracies[4] and to derive Feynman rules for Regge poles.[5]

II. SPIN-ZERO PARTICLES

In order to introduce notation, define phases, and in order to make this paper self-contained, let us first examine the case of elastic scattering of scalar particles of equal mass. This is the original model discussed by Van Hove and Durand.[2] For clarity we will express the model in terms of an effective interaction Hamiltonian between the two external particles B and C, and the exchange particle A of spin J:

$$H_I=g(J)[p(J)]^{1/2}A_{\mu_1\cdots\mu_J}{}^J B(\overleftrightarrow{\partial}_{\mu_1})(\overleftrightarrow{\partial}_{\mu_2})\cdots(\overleftrightarrow{\partial}_{\mu_J})C\,, \quad (1)$$

where

$$\overleftrightarrow{\partial}_\mu\equiv\tfrac{1}{2}(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)$$

and

$$p(J)=\Gamma(2J+2)/2^J\Gamma^2(J+1)\,. \quad (2)$$

For brevity we will write H_I in the form

$$H_I=g(J)[p(J)]^{1/2}A_\mu{}^J B(\overleftrightarrow{\partial}_\mu)^J C\,. \quad (3)$$

We have pulled the factor $[p(J)]^{1/2}$ out of the coupling constant in order to simplify later equations.

Defining the kinematics as in Fig. 1, and writing $t=P^2$, the Feynman amplitude for the elastic scattering of B and C through the direct A-particle pole is

$$F(J)=g^2(J)p(J)(Q_\mu{}')^J(-)^J\Gamma_{\mu;\nu}{}^J(M_A{}^2(J)) \times(Q_\nu)^J[M_A{}^2(J)-t]^{-1}. \quad (4)$$

The symbol μ stands for $\mu_1, \mu_2, \cdots \mu_J$, and similarly for ν. The factor $(-)^J\Gamma_{\mu;\nu}{}^J(M^2)$ is the numerator of the spin-J Feynman propagator. It is given by

$$(-)^J\Gamma_{\mu;\nu}{}^J(M^2)=\sum_\lambda \epsilon_\mu{}^{J,\lambda}\epsilon_\nu{}^{J,\lambda*}\,, \quad (5)$$

where $\epsilon^{J,\lambda}$ is the polarization tensor of the spin-J particle with z component of angular momentum λ, and the sum is carried out over all polarization states. The argument M^2 in $\Gamma_{\mu;\nu}{}^J$ means that the momentum factors appear as $P_\mu P_\nu/M^2$ rather than $P_\mu P_\nu/P^2$, as is described in detail in Ref. 5.

The properties of $\Gamma_{\mu;\nu}{}^J(M^2)$ allow us to write Eq. (4) in closed form in terms of a Legendre polynomial. For later use we define the quantity

$$(-)^J(Q_\mu{}')^J\Gamma_{\mu;\nu}{}^J(M_A{}^2)(Q_\nu)^J\equiv H_J(Q',Q)\,, \quad (6)$$

where

$$H_J=(\bar{Q}'\bar{Q})^J P_J(\bar{z})(2J+1)/p(J)\,, \quad (7)$$

and

$$\bar{Q}^2=-Q_\mu\bar{\theta}_{\mu\nu}Q_\nu\equiv-Q\cdot\bar{\theta}\cdot Q\,,$$
$$\bar{Q}\bar{Q}'\bar{z}=-Q_\mu{}'\bar{\theta}_{\mu\nu}Q_\nu\equiv-Q'\cdot\bar{\theta}\cdot Q\,,$$
$$\bar{\theta}_{\mu\nu}=g_{\mu\nu}-P_\mu P_\nu/M_A{}^2(J)\,.$$

We note that if the A particle is on the mass shell $[P^2=M_A{}^2(J)]$ or if the B and C particles have equal mass $(P\cdot Q=P\cdot Q'=0)$, then

$$\bar{z}=z\,,\quad \bar{Q}^2=\bar{Q}'^2=q^2\,, \quad (8)$$

where z is the t-channel center-of-mass scattering angle and $\mathbf{q}$ is the center-of-mass three-momentum. We also note that for large $\bar{z}$

$$H_J\simeq(-Q'\cdot\bar{\theta}\cdot Q)^J+O((-Q'\cdot\bar{\theta}\cdot Q)^{J-2})\,. \quad (9)$$

It is important for the study of daughter trajectories that the function $H_J(Q,Q')$ is well behaved in the neighborhood of $t=0$, even when the masses of the B and C particles are unequal,[6] but since we are not interested in studying daughter trajectories at this point, we shall take the masses of the B and C particles to be equal in order to emphasize the physical content of this approach.

