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Covariant Angular Momentum Analysis and Analytic Continuation*

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The "little group" decomposition of a two-particle scattering amplitude is carried out using covariant variables throughout. The general covariant expression for the amplitude reproduces, where appropriate, the $O(3)$ and $O(2,1)$ expansions. In the limit of zero momentum transfer, for unequal masses, it goes over correctly to an expansion in the functions of the two-dimensional Euclidean group.

1. INTRODUCTION

A NUMBER of authors[1,2] have discussed the little group decomposition of a covariant two-particle scattering amplitude. For positive total effective square mass, the little group is $O(3)$ and the expansion is the standard one in terms of angular momentum. The fact that the decomposition is of physically observable states of two particles of positive energy, restricts the decomposition to unitary (square integrable) representations of the little group. This restriction leads physically to finite total cross sections.

For negative total effective square mass the little group is $O(2,1)$ and the corresponding expansion of the amplitude over square integrable functions reproduces the familiar background integral of Regge-pole analysis.

Previously these expansions have been made in the appropriate frames—either center-of-mass (c.m.) or brick-wall (B.W.)—which implies the use of noncovariant variables such as c.m. energy and scattering angle. In this paper we use covariant variables *throughout*, thus arriving at a generalized covariant angular momentum expansion for the scattering amplitude which is valid in any frame, both in the energy (positive-square-mass) and in the momentum transfer (negative-square-mass) regions. This is analogous, in the linear-momentum expression, to working with the Mandelstam variables. The main point which we make is that the correct variable to use is not the angular momentum, but the square of the Pauli-Lubanski vector.

For unequal particle masses, the little group for vanishing momentum transfer is the two-dimensional Euclidean group. We show that our general covariant expression correctly reproduces the appropriate expansion of the scattering amplitude. The limit is only well defined if it is approached from the negative-square-mass region.

For equal particle masses, the forward-scattering little group is the homogeneous Lorentz group. This clearly cannot be obtained as a limit from the much smaller little groups for nonzero total mass. A formalism which does give this limit will be developed in a separate paper. We note here that if the particle masses are set equal after the total mass has been taken to zero, then the formalism presented below is ambiguous; but it is simple and well defined if the limit is taken in the other order.

Of course, in the negative total square mass case one is considering the Poincaré decomposition of pseudostates of a particle of positive energy with one of negative energy. Since these are not physically observable, there is no reason why this expansion of the amplitude should be restricted to unitary representations. The phenomenon of Regge poles is precisely the appearance of nonunitary (non-square-integrable) representations in the expansion. In this preliminary paper we confine ourselves to setting up the covariant formalism, using only unitary representations. The appearance of Regge poles (nonunitary representations) and their behavior, with regard to crossing in the forward-scattering limit, will be considered in a separate paper.

2. THE TWO-PARTICLE S MATRIX

Our objective in the next two sections is to write in a covariant form the standard little group expansion of a two-particle scattering amplitude for positive total mass squared. In the c.m. frame this must reduce to the conventional angular-momentum expansion of helicity amplitudes.[3]

We first specify two-particle states in a Poincaré-invariant theory. The generators of the Poincaré group in conventional notation satisfy the commutation relations

$$[P_\mu,P_\nu]=0\,, \tag{2.1}$$

$$[P_\lambda,J_{\mu\nu}]=i(g_{\lambda\mu}P_\nu-g_{\lambda\nu}P_\mu)\,, \tag{2.2}$$

$$[J_{\mu\nu},J_{\rho\pi}]=i(g_{\mu\rho}J_{\nu\pi}+g_{\nu\pi}J_{\mu\rho}-g_{\mu\pi}J_{\nu\rho}-g_{\nu\rho}J_{\mu\pi})\,. \tag{2.3}$$

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[1] M. Toller and L. Sertorio, Nuovo Cimento **33**, 413 (1964); **37**, 631 (1965); CERN Report No. Th. 770, 1967 (unpublished); CERN Report No. 780, 1967 (unpublished); H. Joos, *Lectures in Physics* (University of Colorado Press, Boulder, Colo., 1965), Vol. 7A, p. 152; F. T. Hadjioannou, Nuovo Cimento **44**, 185 (1966).

[2] J. F. Boyce, R. Delbourgo, Abdus Salam, and J. Strathdee, Trieste Report No. IC/67/9 (unpublished).

