Coulomb Interference Corrections in Potential Scattering*

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Corrections to the usual formula which approximates the total phase shift by the sum of the nuclear plus Coulomb phases are considered. The first correction to the phase is of order α (the same order as the Coulomb phase itself) and has been shown to be important in a recent analysis of pion-helium scattering. A convergent expansion in α is derived for the correction terms, and is explicitly exhibited through order α^3. The corrections are finite in each order; previously encountered logarithmic-type divergences are eliminated by a careful treatment of the asymptotic behavior of the Coulomb wave function.

I. INTRODUCTION

In extracting information on strong scattering forces from experimental data, it is usual to assume that any effects of Coulomb scattering can be taken into account by regarding the total phase shift to be given by the sum of the strong plus pure Coulomb phases. It is known, however, that this is only an approximation. Schiff[1] has considered the corrections theoretically and finds that a term of order α (the same order as the Coulomb phase itself) is to be included in the amplitude. Recently, Block[2] has pointed out that this term is important in pion-helium scattering. In fact, he shows that it is crucial if one is to obtain an estimate of the electromagnetic radius of the pion from the data.

Unfortunately, the expression which Schiff gave for the correction is logarithmically divergent. However, as Schiff points out,[3] this expression may still be used to calculate the cross section correctly to order α. On the other hand, by considering the phase shift rather than the total amplitude, Block[2] has removed this divergence by suitably subtracting Born-approximation Coulomb terms from Schiff's equation. The resulting expression leads to good agreement with the experimental data.

Although Block's equation is derived on the basis of very plausible physical arguments, a more fundamental treatment which allows calculation of higher-order terms in a systematic way is desirable. In this paper, we verify Block's conjectured form for the correction to order α and derive a scheme which leads to finite results in higher orders. We employ the Green's-function method, and although the technique is basically simple, the equations beyond first order are rather lengthy.

This problem has also been treated by Antoine[4] and more recently by West.[5] West finds corrections to the scattering amplitude directly, rather than to the phase shifts. It is not obvious that the results are equivalent to those in this paper, although we suspect that they must be. Our treatment is simpler than West's and shows explicitly where the difficulty in Schiff's earlier method arose, namely, in a careful definition of the phase shift for the Coulomb problem. Our method lacks the generality of West's discussion of various potentials, but we believe that it offers an approach which is more intuitive and simpler to apply in the case of Coulomb interference problems.

We begin in Sec. II by reviewing briefly the usual solution to the pure Coulomb-scattering problem. In Sec. III, we develop an iterative technique for calculating the pure Coulomb phase shifts in a convergent expansion in orders of α. The method of Sec. III is easily generalized to include strong plus Coulomb scattering. In Sec. IV, this method is developed, and the corrections through third order in α are explicitly exhibited. Appendices A and B treat details of the derivation to order α and α^3, respectively.

II. REVIEW OF PURE COULOMB SCATTERING

The reduced wave equation[6] for scattering of angular momentum l in a potential $V(r)$ is given by

$$u_l''+\left[k^2-v(r)-\frac{l(l+1)}{r^2}\right]u_l=0\,, \tag{1}$$

where k is the relative wave vector, and

$$v(r)=(2m/\hbar^2)V(r)\,, \tag{2}$$

where m is the reduced mass. The wave function itself is

$$\psi(r)=\sum_l\,[u_l(r)/r]P_l(\cos\theta)\,, \tag{3}$$

and for potentials which fall off faster than $(1/r)$, it goes asymptotically like

$$\psi(r)\sim e^{i\mathbf{k}\cdot\mathbf{r}}+[f(\theta)/r]e^{ikr}\,, \tag{4}$$

where $f(\theta)$ is the scattering amplitude. This leads to

$$ku_l(r)/(2l+1)\sim(i)^l e^{i\delta_l}\sin(kr-\tfrac{1}{2}l\pi+\delta_l)\,, \tag{5}$$

where

$$e^{i\delta_l}\sin\delta_l=\tfrac{1}{2}k\int f(\theta)P_l(\cos\theta)d(\cos\theta)\,. \tag{6}$$

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[1] L. I. Schiff, Progr. Theoret. Phys. (Kyoto) Suppl., Extra Issue, 400 (1965).

[2] M. M. Block, Phys. Letters **25B**, 604 (1967).

[3] L. I. Schiff, Progr. Theoret. Phys. (Kyoto) **37**, 635 (1967).

[4] J. P. Antoine, Nuovo Cimento **44**, 1068 (1966).

[5] G. B. West, J. Math. Phys. **8**, 942 (1967); Phys. Rev. **162**, 1677 (1967).

[6] See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., pp. 114–121.