The total scattering amplitude in this model is given by summing $F(J)$ over J from zero to infinity. Defining $\alpha(t)$ to be the largest root of the equation

$$M_A{}^2(\alpha)-t=0\,, \quad (10)$$

which implies that one is secretly considering $M_A{}^2(J)$ to be the bound-state eigenvalue of some type of equation, the total amplitude F becomes

$$F=\sum_{J=0}^{\infty}F(J)$$
$$=-(2\alpha+1)g^2(\alpha)(\pi/\sin\pi\alpha)(d\alpha/dt)q^{2\alpha}P_\alpha(-z) +\text{a background integral.} \quad (11)$$

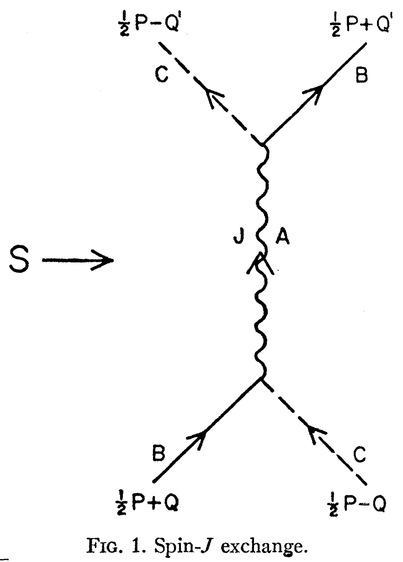

FIG. 1. Spin-J exchange.

[4] R. Blankenbecler, R. L. Sugar, and J. D. Sullivan (to be published).

[5] R. Blankenbecler and R. L. Sugar (to be published).

[6] R. L. Sugar and J. D. Sullivan, Phys. Rev. 166, 1515 (1968).

In terms of the variable s which describes the reaction $B+\bar{C}\to B+\bar{C}$, we have, for large s,

$$F\simeq -p(\alpha)g^2(\alpha)(\pi/\sin\pi\alpha)\frac{d\alpha}{dt}[\tfrac{1}{4}(s-u)]^\alpha + O((s-u)^{\alpha-1}). \quad (12)$$

It has been emphasized by Durand[2] that in obtaining Eq. (11), one has assumed that the functions $g^2(J)$ and $M^2(J)$ are in some sense smooth in J. Also, if one requires that the asymptotic behavior given in Eq. (12) holds for all complex s, which means that F is polynomially bounded, then the functions $g^2(J)$ and $[dM^2(J)/dJ]^{-1}$ cannot decrease too fast for large J. The importance of this restriction on the asymptotic behavior has been stressed by Feynman.[7]

At this time instead of presenting a general mathematical discussion of the restrictions necessary to achieve an F which is polynomially bounded, we prefer to simply give some examples which the reader can easily verify, and which we believe indicate the necessary restrictions. The examples bear some resemblance to phenomenological fits to experiment, and they will be useful in our later discussions. For large values of (z) we find

$$\sum_{n=N}^{\infty}(-z)^n(n-\alpha)^{-1} \simeq \frac{-\pi z^\alpha}{\sin\pi\alpha}+\frac{(-)^{N+1}z^{N-1}}{(1+\alpha-N)}+O(z^{N-2}),$$

$$\sum_{n=N}^{\infty}(-z)^n(n-\alpha)^{-1}\Gamma^2(n+1)/\Gamma(n+N+1)\Gamma(n-N+1) \simeq \frac{-\pi z^\alpha}{\sin\pi\alpha}\Gamma^2(\alpha+1)/\Gamma(\alpha+N+1)\Gamma(\alpha-N+1) + \frac{(-)^{N+1}N}{(\alpha+1)}\frac{\ln z}{z}+O\left(\frac{1}{z^2}\right),$$

$$\sum_{n=0}^{\infty}h^nP_n(-z)(n-\alpha)^{-1}\simeq -\frac{\pi h^\alpha P_\alpha(z)}{\sin\pi\alpha} - \frac{1}{(\alpha+\frac{1}{2})(2zh)^{1/2}}+O\left(\frac{1}{(zh)^{3/2}}\right),\quad \alpha>-\tfrac{1}{2} \quad (13)$$

$$\sum_{n=0}^{\infty}(2n+1)h^nP_n(-z)(n-\alpha)^{-1} \simeq \frac{\pi(2\alpha+1)}{\sin\pi\alpha}h^\alpha P_\alpha(z)+O\left(\frac{1}{(zh)^{3/2}}\right).$$

These sums yield functions of the required behavior, whereas, for example,

$$\sum_{n=0}^{\infty}(-z)^n/(n-\alpha)n!\simeq \frac{-\pi z^\alpha}{\sin\pi\alpha\Gamma(\alpha+1)}-\frac{e^{-z}}{z}+O\left(\frac{e^{-z}}{z^2}\right) \quad (14)$$

does not. It seems clear from these examples that in addition to requiring that $g^2(J)$ and $M^2(J)$ be analytic in the right-half J plane, one must also require that the product $g^2(J)[dM^2/dJ]^{-1}$ falls off more slowly than $1/J!$ for large J.