[3] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

The generators of the little group of P_μ can be defined in terms of the three independent components of the Pauli-Lubanski vector[4]

$$W_\mu = -\tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}J^{\nu\lambda}P^\rho, \tag{2.4}$$

which satisfies

$$P^\mu W_\mu = 0, \tag{2.5}$$

and the commutation relations

$$[W_\mu, P_\nu] = 0, \tag{2.6}$$

$$[W_\mu, W_\nu] = i\epsilon_{\mu\nu\lambda\rho}W^\lambda P^\rho. \tag{2.7}$$

If we define

$$\tfrac{1}{2}\epsilon_{0ijk}J^{jk} = J_i \tag{2.8}$$

and

$$J_{0i} = K_i, \tag{2.9}$$

then

$$W_0 = \mathbf{J}\cdot\mathbf{P} \tag{2.10}$$

and

$$W_i = \epsilon_{ijk}K_jP_k + J_iP_0. \tag{2.11}$$

Single-particle states are specified in terms of the eigenvalues of the operators P^2, W^2, $\mathbf{P}$, and W_a, where W_a is some component of W_μ to be chosen later. The covariant normalization is[5]

$$\langle m^2, W^2, p_\mu, W_a | m^2, W^2, p_\mu', W_a'\rangle \Delta^+(p) = (2\pi)^4\delta^4(p_\mu - p_\mu')\delta(W_a - W_a'), \tag{2.12}$$

where

$$\Delta^+(p) = 2\pi\theta(p_0)\delta(p^2 - m^2). \tag{2.13}$$

The masses m_i^2 and spins W_i^2 of the particles (i is a particle label) are part of any representation and will not be repeated below. The remaining labels specifying a one-particle state, W_a and p_μ, subject to the condition

$$p^2 - m_i^2 = 0,$$

provide four independent parameters. A two-particle state can be expressed as the product of one-particle states, and therefore needs eight labels in addition to masses and spins. In place of the individual four-momenta p_μ^1 and p_μ^2, we introduce the total-momentum operator

$$P_\mu = p_\mu^1 + p_\mu^2 \tag{2.14}$$

and the relative-momentum operator

$$q_\mu = \tfrac{1}{2}(p_\mu^1 - p_\mu^2). \tag{2.15}$$

The normalization of the corresponding states is

$$\langle P_\mu', q_\mu', W_{1a}', W_{2a}' | P_\mu, q_\mu, W_{1a}, W_{2a}\rangle \Delta^+(p_1)\Delta^+(p_2) = (2\pi)^8\delta^4(P - P')\delta^4(q - q') \times \delta(W_{1a} - W_{1a}')\delta(W_{2a} - W_{2a}'). \tag{2.16}$$

It is convenient to think of all four components of P_μ and two of q_μ as being independent; the other two components of q_μ are determined by the two δ functions in the Δ^+ factors. We signify this by writing $P_\mu^{[4]}$, $q_\mu^{[2]}$. In the c.m. frame ($\mathbf{P}=0$), P_0 is the total c.m. energy, and the two independent components in $q_\mu^{[2]}$ fix the solid angle of the relative-momentum vector in the c.m. system.

If we introduce

$$P^2 \equiv t, \tag{2.17}$$

and

$$\Delta(t,1,2) = [t^2 + m_1^4 + m_2^4 - 2tm_1^2 - 2tm_2^2 - 2m_1^2m_2^2]^{1/2}, \tag{2.18}$$

the completeness relation (neglecting the spin factor) for two-particle states is

$$\begin{aligned} 1 &= |P,q\rangle \Delta^+(p_1)\Delta^+(p_2)\frac{d^4P}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\langle P,q| \\ &= |P,q^{[2]}\rangle \frac{d^4P}{(2\pi)^4}\frac{\Delta(t,1,2)}{(4\pi)^2 2t}d\Omega(q^{[2]})\langle P,q^{[2]}|. \end{aligned} \tag{2.19}$$

The labels W_{ia} ($i=1, 2$) are replaced by

$$\lambda^i = \frac{2W_\mu^iP^\mu}{\Delta(t,1,2)}\frac{t}{|t|}. \tag{2.20}$$

We shall refer to this covariant specification of particle spin component as covariant helicity. By (2.4), it is easily checked that this reduces to the conventional definition of helicity in the c.m. frame. (The factor $t/|t|$ is included in anticipation of the discussion of the negative-t case.)

The S-matrix element for scattering of particles 1 and 2 into 3 and 4 is

$$\langle P_\mu', q_\mu', \lambda^3, \lambda^4 | S | P_\mu, q_\mu, \lambda^1, \lambda^2\rangle, \tag{2.21}$$

where

$$P_\mu' = p_\mu^3 + p_\mu^4 \tag{2.22}$$

and

$$q_\mu' = \tfrac{1}{2}(p_\mu^3 - p_\mu^4). \tag{2.23}$$

Since S is invariant with respect to displacements (generated by P_μ), the S matrix must be diagonal in P_μ. Thus we write, as usual,

$$\langle P', q', \lambda^3, \lambda^4 | S | P, q, \lambda^1, \lambda^2\rangle \equiv i(2\pi)^4\delta^4(P - P') \times \langle P', q', \lambda^3, \lambda^4 | T(t) | P, q, \lambda^1, \lambda^2\rangle. \tag{2.24}$$

The whole discussion below will be applicable to some fixed value of $P^{[4]}$, which will be omitted from the states and completeness relations from now on [i.e., drop $(2\pi)^4\delta^4(P-P')$ in (2.16) and $d^4P/(2\pi)^4$ in (2.19)].