However, in the Coulomb case where

$$v(r)=\frac{2m}{\hbar^2}\frac{Z_1Z_2e^2}{r}\equiv\frac{2\xi k}{r}, \tag{7}$$

the amplitude has an additional logarithmic dependence which is conventionally subtracted from the definition of the phase shift. Thus, for Coulomb scattering,

$$ku_l(r)/(2l+1)\sim(i)^l e^{ip(r)}\sin[kr-\tfrac{1}{2}l\pi+p(r)], \tag{8}$$

where

$$p(r)\sim\eta_l-\xi\ln 2kr. \tag{9}$$

In Eq. (9), η_l is defined to be the Coulomb phase shift and is given by

$$\eta_l=\arg[\Gamma(l+1+i\xi)]. \tag{10}$$

III. COULOMB PHASE SHIFTS FROM PERTURBATION THEORY

A straightforward approach to potential scattering is to write

$$u_l(r)=krj_l(kr)+\int_0^\infty G_l(r,r')v(r')u_l(r')dr', \tag{11}$$

where $G_l(r,r')$ is the Green's function for the $v=0$ case, with scattering boundary conditions[7]

$$G_l(r,r')=krj_l(kr_<)r_>[n_l(kr_>)-ij_l(kr_>)]. \tag{12}$$

Unfortunately, in the Coulomb case this equation does not have the correct asymptotic behavior. Both the incoming and outgoing waves must be modified by logarithmic phase factors. The equation can be used, however, if we are careful in handling the logarithmic divergences.

To treat the Coulomb potential, we define[8]

$$\begin{aligned} v_R(r)&=2\xi k/r, & r&<R \\ &=0, & r&>R. \end{aligned} \tag{13}$$

We consider the equation

$$u_R(r)=krj_l(kr)+\int_0^\infty G(r,r')v_R(r')u_R(r')dr', \tag{14}$$

with the phase shift given by

$$e^{i\delta}\sin\delta=-\int_0^\infty rj(kr)v_R(r)u_R(r)dr. \tag{15}$$

By matching the regular Coulomb solution at $r=R$ with the free solution for $r>R$, we can see that for large R the phase shift behaves like

$$\delta\sim\eta_l-\xi\ln 2kR+O(1/kR). \tag{16}$$

Expanding δ in a power series in ξ as

$$\begin{aligned}\delta&=\eta_l-\xi\ln 2kR\cong\xi(\eta_l'-\ln 2kR)+\xi^3(\eta_l'''/3!)+O(\xi^5)\\ &\equiv a\xi+c\xi^3+\cdots,\end{aligned}$$

we find

$$e^{i\delta}\sin\delta=a\xi+ia\xi^2+(c-\tfrac{2}{3}a^3)\xi^3+\cdots. \tag{17}$$

A term-by-term expansion of Eq. (14) used in Eq. (15) can now be compared with Eq. (17). For example, to lowest order, $u_R(r)$ is given simply by $krj(kr)$. Thus

$$a=\lim_{R\to\infty}-2\int_0^{kR}xj^2(x)dx+O(1/kR), \tag{18}$$

which is easily seen to be

$$a=\eta_l'-\ln 2kR+O(1/kR), \tag{19}$$

as expected. In Appendix A, we show that Eq. (19) follows from Eq. (18), and in Appendix B, the subtraction of divergences is carried out explicitly to order ξ^3. The coefficient c of Eq. (17) is given in Eq. (B5).

The essential element of this technique is to correctly define the phase shift relative to the logarithmic terms [Eq. (19)] in the limit $R\to\infty$. We generalize the method to include strong potentials in the next section.

IV. COULOMB CORRECTIONS TO STRONG SCATTERING

When the potential has a strong, short-range part plus the Coulomb potential, we consider

$$u_R(r)=krR_l(kr)+\int_0^\infty G_l^S(r,r')v_R(r')u_R(r') \tag{20}$$

instead of Eq. (14). Here again $v_R(r)$ is given by Eq. (13), but now

$$G_l^S(r,r')=kr_<R_l(kr_<)r_>[I_l(kr_>)-iR_l(kr_>)], \tag{21}$$

where R_l and I_l are the regular and irregular solutions of the strong-scattering problem ($v_R=0$), respectively. Asymptotically, they behave analogously to j_l and n_l:

$$\begin{aligned} R_l(kr)&\sim(1/kr)\sin(kr-\tfrac{1}{2}l\pi+\delta_S),\\ I_l(kr)&\sim-(1/kr)\cos(kr-\tfrac{1}{2}l\pi+\delta_S), \end{aligned} \tag{22}$$

where δ_S is the strong phase shift.

As in Eq. (16), the phase shift of u_R in Eq. (21) behaves like

$$\delta\sim\eta_l-\xi\ln 2kR+\Delta_l+O(1/kR) \tag{23}$$

for large kR and is given by

$$e^{i\delta}\sin\delta=-\int_0^\infty rR_l(kr)v_R(r)u_R(r)dr. \tag{24}$$

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[7] We normalize as in Ref. 6, pp. 77–79. See also Eq. (22) of this paper.