In many applications, for example the study of daughter[6] and conspirator[4] trajectories, it is important to have models in which the scattering amplitude satisfies elastic unitarity in the t channel. Such models can easily be obtained by summing the self-energy bubbles of the A particle. Using the effective interaction defined by Eq. (1), and the Born term $F(J)$, given by Eq. (4), the integral equation for the total amplitude T^J, which includes the effect of bubbles, is

$$T^J(Q',Q)=F^J(Q',Q) - i\int\frac{d^4k}{(2\pi)^4}\frac{F^J(Q',k)T^J(k,Q)}{[M_C^2-(\frac{1}{2}p+k)^2][M_C^2-(\frac{1}{2}p-k)^2]}. \quad (15)$$

Defining the amplitude f^J by

$$T^J=p(J)(-)^J(Q_\mu')^J\Gamma_{\mu;\nu}{}^J(Q_\nu)^Jf^J, \quad (16)$$

the integral equation is easily solved and one finds

$$f^J=g^2(J)[M_A^2(J)-t+g^2(J)I^J(t)]^{-1}, \quad (17)$$

where

$$I^J(t)=i\int\frac{d^4k}{(2\pi)^4}\frac{(\mathbf{k}^2)^J}{[M_C^2-(\frac{1}{2}p+k)^2][M_C^2-(\frac{1}{2}p-k)^2]}, \quad (18)$$

and

$$\mathbf{k}^2\equiv -k^2+(k\cdot p)^2/p^2.$$

The integral for $I^J(t)$ is infinite for positive J, and we should make it convergent by introducing cutoff functions, which depend on the relative momentum of the B and C particles, at each vertex. We will not bother to write in such cutoff functions explicitly. In most cases, once the mass and coupling constant renormalization has been performed and the sum over J has been done, it will be possible to let the cutoff functions go to 1. (See, for example, Ref. 6.)

If one now sums T^J over J, the leading power of s is the largest value of J which causes the denominator of Eq. (17) to vanish. We now see that this power will certainly be complex for t greater than $4M_C^2$. Also, we note that any fixed poles in $g^2(J)$ will be turned into moving poles by the form of the denominator.

Finally let us consider f^J in the limit that $M_A^2(J)$ gets very large while f^J remains finite. To that end we define

$$g^2(J)\equiv\Lambda^2(J)M_A^2(J), \quad (19)$$

and find that

$$f^J\to\Lambda^2(J)[1+\Lambda^2(J)I^J(t)]^{-1}. \quad (20)$$

We will return to this result in just a moment.

[7] R. P. Feynman (private communication).

It should be emphasized that the introduction of the cutoff functions discussed above, or the introduction of the usual vertex function, which arises from diagrams of higher order in $g(J)$, will not basically alter our results. The vertex and cutoff functions only affect the form of the Regge residue. What is important for the coupling of a Regge pole to scalar particles of unequal mass, and to the study of daughter trajectories, is the form of the function $H_J(Q',Q)$ defined in Eqs. (6) and (7). This depends only on the form of the spin-J Feynman propagator, and on our assumption that the Regge pole arose from the Feynman diagrams containing the A-particle pole in the first place. In the next section we shall show how even this assumption can be relaxed.

III. FOUR-POINT INTERACTION

In order to illustrate the fact that it is easy to construct Feynman-diagram models of Regge poles which are more general than the one-particle-exchange model considered in the last section, let us consider the four-point interaction given by

$$H_I^J = -\Lambda^2(J)(-)^J p(J)[B(\overleftrightarrow{\partial}_\mu)^J C]^2. \tag{21}$$

The first-order scattering amplitude is given by

$$F(J) = \Lambda^2(J) p(J)(-Q'\cdot Q)^J, \tag{22}$$

which contributes to scattering in the state J, $J-2$, $J-4$, etc. Defining the sum of the four-point bubbles in the t channel to be T^J (see Fig. 2), we introduce the amplitude f^J by the equation

$$T^J = p(J)(-)^J (Q_\mu')^J \Gamma_{\mu;\nu}{}^J (Q_\nu)^J f^J + (Q'\cdot Q)^{J-2} \text{ terms}. \tag{23}$$

f^J is easily computed to be

$$f^J = \Lambda^2(J)[1 + i\Lambda^2(J) I^J(t)]^{-1}. \tag{24}$$

This is identical to the result found previously, Eq. (20), in the limit $M_A{}^2 \to \infty$. Thus it is possible to replace a three-field interaction by a four-point interaction and not change the detailed character of the kinematic behavior of the amplitude. The kinematic structure of the amplitude is all that we wish to study at the moment. It should now be clear that in the model discussed in the last section it is not necessary to require that the mass spectrum $M_A{}^2(J)$ go to infinity, even though this seems to be implied by our effective Hamiltonian. In other words, it is not necessary to consider only infinitely rising trajectories.

It should also be clear by now that the essential feature of our models is that a Regge pole is given by the exchange of objects of spin 0, 1, 2, $\cdots$, each of which gives rise to a pole in the scattering amplitude. It is irrelevant whether these poles came from one-particle-exchange diagrams or arise dynamically from the interaction of two or more particles. What does seem to be important is that we ensure that the contribution of the spin-J pole has the proper analyticity properties.

We shall return to the four-point interaction model at a later time in a discussion of conspiracies.[4] This model has the advantage of allowing a wider range of coupling schemes than the one-particle-exchange model. In particular it allows couplings which give rise to parity-doubled conspiracies in a natural way, while the one-particle-exchange model does not.