Following Jacob and Wick,[3] it is convenient to express the final state as the Lorentz transform (pure rotation in the c.m. frame) of a state with relative momentum in the same direction as the initial state. We denote such a state by

$$|q^{[2]}, \lambda^3, \lambda^4\rangle.$$

[4] We use the convention $X_\mu = (X_0, \mathbf{X})$ and $X^\mu = (X_0, -\mathbf{X})$, with $\mathbf{X} = (X_1, X_2, X_3)$ and $\epsilon_{0123} = +1$.

[5] The eigenvalues of the operator W_a take on discrete values (integer or half-integer), since W_a will refer to a rotation about some axis either fixed in space or relative to the particle direction. Thus the quantity $\delta(W_a - W_a')$ is really a Kronecker δ.

TABLE I. Evaluation of various quantities in the particular frames convenient for the different ranges of t.

General	c.m.	B.W.	Lightlike
	$t>0$	$t<0$	$t\to 0^-$
P_μ	$(\sqrt{t},0,0,0)$	$(0,0,0,\sqrt{\|t\|})$	$(\omega,0,0,\omega)$
q_μ	$(q_{C0},0,0q_{C3})$	$(q_{B0},0,0,q_{B3})$	$(q_{L0},0,0q_{L3})$
q_μ'	$(q_{C0}',q_{C1}',0,q_{C3}')$	$(q_{B0}',q_{B1}',0,q_{B3}')$	$(q_{L0}',q_{L1}',0,q_{L3}')$
	$q_{C3}=\dfrac{\Delta(t,1,2)}{2\sqrt{t}}$	$q_{B0}=\dfrac{\Delta(t,1,2)}{2\sqrt{\|t\|}}$	$\Delta(0,1,2)=(m_1^2-m_2^2)=2\omega(q_{L0}-q_{L3})$
α^μ	$(0,1,00)$	$(0,-1,00)$	$(0,-1,00)$
β^μ	$(0,0,1,0)$	$(0,0,-1,0)$	$(0,0,-1,0)$
γ^μ	$(0,0,0,1)$	$(1,0,0,0)$	null
$M_{\alpha,\beta,\gamma}$	(J_{23},J_{31},J_{12})	$(iJ_{02},-iJ_{01},J_{12})$	$(\Pi_2/\sqrt{t},\Pi_1/\sqrt{t},J_{12})$
W^2	$-t(J_{23}^2+J_{31}^2+J_{12}^2)$	$-t(-J_{02}^2-J_{01}^2+J_{12}^2)$	$-(\Pi_1^2+\Pi_2^2)$
χ	θ	$i\beta$	$-\zeta\sqrt{t}$
$U=\exp(-iM_\beta\chi)$	$\exp(-iJ_{31}\theta)$	$\exp(-iJ_{01}\beta)$	$\exp(-i\Pi_1\zeta)$
$\int_{C_\lambda}\cdots\dfrac{d(-W^2)}{2i\tan\pi(j-\|\lambda\|)}$	$\sum_{i\geq\|\lambda\|}$	$\int_{C_\lambda}\cdots\dfrac{d(-W^2)}{2i\tan\pi(j-\|\lambda\|)}$	$\int_0^\infty\cdots d(-W^2)$
$\lambda^{(i)}$	$J_{12}^{(i)}$	$J_{12}^{(i)}$	$J_{12}^{(i)}$

Since this has the same total four-momentum P_μ as $|q',\lambda^3,\lambda^4\rangle$, the two states are in general related by a transformation of the little group of P_μ. We define an orthogonal system of unit four-vectors[6]

$$n^\mu=P^\mu(\sqrt{|t|})/t, \tag{2.25}$$

$$\gamma^\mu=-2(P^2q^\mu-(P\cdot q)P^\mu)/(\Delta(t,1,2)\sqrt{|t|}), \tag{2.26}$$

$$\beta^\mu\sim\epsilon^{\mu\nu\pi\rho}q_\nu q_\pi' P_\rho, \tag{2.27}$$

$$\alpha^\mu\sim\epsilon^{\mu\nu\pi\rho}\beta_\nu\gamma_\pi P_\rho. \tag{2.28}$$

Note that

$$n^\mu n_\mu=-\gamma^\mu\gamma_\mu=t/|t|, \tag{2.29}$$

so that if P_μ is timelike, γ_μ is spacelike (and vice versa). However, since α_μ and β_μ are orthogonal to both n^μ and γ^μ, they are always spacelike. The generators of the little group of P_μ are

$$(M_\alpha,M_\beta,M_\gamma)\equiv(W_\alpha/\sqrt{t},W_\beta/\sqrt{t},W_\gamma/\sqrt{|t|}), \tag{2.30}$$

where

$$W_\alpha=W_\mu\alpha^\mu,\text{ etc.} \tag{2.31}$$

The vector β^μ is normal to q and q', so that[7]

$$|q',\lambda^3,\lambda^4\rangle=e^{iM_\beta\chi}|q^{[2]},\lambda^3,\lambda^4\rangle, \tag{2.32}$$

where χ is the angle between q and q' which is given by

$$\cos\chi=\frac{2st+t^2-(m_1^2+m_2^2+m_3^2+m_4^2)t+(m_1^2-m_2^2)(m_3^2-m_4^2)}{\Delta(t,1,2)\Delta(t,3,4)}, \tag{2.33}$$

where

$$s=(p_1-p_3)^2=(p_2-p_4)^2. \tag{2.34}$$

In the c.m. system ($\mathbf{P}=0$), χ is the angle θ between the three-vectors $\mathbf{q}$ and $\mathbf{q}'$. (See Table I.)