IV. SPIN-ONE PARTICLES

In this section we wish to discuss the elastic scattering of a spin-one particle and a spin-zero particle. We start by writing down the effective coupling as was done in the previous case. Several types of coupling are possible, however, depending upon how we couple the spin to the orbital angular momentum to form a state of angular momentum J. We shall restrict ourselves for the moment to the coupling which involves the minimum number of derivatives. This coupling will be quite interesting from a theoretical point of view. In any case, let us examine the interaction

$$H_I = g(J,1)[3p(J)]^{1/2}[J/(J+1)] A_\mu{}^J B(\overleftrightarrow{\partial}_\mu)^{J-1} C_\mu. \tag{25}$$

The one in the coupling constant is to emphasize that $g(J,1)$ describes the coupling of a spin-J particle to a spin-one and a scalar particle. The factor $[3p(J)]^{1/2}$ has been put in to simplify later equations, and the factor $J/(J+1)$ to ensure that the number of derivatives is a positive number or, in other words, to control the nonsense-state coupling. The reason for choosing a factor of J rather than, say, $J^{1/2}$ to kill the nonsense coupling at $J=0$ is to make sure that there is no branch point in the angular-momentum plane at $J=0$, for inelastic reactions. The reaction scalar+spin $1\to 2$ scalar would exhibit such a singularity, which would give rise to a $\ln s$ asymptotic behavior which we do not like.

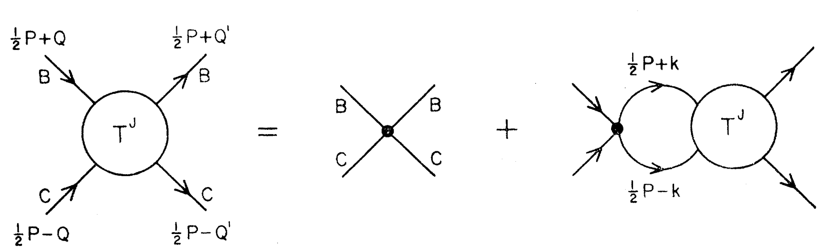

FIG. 2. Bubble graphs.

Introducing the polarization vectors ϵ_μ^* and ϵ_ν to describe helicity states M' and M of the C particle, and proceeding as before, the Feynman amplitude is

$$F^{M'M}(J)=g^2(J,1)3p(J)[J/(J+1)]^2(M^2(J)-t)^{-1} \times\epsilon_\mu^*(Q_\mu')^{J-1}(-)^J\Gamma_{\mu;\nu}{}^J(M^2(J))(Q_\nu)^{J-1}\epsilon_\nu. \quad (26)$$

There are several methods which can be used to evaluate this expression in closed form. Perhaps the simplest is to use the fact that $\Gamma_{\mu;\nu}{}^J$ is symmetric in the μ indices and in the ν indices which allows the replacement

$$\epsilon_\mu^*(Q_\mu')^{J-1}\rightarrow(1/J)(\epsilon^*\cdot\partial')(Q_\mu')^J, \quad (27)$$

where ∂_μ' means derivative with respect to Q_μ'. Thus one finds

$$F^{M'M}(J)=3p(J)g^2(J,1)(J+1)^{-2}[M^2(J)-t]^{-1} \times(\epsilon^*\cdot\partial')(\epsilon\cdot\partial)H_J(Q',Q), \quad (28)$$

where we have again assumed that the B and C particles have the same mass and hence $P\cdot Q=P\cdot Q'=0$. The sum over J can be performed as before, and we find for the leading term

$$F^{M'M}\simeq-3p(\alpha)g^2(\alpha,1)(\pi/\sin\pi\alpha)(d\alpha/dt) \times[(\epsilon^*\cdot\partial')(\epsilon\cdot\partial)/(\alpha+1)^2]H_\alpha(-Q',Q). \quad (29)$$

The derivatives can readily be evaluated exactly, but the asymptotic limit of this expression is particularly easy to work out explicitly:

$$F^{M'M}\simeq-3g^2(\alpha,1)(\pi/\sin\pi\alpha)(d\alpha/dt)\alpha p(\alpha)(\alpha+1)^{-2} \times(Q'\cdot Q)^{\alpha-1}[\epsilon^*\cdot\epsilon+(\alpha-1)\epsilon^*\cdot Q\epsilon\cdot Q'/Q\cdot Q'], \quad (30)$$

where it is now permissible to set Q and Q' equal to their mass-shell values. We will return to the problem of evaluating the exact $F^{M'M}$ in a more general situation. Equation (30) can be rewritten in terms of the asymptotic amplitude for scalar-scalar scattering:

$$F=-p(\alpha)g^2(\alpha)(\pi/\sin\pi\alpha)(d\alpha/dt)(Q\cdot Q')^\alpha. \quad (31)$$

After replacing $g(\alpha)$ by $g(\alpha,1)$ we have

$$F^{M'M}=-3Q^{-2}FS^{M'M}, \quad (32)$$

where

$$S^{M'M}=-[\alpha Q^2/(\alpha+1)^2] \times[\epsilon^*\cdot\epsilon+(\alpha-1)\epsilon^*\cdot Q\epsilon\cdot Q'/Q\cdot Q']/Q\cdot Q'. \quad (33)$$

Let us now evaluate this spin factor for states of definite helicity. We need consider only the cases with M and M' positive. Using the explicit polarization vectors introduced in the Appendix, the leading terms of S are found to be

$$S^{11}\simeq\alpha^2/2(1+\alpha)^2,\quad S^{10}\simeq-i\sqrt{t}S^{11}/M_C,\quad S^{00}\simeq tS^{11}/2M_C^2. \quad (34)$$

This type of ghost killing is the noncompensating mechanism.[8]

An interesting point here is the fact that the polarization vectors are wholly responsible for the kinematic singularities in S^{10}, which lead in turn to singularities in F^{10}

V. HIGHER SPIN COUPLINGS

Let us now turn to the problem of coupling a spin-J particle to a spin-S and a spin-zero particle. We will again choose the minimal derivative coupling. A method must be devised for killing the nonsense couplings which preserves the requisite analyticity in J required by the model. The appropriate interaction turns out to be

$$H_I=g(J,S)d(J,S)A_\mu{}^JB(\overleftrightarrow{\partial}_\mu)^{J-S}C_\mu{}^S, \quad (35)$$

where

$$d(J,S)=[p(J)p(S)]^{1/2} \times\Gamma^2(J+1)/\Gamma(J+S+1)\Gamma(J-S+1). \quad (36)$$

Again the factor $d(J,S)$ has been chosen to vanish linearly at the nonsense points to avoid branch points in inelastic reactions. For the values $S=0$ and 1, this coupling reduces to the previously considered vector and scalar cases.

The derivative trick can again be used to explicitly evaluate the Feynman amplitude. We make the replacement

$$\epsilon_\mu(S,M)(Q_\mu)^{J-S}\rightarrow(\epsilon\cdot\partial^S)(Q_\mu)^J \times\Gamma(J-S+1)/\Gamma(J+1), \quad (37)$$

where $\epsilon_\mu(S,M)=\epsilon_{\mu_1\mu_2\cdots\mu_s}(S,M)$ is the polarization tensor of the spin-S particle corresponding to helicity M. The S derivatives are to be taken on the right-hand side of Eq. (37). The spin-J scattering amplitude now becomes

$$F^{M'M}(J)=d^2(J,S)\Gamma^2(J-S+1)/p(J)\Gamma^2(J+1) \times(\epsilon^*\cdot\partial'^S)(\epsilon\cdot\partial^S)F(J), \quad (38)$$

where $F(J)$ is the amplitude for zero-spin external particles, but with $g(J)$ replaced by $g(J,S)$. One again does the sum over J, and in the asymptotic limit, the total amplitude simplifies to

$$F^{M'M}\simeq FS^{M'M}, \quad (39)$$

[8] C. B. Chiu, S. Y. Chu, and L. L. Wang, Phys. Rev. **161**, 1563 (1967). It is possible to use other nonsense killing factors. For example, the nonsense-choosing or Gell-Mann mechanism can be obtained by introducing a factor of $[J/(J+1)]^{1/2}$ instead of $J/(J+1)$ in the effective interaction, but then one must consider only elastic scattering to avoid branch points at $J=0$. This does not seem as esthetic as our choice, especially when one tries to treat higher spins, as in the next section. Also, there is some slight experimental preference for the noncompensating mechanism, as discussed by the above authors.

where F is given by Eq. (31), with $g(\alpha)$ replaced by $g(\alpha,S)$ and

$$S^{M'M}=\frac{d^2(\alpha,S)\Gamma^2(\alpha-S+1)}{p(\alpha)\Gamma^2(\alpha+1)(Q\cdot Q')^\alpha}(\epsilon^*\cdot\partial'^S)(\epsilon\cdot\partial^S)(Q\cdot Q')^\alpha$$

$$=\frac{d^2(\alpha,S)\Gamma^2(\alpha-S+1)}{p(\alpha)\Gamma(\alpha+1)}\sum_{k=0}^{S}\frac{\Gamma^2(S+1)}{\Gamma^2(k+1)\Gamma(S-k+1)\Gamma(\alpha-S-k+1)}\frac{(\epsilon^*\cdot Q^k)\cdot(\epsilon\cdot Q'^k)}{(Q\cdot Q')^{S+k}}. \quad (40)$$

This sum can be worked out easily for any given polarization vectors. The connection between this approach and the helicity formulation of Jacob and Wick is through the fact that the $S^{M'M}$ are closely related to the limiting behavior of the functions $d_{M'M}{}^\alpha$. This relation will be examined in detail in a later paper because it leads to a simple characterization of the kinematic singularities of the helicity amplitudes for any value of the spin. An example of this relation is the vector case treated above, especially Eq. (34).