Thus

$$\langle q',\lambda^3,\lambda^4|T(t)|q,\lambda^1,\lambda^2\rangle =\langle q^{[2]},\lambda^3\lambda^4|e^{-iM_\beta\chi}T(t)|q^{[2]},\lambda^1,\lambda^2\rangle. \tag{2.35}$$

The little group decomposition of T is obtained by introducing a representation in this equation for the operator M_β, which diagonalizes W^2.

Before proceeding, we remark that $|q^{[2]},\lambda^1\lambda^2\rangle$ is an eigenstate of M_γ. This follows in general since (note that 1, 2 are particle labels)

$$W_\mu=-\tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}(J_1+J_2)^{\nu\lambda}(p_1+p_2)^\rho =W_\mu{}^1+W_\mu{}^2-\tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}(J_1{}^{\nu\lambda}p_2{}^\rho+J_2{}^{\nu\lambda}p_1{}^\rho). \tag{2.36}$$

Hence, after some elementary algebra and using the fact that P and q satisfy (2.14) and (2.15),

$$W_\mu q^\mu|q^{[2]},\lambda^1,\lambda^2\rangle=(-W_\mu{}^1+W_\mu{}^2)P^\mu|q^{[2]},\lambda^1,\lambda^2\rangle, \tag{2.37}$$

so that

$$M_\gamma|q^{[2]},\lambda^1,\lambda^2\rangle=(\lambda^1-\lambda^2)|q^{[2]},\lambda^1,\lambda^2\rangle. \tag{2.38}$$

[6] We have not bothered to normalize the right-hand sides of Eqs. (2.27) and (2.28) so that $\alpha^\mu\alpha_\mu=\beta^\mu\beta_\mu=-1$. It is not necessary for the later discussions.

[7] It must be emphasized that $e^{iM_\beta\chi}$ is indeed a little group transformation since $[M_\beta,P_\mu]=0$ and β_μ is a c number depending on q, q', and p, the eigenvalues associated with the incoming and outgoing states.

3. COVARIANT ANGULAR MOMENTUM DECOMPOSITION

An alternative complete commuting[8] set of operators for two-particle states which includes W^2 is

$$P_\mu^{[4]},\ W^2,\ M_\gamma,\ W_\mu{}^3P^\mu,\ W_\mu{}^4P^\mu. \tag{3.1}$$

As corresponding labels for the states we take[9] (again dropping P_μ)

$$|W^2,M,\lambda^3,\lambda^4\rangle. \tag{3.2}$$

To define the orthonormality and completeness in a manner which will continue analytically to negative values of t, we must treat W^2 (but not M) as a continuous complex variable. To this end we use the Sommerfeld-Watson technique. Introduce a continuous variable j corresponding to the conventional c.m. angular momentum,

$$-W^2\equiv +tj(j+1) \tag{3.3}$$

or, equivalently,

$$-W^2=-\tfrac{1}{4}t+t(j+\tfrac{1}{2})^2. \tag{3.4}$$

Consider the function

$$1/\tan\pi(j-|\lambda|), \tag{3.5}$$

where λ is an integer or half-integer. This function has a branch point in the $-W^2$ plane at $-\frac{1}{4}t$. The physical sheet of this function is defined as the cut plane with the cut running from $-\frac{1}{4}t$ to $(-t/|t|)\infty$. In the j plane the physical region is to the right of the line $\mathrm{Re}(j+\frac{1}{2})>0$.

The function (3.5) has poles on the real axis at points

$$0\leqslant j=n+|\lambda|, \tag{3.6}$$

where n is an integer. The completeness of states is given by

$$\sum_{M,\lambda^3,\lambda^4}\int_{C_\lambda}|W^2,M,\lambda^3,\lambda^4\rangle\frac{d(-W^2)}{2i\tan\pi(j-|\lambda_{34}|)}\times\langle W^2,M,\lambda^3,\lambda^4|=1, \tag{3.7}$$

where

$$\lambda_{34}=\lambda^3-\lambda^4, \tag{3.8}$$

and the contour C_λ is a loop about the real axis coming in from $+\infty$ above the axis and crossing the real axis between the points corresponding to $j=|\lambda_{34}|$ and $j=|\lambda_{34}|-1$. For positive t this contour encloses the poles at allowed (discrete) values of j, and the integral over $-W^2$ is equivalent to a discrete sum over the values of j allowed by the covariant helicities. The sum over M is taken over the values allowed by j. The normalization consistent with (3.7) is

$$\langle W'^2,M',\lambda_3',\lambda_4'|W^2,M,\lambda_3,\lambda_4\rangle =2i\tan\pi(j-|\lambda_{34}|)\,\delta(W^2-W'^2)\times\delta_{MM'}\delta_{\lambda_3\lambda_3'}\delta_{\lambda_4\lambda_4'}. \tag{3.9}$$

(It is tacitly assumed here, and below, that δ functions in W^2 are to be evaluated on the Sommerfeld-Watson contour C_λ.)