VI. BI-REGGE COUPLINGS

Let us now turn to a direct extension of the previous coupling to the case where both the spins involved in the effective interaction Reggeize. The diagram which we wish to consider is shown in Fig. 3. The reaction that we wish to finally consider has the particles with momentum $\frac{1}{2}P+Q$ and $\frac{1}{2}P'+Q'$ incident and the other three outgoing. Hence the square of the total incident energy is

$$s_i=(\tfrac{1}{2}P+Q+\tfrac{1}{2}P'+Q')^2. \quad (41)$$

All the particles have mass M except for B, which has mass μ. The asymptotic behavior that we wish to examine occurs when the square of the relative energies between the final pairs are large. That is, we are interested in large values of

$$s=(P'+\tfrac{1}{2}P+Q)^2,$$
$$s'=(P+\tfrac{1}{2}P'+Q')^2, \quad (42)$$

for fixed values of $t=P^2$ and $t'=P'^2$. To complete all the kinematic preliminaries that will be needed, note that the variables

$$x\equiv Q\cdot P'=\tfrac{1}{2}(s-M^2)-\tfrac{1}{4}(\mu-t+t')$$
and
$$y\equiv Q'\cdot P=\tfrac{1}{2}(s'-M^2)-\tfrac{1}{4}(\mu-t'+t) \quad (43)$$

get large in the expected Regge region. Finally, it is convenient to introduce the quantity

$$W\equiv\mu^2\, Q\cdot Q'/xy, \quad (44)$$

which, as s and s' get large, approaches the value

$$W\simeq(1-\hat{Q}\cdot\hat{Q}') \quad (45)$$

in the rest system of the B particle defined by $P+P'=(\mu,\mathbf{0})$. Thus W is a measure of the angle between the $\mathbf{Q}$ and $\mathbf{Q}'$ planes, which is essentially the angular variable $\cos\omega$ introduced in Ref. 7.

The model is now completely defined by giving the couplings between the particles. We shall adopt for the moment minimum derivative couplings and explore their consequences. The effective interaction given by Eq. (35) will be used at all the vertices. However, to restore the symmetry of the coupling between A, B, and C, we will also add an interaction of the same form and coupling constant but with C and A interchanged so that values of J' greater than J are now allowed. This model could be considerably generalized, but to illustrate the physics involved, we have chosen to remain as simple as possible.

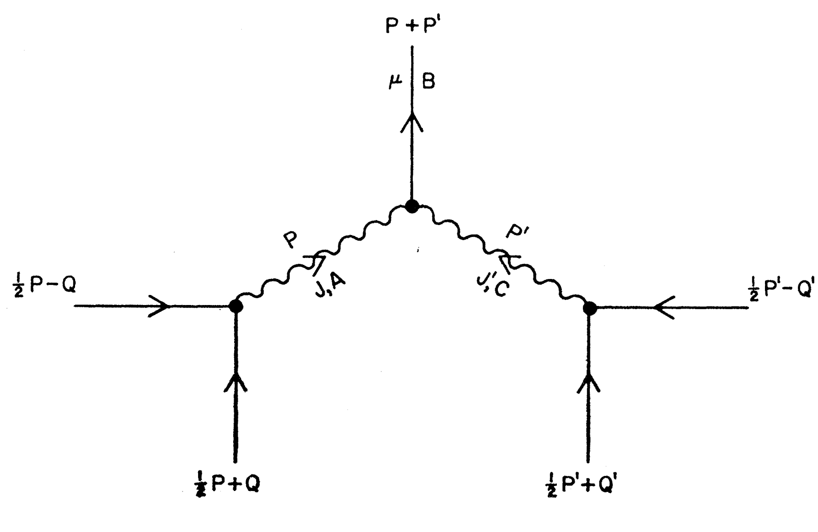

FIG. 3. Multiple Regge diagram.

For a fixed value of J and J', $J\geq J'$; the contribution of the graph given in Fig. 3 to the Feynman amplitude is

$$F_1(J,J')=-g(J,0)g(J,J')g(J'0)d(J,0)d(J,J')d(J',0)\times f_1[M_A{}^2(J)-t]^{-1}[M_C{}^2(J')-t']^{-1}, \quad (46)$$

where

$$f_1=(Q_\mu)^J\Gamma_{\mu;\nu}{}^J(M_A{}^2)(P_\nu'+\tfrac{1}{2}P_\nu)^{J-J'}\times\Gamma_{\nu;\sigma}{}^{J'}(M_C{}^2)(-Q_\sigma')^{J'}. \quad (47)$$

Using the derivative trick again, this product can be written as

$$f_1=\frac{\Gamma(J-J'+1)}{\Gamma(J+1)\Gamma(J'+1)}[(\partial_\nu')^{J'}H_J(-Q,P')]\times[(\partial_\nu)^{J'}H_{J'}(Q',P)]. \quad (48)$$

The advantage of this form is that it is a simple matter to expand the H_J functions and to discuss the problem of daughter trajectories. The presence of the projection operators $\bar{\theta}$ in H leads to daughters, just as in the case of a single Regge coupling. The interested reader can work out for himself the terms corresponding to the leading trajectory on one side of the diagram coupling to the leading daughter on the other. Because of the masses chosen in the model, the first daughter is down two units from the leading trajectory.

For large x and y the leading terms in Eq. (47) can easily be evaluated. Since the external incoming particles have equal mass, $Q\cdot P=Q'\cdot P'=0$, and one finds

$$f_1\simeq(Q\cdot P')^{J-J'}(-Q\cdot Q')^{J'}=x^J(-yw)^{J'}. \quad (49)$$

The contribution of the coupling for $J'>J$ will be called F_2, and it is the same as Eq. (46), except J and J' are interchanged in the g and d factors, and f_2 is given by

$$f_2=(-Q_\mu)^J\Gamma_{\mu;\nu}{}^J(M_A{}^2)(P_\nu+\tfrac{1}{2}P_\nu')^{J'-J}\times\Gamma_{\nu;\sigma}{}^{J'}(M_C{}^2)(Q_\sigma')^{J'}, \quad (50)$$

$$f_2\simeq(-xW)^Jy^{J'}.$$

The sum over J and J' can now be carried out, and the leading terms are of the multi-Regge form. Defining the trajectories by

$$M_A{}^2(\alpha)=t,\quad M_C{}^2(\alpha')=t', \quad (51)$$

the asymptotic amplitude is expressible in the form

$$F=\sum_{J,J'}[F_1(J,J')+F_2(J',J)],$$
$$F\simeq\frac{\pi^2g(\alpha,0)g(\alpha',0)}{\sin\pi\alpha\,\sin\pi\alpha'}\frac{d\alpha}{dt}\frac{d\alpha'}{dt'}H(x,y,W), \quad (52)$$

where

$$H(x,y,W)=d(\alpha,0)d(\alpha',0)x^\alpha y^{\alpha'}\times[g(\alpha,\alpha')d(\alpha,\alpha')W^{\alpha'}e^{-i\pi\alpha}+g(\alpha',\alpha)d(\alpha',\alpha)W^\alpha e^{-i\pi\alpha'}]. \quad (53)$$

This form has several amusing consequences which could be used as a test to see if minimal derivative coupling is the dominant coupling chosen by nature. One should note that by choosing a particular model we have definite predictions to be looked for in the experimental data. Perhaps the two most striking properties of Eq. (53) are the power dependence on W and the zeroes introduced by the $d(\alpha',\alpha)$ functions if $\alpha=\alpha'+1$.

If α and α' are positive, then the reaction should be depressed for small W, which occurs when $\hat{Q}$ and $\hat{Q}'$ are parallel. Therefore planar events are depressed. However, if α and α' are negative, which should happen for large enough values of t and t', planar events should dominate. A word of caution is appropriate here since in deriving H it was assumed that the quantity (xyW) was large, so that W cannot be allowed to get too small. Since x and y are large, this should be no real restriction

If $\alpha=\alpha'+1$, which could be achieved by an appropriate choice of t, t' and the trajectories (the Pomeranchuk and pion differ by almost one unit), then the second term in H vanishes by nature of the $d(\alpha',\alpha)$ function. In the first term one finds that the factors become

$$d(\alpha,0)d(\alpha-1,0)d(\alpha,\alpha-1)=2\alpha^2p(\alpha). \quad (54)$$

Thus the nonzero term vanishes very rapidly when α approaches zero. The particular power of α depends crucially on our choice of minimal derivative coupling, but the vanishing is a general phenomena since the coupling presumably cuts off the nonsense states in some way if α and α' have the same signature. This would be an interesting effect to study experimentally. In any event, the predicted rather striking dependence of the bi-Regge residue function on α and α' due to the Γ functions is particular to our model approach and would probably not follow from general kinematic arguments.

VII. MAXIMUM DERIVATIVE COUPLINGS

In this case, which is the opposite extreme from the previously discussed coupling, Eq. (35), the object is to couple all of the indices on the field tensors to derivatives of the other fields. Thus it is not necessary to introduce the $d(J,S)$ function to handle the nonsense states. The coupling is chosen to be

$$H_I=G(J,S)(p(J)p(S))^{1/2}A_\mu{}^J(-\overleftrightarrow{\partial}_\nu)^SB(\overleftrightarrow{\partial}_\mu)^JC_\nu{}^S. \quad (55)$$

The amplitude for the multi-Regge graph of Fig. 3 is

$$F(J,J')=-g(J,0)G(J,J')g(J',0)p(J)p(J')\times[M_A{}^2(J)-t]^{-1}[M_C{}^2(J')-t']^{-1}\times(Q_\mu)^J\Gamma_{\mu;\nu}{}^J(M_A{}^2)(-P_\nu'-\tfrac{1}{2}P_\nu)^J\times(Q_\lambda')^{J'}\Gamma_{\lambda;\sigma}{}^{J'}(M_C{}^2)(-P_\sigma-\tfrac{1}{2}P_\sigma')^{J'}. \quad (56)$$

The sum over J and J' can be carried out as before to yield the asymptotic behavior of the total amplitude F:

$$F\simeq-\frac{\pi^2g(\alpha,0)G(\alpha,\alpha')g(\alpha',0)}{\sin\pi\alpha\,\sin\pi\alpha'}\frac{d\alpha}{dt}\frac{d\alpha'}{dt'}p(\alpha)p(\alpha')x^\alpha y^{\alpha'}. \quad (57)$$

Thus the maximal derivative coupling is characterized by the fact that there is no dependence on W and by the fact that there is no necessity for the strong dependence on $(\alpha\pm\alpha')$ introduced by the Γ functions in the minimal-coupling case. However, factors of $p(\alpha)$ and $p(\alpha')$ are present, and it is amusing to note that for the case $\alpha=\alpha'+1$,

$$p(\alpha)p(\alpha-1)=\alpha p^2(\alpha)/(2\alpha+1)\,, \quad (58)$$

which vanishes linearly with α.