Using this representation, (2.32) can be rewritten as

$$\langle W^2,M,\lambda^3,\lambda^4|q',\lambda^3,\lambda^4\rangle =\int_{C_\lambda}\langle W^2,M,\lambda^3,\lambda^4|e^{-iM_\beta\chi}|W'^2,M',\lambda^3,\lambda^4\rangle \times\frac{d(-W'^2)}{2i\tan\pi(j-|\lambda_{34}|)}\langle W'^2,M',\lambda^3,\lambda^4|q^{[2]},\lambda^3,\lambda^4\rangle. \tag{3.10}$$

Define the functions $d_{MM'}{}^{W^2}(\chi)$ by the relation

$$\langle W^2,M,\lambda^3,\lambda^4|e^{-iM_\beta\chi}|W'^2,M',\lambda^3,\lambda^4\rangle \equiv 2i\tan\pi(j-|\lambda_{34}|)\,\delta(W^2-W'^2)d_{MM'}{}^{W^2}(\chi). \tag{3.11}$$

In the final factor, by (2.38) and (3.1), $M'=\lambda_{34}$, and we write

$$\langle W'^2,\lambda_{34},\lambda^3,\lambda^4|q^{[2]},\lambda^3,\lambda^4\rangle=C_{34}(t). \tag{3.12}$$

This is a normalization factor which can be chosen to make the conventional normalization of the $d_{MM'}{}^{W^2}(\chi)$ consistent with that already specified for the states. Then

$$d_{MM'}{}^{W^2}(0)=\delta_{MM'} \tag{3.13}$$

and

$$\langle W^2,M,\lambda^3,\lambda^4|q',\lambda^3,\lambda^4\rangle=d_{M,\lambda_{34}}{}^{W^2}(\chi)C_{34}(t). \tag{3.14}$$

We now define generalized partial-wave amplitudes by

$$\langle W^2,M,\lambda^3,\lambda^4|T(t)|W'^2,M',\lambda^1,\lambda^2\rangle =2i\tan\pi(j-|\lambda_{\min}|)\,\delta(W^2-W'^2)\delta_{MM'}\times\langle\lambda^3,\lambda^4|T(W^2,t)|\lambda^1,\lambda^2\rangle, \tag{3.15}$$

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[8] Yet another possibility is to take the complete commuting set

$$P^2,\ W^2,\ J^{\mu\nu}J_{\mu\nu},\quad \epsilon^{\mu\nu\pi\rho}J_{\mu\nu}J_{\pi\rho},\ J^2,\ J_3,\ \lambda^3,\ \lambda^4.$$

Eigenstates of this set of operators do not specify P_μ, but instead, through J^2 and J_3, pick eigenvalues of the angular momentum of the entire scattering system about some arbitrary origin. The internal angular momentum W^2 (which is the variable of interest) contributes to this in just the same way as the spin **s** of a single particle contributes to its total angular momentum **J**. The relation between the two is not simple and this appears to be a very awkward way of performing the W^2 decomposition.

An exceptional case is when $P_\mu=0$ in Eq. (3.1) (elastic forward scattering). Then the total angular momentum coincides with the internal angular momentum. Algebraically, $P_\mu=0$ is only three conditions, $\lambda^3=\lambda^4$. The extra parameters allowed can be taken as k_0, C and $M^2=J^2$, $M_\gamma=J_3$, so the two representations merge.

[9] It is worth noting (by examining Table I) what these states [Eq. (3.2)] and those defined in the previous section reduce to in the c.m. system. The incoming states $|P_\mu,q_\mu,\lambda^1,\lambda^2\rangle$ are the linear-momentum states in which the direction of q is chosen as the z axis and, of course, $\mathbf{P}=0$, and λ^1 and λ^2 are the helicities of the incoming particles. The outgoing states $|P_\mu,q_\mu',\lambda^3,\lambda^4\rangle$ also have $\mathbf{P}=0$ and for convenience q' is chosen to have an x and z component only; λ^3 and λ^4 are the helicities of the outgoing particles. The states defined by (3.2) reduce to angular momentum states in the c.m. system where $W^2\to J^2$ and $M_\gamma\to J_3$ (where the 3-axis is the direction of the incoming relative momentum q). This is of course consistent with the fact that the angular momentum commutes with the total momentum in the c.m. frame. That is, $[J_i,P_j]=0$ when acting on states with $\mathbf{P}=0$.

where $\lambda_{\min}$ is the least of λ_{12} and λ_{34}. This expresses the fact that $T(t)$ commutes with W^2 and cannot depend on the value of the generator M_γ.