VIII. VECTOR-PARTICLE PRODUCTION

Let us now turn to the case in which the B particle has spin one. In the minimal-derivative case, the effective interaction will be written as

$$H_I=g(J,S)d(J,S)\left(\frac{J-S}{J+S+1}\right)A_\mu{}^J B_\mu{}^1(\overleftrightarrow{\partial}_\mu)^{J-S-1}C_\mu{}^S, \quad (59)$$

with a similar term involving the interchange of A and C. The amplitude in the multi-Regge case is similar to Eqs. (46) and (47), except that a factor of $-i(P_\nu'+\frac{1}{2}P_\nu)$ is replaced by the polarization vector $\epsilon_\nu{}^{M*}$ of the B particle. In the asymptotic limit the partial amplitude coming from the interaction of Eq. (59) is

$$F_1{}^M(J,J')=-g(J,0)g(J,J')g(J',0)\times d(J,0)d(J,J')d(J',0)\left(\frac{J-J'}{J+J'+1}\right)\times F_1{}^M[M_A{}^2(J)-t]^{-1}[M_C{}^2(J')-t']^{-1}, \quad (60)$$

where

$$F_1{}^M\simeq i\epsilon^*\cdot Q(Q\cdot P')^{J-J'-1}(-Q\cdot Q')^{J'}\simeq\left(\frac{i\epsilon^*\cdot Q}{P'\cdot Q}\right)F_1. \quad (61)$$

The total amplitude can be expressed in terms of the scalar B-particle amplitude F_1, given by Eq. (52) and the first term of Eq. (53):

$$F_1{}^M\simeq F_1[(\alpha-\alpha')/(\alpha+\alpha'+1)](i\epsilon^*\cdot Q/Q\cdot P')\,. \quad (62)$$

In this case we see that the minimal derivative coupling forces the amplitude to vanish if $\alpha=\alpha'$. This will also be true for the other coupling $F_2{}^M$. In addition, the polarization of the B particle is predicted. For t and t' small compared to s_i, the ratio of the amplitudes for transversely and longitudinally polarized B's is easily shown to be

$$F_1{}^1/F_1{}^0\simeq(-2\mu^2 t)^{1/2}/(\mu^2+t-t')$$

and

$$F_2{}^1/F_2{}^0\simeq(-2\mu^2 t')^{1/2}/(\mu^2+t'-t)\,. \quad (63)$$

Thus, for small t and t', the model predicts that the B's will be longitudinally polarized.

The maximal derivative coupling also yields an amplitude of the same form as the scalar case, but with an additional factor of $\epsilon^*\cdot\frac{1}{2}(P-P')$. This produces *only* longitudinally polarized B's.

It should be possible to test this behavior by looking at, for example, ρ^0 production, using the decay (charged) pions to measure the polarization.

Finally, we note that minimal derivative coupling will suppress the production of particles with spin due to the factor of $(\alpha-\alpha')$ which occurs in the residue. Thus the production of vector and tensor particles by the exchange of particles with similar trajectories will be down from the expected rate. An example of this effect could be the experimental lack of double Pomeranchuk exchange production of the f_0. There are many more possible reactions where this mechanism could play a role.

APPENDIX

Our conventions for determining the polarization vectors of the C particle when it has spin one are as follows. In the t-channel center-of-mass system the momentum of an incoming C particle is $\frac{1}{2}P-Q=(\frac{1}{2}\sqrt{t};\,-\mathbf{Q})$ and that of an outgoing C particle is $\frac{1}{2}P-Q'=(\frac{1}{2}\sqrt{t};\,-\mathbf{Q}')$. Taking the polar axis to be along $\mathbf{Q}$ and writing $q=|\mathbf{Q}|$, we have

$$Q=(0;\mathbf{Q})=(0;0,0,q)\,, \quad (A1)$$
$$Q'=(0;\mathbf{Q}')=(0;-q\sin\theta,0,q\cos\theta)\,,$$

where we have again assumed that the masses of the B and C particles are equal.

The polarization vectors of the incoming C's are now

$$\epsilon^1=-(\sqrt{\tfrac{1}{2}})(0;1,i,0)\,,$$
$$\epsilon^{-1}=(\sqrt{\tfrac{1}{2}})(0;1,-i,0)\,, \quad (A2)$$
$$\epsilon^0=\left(-\frac{q}{M_C};0,0,\frac{\sqrt{t}}{2M_C}\right),$$

with $\epsilon=(\epsilon_0;\boldsymbol{\epsilon})$. For the outgoing particles we have

$$\epsilon^1=-(\sqrt{\tfrac{1}{2}})(0;\cos\theta,i,\sin\theta)\,,$$
$$\epsilon^{-1}=(\sqrt{\tfrac{1}{2}})(0;\cos\theta,-i,\sin\theta)\,, \quad (A3)$$
$$\epsilon^0=\left(-\frac{q}{M_C};-\frac{\sqrt{t}}{2M_C}\sin\theta,0,\frac{\sqrt{t}}{2M_C}\cos\theta\right).$$

With these definitions, it is a simple matter to work out the results reported in the text.