Then from (2.35), using Eqs. (3.7), (3.11), (3.14), and (3.15), we have the generalized partial-wave expansion

$$\langle q',\lambda^3,\lambda^4|T(t)|q,\lambda^1,\lambda^2\rangle = C_{34}^*(t)C_{12}(t)$$
$$\times\int_C d_{\lambda_{34},\lambda_{12}}{}^{*W^2}(\chi)\langle\lambda^3,\lambda^4|T(W^2,t)|\lambda^1,\lambda^2\rangle \times\frac{d(-W^2)}{2i\tan\pi(j-|\lambda_{\max}|)}. \quad (3.16)$$

For conventional normalization[10] of $d_{MM'}{}^{W^2}(\chi)$,

$$C_{ij}(t)=[8\pi/\Delta(t,i,j)]^{1/2}. \quad (3.17)$$

Equation (3.16) is our main result. Before closing this section, we make some further remarks about the little group and the functions $d^{W^2}(\chi)$.

The little group generators, defined in Eq. (2.30), satisfy the commutation relations

$$[M_\alpha,M_\beta]=-i\epsilon_{\alpha\beta\gamma}M^\gamma \quad (3.18)$$

and cyclic permutations, where

$$\epsilon_{\alpha\beta\gamma}=+1, \quad (3.19)$$

and, as usual,

$$M^\gamma=M_\gamma \quad (\gamma^\mu \text{ timelike}),$$
$$M^\gamma=-M_\gamma \quad (\gamma^\mu \text{ spacelike}).$$

Further, since

$$n^\mu W_\mu=0, \quad (3.20)$$

we have

$$W^2=W_\alpha W^\alpha+W_\beta W^\beta+W_\gamma W^\gamma$$
$$=-t(M_\alpha^2+M_\beta^2+M_\gamma^2). \quad (3.21)$$

If we now consider the transformation in Eq. (2.35),

$$U(\chi)\equiv e^{iM_\beta\chi}, \quad (3.22)$$

then

$$U^{-1}M_\gamma U=M_\gamma\cos\chi+M_\alpha\sin\chi,$$

and by the usual manipulations[2] we have the operator equation

$$-U\frac{W^2}{t}=-\frac{d^2U}{d\chi^2}-\cot\chi\frac{dU}{d\chi}$$
$$+\frac{UM_\gamma^2-2\cos\chi\, M_\gamma UM_\gamma+M_\gamma^2U}{\sin^2\chi}. \quad (3.23)$$

Taking the appropriate matrix element,

$$\left[\frac{d^2}{d\chi^2}+\cot\chi\frac{d}{d\chi}-\frac{W^2}{t}-\frac{M^2-2MM'\cos\chi+M'^2}{\sin^2\chi}\right]$$
$$\times d_{MM'}{}^{W^2}(\chi)=0, \quad (3.24)$$

which is the differential equation for the functions $d_{MM'}{}^{W^2}(\chi)$.

4. ANALYTIC CONTINUATION

A. Spacelike Case

The expansion (3.16) for the scattering amplitude in terms of its little group decomposition for $t>0$ [little group $SO(3)$] clearly reproduces the conventional partial-wave expansion in terms of helicity amplitudes.

If $t<0$, by the substitution law we are considering the process

$$1+\bar{3}\to\bar{2}+4 \quad (4.1)$$

expanded in terms of the little group $SO(2,1)$ of imaginary mass states of particles 1 and 2, or 3 and 4. To make the analytic continuation, one can simply replace $\Delta^+(p)$ throughout by

$$\Delta(p)=2\pi\delta(p^2-m^2) \quad (4.2)$$

and interpret all variables carrying particle labels 2 and 4 as signifying the negative of their physical values. With this interpretation the entire discussion of the little group and the functions $d^{W^2}(\chi)$ remains valid. In the brick wall (B.W.) frame[11] the "relative" momenta q and q' are related by a Lorentz boost. [An operation in $O(2,1)$ with pure imaginary parameter $\chi=i\beta$ still given[12] by Eq. (2.33) (see Table I).]

The expansion (3.16) remains valid provided we assume that the singularities in the $-W^2$ plane come only from the kinematic factors. The contour C_λ in the $-W^2$ plane is held fixed as a loop about the real axis going off to $+\infty$, but owing to the change in sign of t, it now encloses the cut and the poles from $\tan\pi(j-|\lambda|)$ at those negative values of j corresponding to $|j|$ which were excluded in the c.m. case. The integral thus

[10] Evaluating Eqs. (2.19) and (3.9),

$$\langle W^2,M,\lambda_{34}|q^{[2]},\lambda_{34}\rangle\frac{\Delta(t,3,4)}{2(4\pi)^2t}d\Omega\langle q^{[2]},\lambda_{34}|W'^2,M',\lambda_{34}\rangle$$
$$=2i\tan\pi(j-|\lambda|)\,\delta(W^2-W'^2)\delta_{MM'},$$

and using (3.14) with this choice of $C(t)$ gives

$$\int d_{M\lambda}{}^{*W^2}(\chi)d_{M'\lambda}{}^{W'^2}(\chi)\frac{d\Omega(\chi)}{4\pi t}=2i\tan\pi(j-|\lambda|)\,\delta(W^2-W'^2)\delta_{MM'}.$$

Similarly, expanding

$$\langle q^{[2]},\lambda_{34}|q'^{[2]},\lambda_{34}\rangle=(4\pi)^2\frac{2t}{\Delta(t,3,4)}\delta(\cos\chi-\cos\chi')\delta(\phi-\phi')$$

and utilizing (3.7) gives

$$\int_{C_\lambda}d_{M\lambda}{}^{*W^2}(\chi)d_{M\lambda}{}^{W^2}(\chi')\frac{d(-W^2)/t}{2i\tan\pi(j-|\lambda|)}=2\delta(\cos\chi-\cos\chi').$$

[11] By the brick-wall (B.W.) frame we specifically mean that frame in which P_μ has only space components (in particular, only a 3-component, see Table I). In fact, this reduces to the conventionally defined brick-wall frame only in the equal-mass case.

[12] It is straightforward to show (using the coordinate system of Table I) that, indeed, $\tanh\beta=v$, where v is the magnitude of the velocity of the Lorentz transformation which takes q_μ into q_μ'.

incorporates the continuous and certain of the discrete unitary representations of $O(2,1)$ and reproduces the background integral of Regge-pole theory.

B. Lightlike Case

We restrict the discussion to the case in which the masses of the scattered particles are all unequal. From Eq. (2.33), it follows that[13]

$$\sin^2\chi=\frac{4t\phi(s,t,u)}{\Delta^2(t,1,2)\Delta^2(t,3,4)}, \tag{4.3}$$

where

$$u=\sum_i m_i^2-s-t\equiv K-s-t, \tag{4.4}$$

$$\phi(s,t,u)=stu-(as+bt+cu), \tag{4.5}$$

and, for example,

$$b=(m_1^2m_2^2-m_3^2m_4^2)(m_1^2+m_2^2-m_3^2-m_4^2)/K. \tag{4.6}$$

The boundary of all three physical regions in the s, t plane is

$$\phi(s,t,u)=0, \tag{4.7}$$

with ϕ positive inside the physical region. The line

$$t=0 \tag{4.8}$$

is an asymptote to the physical region, which cuts the boundary curve, Eq. (4.7), at

$$t=0,\quad s=cK/(c-a)\equiv s_t. \tag{4.9}$$

By considering (4.7) for large s and small $|t|$, it is easy to show that if

$$c-a\equiv -K(m_1^2-m_2^2)(m_3^2-m_4^2)>0, \tag{4.10}$$

the curve approaches the asymptote $t=0$ at large s for $t>0$, and hence, that $s=s_t$, $t=0$ lies on the boundary of the s-channel (*not* the u-channel) region. In this case, part of the asymptote, and also lines of constant t (>0) for the range

$$0<t<\min[(m_1-m_2)^2,(m_3-m_4)^2], \tag{4.11}$$

lie inside the s-channel region. We consider this situation. For $t=0$ ($P_\mu\neq 0$), the little group is the Euclidean group in 2-dimensions $SO(2)\sim T(2)$. The physical process is the same as that considered immediately above and the little-group expansion is given by the formalism, though the limit is most simply defined if it is taken from the negative-t region. We work in a frame in which

$$P_\mu=(\omega,0\,0,\omega). \tag{4.12}$$

Since ω defines a particular frame, it should not appear in any invariant expression. The little-group generators, Eq. (2.30), are then

$$(M_\alpha,M_\beta,M_\gamma)=(\Pi_2/\sqrt{t},\Pi_1/\sqrt{t},J_{12}), \tag{4.13}$$

where

$$\Pi_1=(J_{10}-J_{13})\omega,$$
$$\Pi_2=(J_{20}-J_{23})\omega. \tag{4.14}$$

Since the commutation relations (3.18) are insensitive to a common factor in Π_1 and Π_2, the limit is well defined and leads correctly to

$$[J_{12},\Pi_1]=i\Pi_2,$$
$$[J_{12},\Pi_2]=-i\Pi_1, \tag{4.15}$$
$$[\Pi_1,\Pi_2]=0.$$

Also, (3.21) gives

$$W^2=\lim_{t\to 0}\left(-(\Pi_1^2+\Pi_2^2)+(\sqrt{|t|})J_{12}^2\right)=-\Pi^2. \tag{4.16}$$

From Eq. (4.3) we have that for small t

$$\chi=-\zeta\sqrt{t}, \tag{4.17}$$

where

$$\zeta^2=\frac{4\phi(s,0)}{\Delta^2(0,1,2)\Delta^2(0,3,4)}$$
$$=\frac{-4(s-s_t)}{(m_1^2-m_2^2)(m_3^2-m_4^2)}. \tag{4.18}$$

Explicitly, by Eq. (4.10) this is positive inside the physical s-channel region. (This also follows quite generally from the fact that ϕ is positive inside any physical region.) The transformation in (2.35) is

$$\lim_{t\to 0} e^{-iM_\beta\chi}=e^{-i\Pi_1\zeta}. \tag{4.19}$$

Making the change of variable to ζ in (3.24) and taking the limit so that

$$\cos\chi\to 1,$$
$$\sin\chi\to -(\sqrt{t})\zeta, \tag{4.20}$$

[13] T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960). Note that we have *not* assumed any ordering of the masses and have taken the channels to be

$$s=\bar{3}1\to\bar{2}4;\quad t=12\to 34;\quad u=2\bar{3}\to\bar{1}4.$$

We are indebted to Dr. Hugh Jones and Dr. R. Delbourgo for discussions on relativistic kinematics.

reproduces the correct equation for the d functions[14]:

$$\left[\frac{d^2}{d\zeta^2}+\frac{1}{\zeta}\frac{d}{d\zeta}-W^2-(M-M')^2\right]D_{MM'}{}^{W^2}(\zeta)=0\,, \quad (4.21)$$

where

$$D_{MM'}{}^{W^2}(\zeta)=\lim_{t\to 0} d_{MM'}{}^{W^2}(\chi)\,. \quad (4.22)$$

Since by Eq. (4.16) $-W^2$ remains finite at $t\to 0^-$, from (3.3) we see that $j\to\pm i\infty$ as $-W^2\to+\infty$, depending on whether $(-W^2)$ is above or below the cut (which is now along the positive real axis). The corresponding limits of (3.5) are

$$\lim_{t\to 0^-} i\tan\pi j=\mp 1\,, \quad (4.23)$$

so the integral in the completeness relation is just along the real axis[15] and the corresponding normalization is

$$\langle W'^2,M',\lambda_3',\lambda_4'|W^2,M,\lambda_3,\lambda_4\rangle = \delta(W^2-W'^2)\delta_{MM'}\delta_{\lambda_3\lambda_3'}\delta_{\lambda_4\lambda_4'}\,. \quad (4.24)$$

[14] In the limit $t\to 0$,

$$\langle q^{[2]}|q'^{[2]}\rangle=\frac{(4\pi)^2}{\Delta(0,3,4)}\frac{2}{\zeta}\delta(\zeta-\zeta')\delta(\phi-\phi'),$$

$$\int|q^{[2]}\rangle\frac{\Delta(0,3,4)\zeta d\zeta d\phi}{2(4\pi)^2}\langle q^{[2]}|=1,$$

$$\langle W^2,M|W'^2,M'\rangle=\delta(W^2-W'^2)\delta_{MM'},$$

$$\sum_M\int_0^\infty|W^2,M\rangle d(-W^2)\langle W^2,M|=1.$$

Hence, using (4.22), we have

$$\int D^{*W^2}(\zeta)D^{W^2}(\zeta)\zeta d\zeta d\phi=4\pi\delta(W^2-W'^2),$$

$$\int D^{*W^2}(\zeta)D^{W^2}(\zeta)d(-W^2)=(2/\zeta)\delta(\zeta-\zeta').$$

[15] The contour in the $-W^2$ plane for $t>0$ is about poles. The distance of the first from the origin and the distance between adjacent poles are both of order tj. Since $tj(j+1)$ remains finite as $t\to 0$, in the limit $t\to 0^+$, this line of poles will coalesce to form a cut along the real axis. However, this limit of $\tan\pi j$ is not well defined.

One can evaluate the Sommerfeld-Watson integral for $t>0$ to give a sum over discrete values of j, and then take the above limit, which converts the sum back to an integral along the positive real axis. This also leads to Eq. (4.25).

Then (3.16) becomes

$$\langle q',\lambda^3,\lambda^4|T(0)|q,\lambda^1,\lambda^2\rangle=\frac{4(4\pi)}{|m_1^2-m_2^2|\,|m_3^2-m_4^2|}\times\int_0^\infty D_{\lambda_{34}\lambda_{12}}{}^{*W^2}(\zeta)\langle\lambda^3,\lambda^4|T(W^2)|\lambda^1,\lambda^2\rangle\times d(-W^2)\,. \quad (4.25)$$

All the above remarks (summarized in Table I) can be checked against the excellent review paper by Strathdee *et al.*[2]

5. CONCLUSIONS

The little group expansion of a two-particle scattering amplitude has been written using covariant variables throughout, so that the final expression, Eq. (3.16), is valid in any frame of reference. It emerges very clearly that, just as the Mandelstam variables t and s are appropriate for the general covariant form, it is the variable W^2 which plays the crucial role in the covariant little group expansion. It is also important that the spin be described covariantly in terms of covariant helicity, rather than the usual noncovariant helicity states.

A very important aspect of the formalism is that for unequal particle masses, it correctly reproduces the zero momentum transfer limit with little group (the Euclidean group) in two dimensions.

In this paper we have confined our attention to the unitary representations. Thus, we are not concerned with Regge poles. The formalism clearly provides a framework for a manifestly covariant Regge-pole theory which should facilitate a discussion of its crosssing properties and zero momentum transfer limit. These topics will be discussed in a separate paper.

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