[8] The parameter R is introduced for formal convenience. We shall find results which are independent of R as $R\to\infty$. In what follows, we shall often omit the subscript l, but it is understood to index all quantities referring to a particular partial wave.

Δ_l is the correction to the phase shift due to interference of the Coulomb and the strong scattering which we wish to calculate. Again we expand δ in a power series, which yields

$$\delta \cong \xi(\eta_l' - \ln 2kR + \Delta_l') + \xi^2 \frac{\Delta_l''}{2!} + \xi^3 \frac{(\eta_l''' + \Delta_l''')}{3!} + \cdots$$
$$\equiv \bar{a}\xi + \bar{b}\xi^2 + \bar{c}\xi^3 + \cdots ,$$

and thus

$$e^{i\delta} \sin\delta \cong \bar{a}\xi + (\bar{b} + i\bar{a}^2)\xi^2 + (\bar{c} - \tfrac{2}{3}\bar{a}^3 + 2i\bar{a}\bar{b})\xi^3 + \cdots . \quad (25)$$

Note that we have kept a term of order ξ^2 in the expansion of Δ_l. We know that such a term is absent from η_l, and this can be shown explicitly by using the expressions in Sec. III. We have not been able to obtain an analogous result for Δ_l and cannot see any physical reason for it to be zero.

We can now proceed to subtract logarithmic infinites as in Sec. III. Using lowest order to illustrate this procedure, we find

$$\bar{a} = -2\int_0^{kR} x R_l^2(x) dx . \quad (26)$$

Comparing this with Eqs. (18) and (19) suggests writing

$$\bar{a} = -2\int_0^{kR} x[R_l^2(x) - j_l^2(x)]dx - 2\int_0^{kR} x j_l^2(x) dx .$$

Since the last term behaves like $a = \eta_l' - \ln 2kR$, we can identify

$$\Delta_l' = -2\int_0^{\infty} x[R_l^2(x) - j_l^2(x)]dx \quad (27)$$

from the definition of $\bar{a}$. This is exactly the result proposed by Block[2]: We have let $kR \to \infty$ because it is finite in this limit.

Continuing to order ξ^2, we can easily identify the $(i\bar{a}^2)$ term of Eq. (25) and find (after rearranging orders of integration)

$$\bar{b} = -8\int_0^{\infty} x R_l^2(x) dx \int_x^{\infty} y I_l(y) R_l(y) dy . \quad (28)$$

(Again, since this term is finite, we have taken kR to infinity.) As mentioned above, it is possible that this term is zero, but this has not been shown. The general procedure should now be clear. $\bar{c}$ is given by Eq. (B5), with j_l and n_l replaced by R_l and I_l, respectively.

To order ξ^3 in the interference terms, the total phase shift for the scattering is

$$\delta_T = \delta_S + \eta_l + \xi\Delta_l' + \xi^2(\Delta_l''/2!) + \xi^3(\Delta_l'''/3!) + \cdots , \quad (29)$$

where η_l is given by Eq. (10), Δ_l' by Eq. (27), $\Delta_l''/2!$ $(=\bar{b})$ by Eq. (28), and $\Delta_l'''/3!$ $(=\bar{c} - \eta_l'''/3!)$ by Eq. (B5). Note that the interference terms Δ_l are well behaved for large l. As is well known, we must keep all the Coulomb phases η_l because they diverge like $\xi \ln(l+1)$ for large l. It is easily seen that this is not the case for Δ_l. In fact, we expect it to vanish for large l at about the same rate as δ_S, the strong phase. This allows us to write the total amplitude as

$$(e^{2i\delta_T} - 1) = (e^{2i\eta_l} - 1) + e^{2i\eta_l}(e^{2i\delta_S} - 1) + e^{2i\eta_l} e^{2i\delta_S}(e^{2i\Delta_l} - 1) , \quad (30)$$

where we must keep all terms in the partial-wave sum [Eq. (6)] for the first term, but only a finite number need be retained in the last two. Thus Eq. (29) is of practical value in calculating Δ_l. For completeness, we note that with our normalization,

$$(1/2ik)\sum_l (2l+1)(e^{2i\eta_l} - 1)P_l(\cos\theta) = \frac{-\xi}{2k \sin^2(\theta/2)} e^{i\{2\eta_0 - \xi \ln[\sin^2(\theta/2)]\}} . \quad (31)$$

From the point of view of fitting data, the parameter ξ is small enough in current experiments to allow one to neglect all but the first three terms in Eq. (29). (The ξ^2 correction term is of interest in very-low-momentum experiments.) In applying this analysis to pion-helium scattering, Block and Koetke[9] have found that the part of the total amplitude due to the interference phase may be as much as three times larger than the pure Coulomb amplitude for scattering in the backward hemisphere. This corresponds to Δ_l on the order of 5% of the strong phase δ_S. Of even more interest is the case when a strong, complex optical potential is used. The interference phases develop complex parts and lead to a 6–7% difference in the $\pi^{\pm}$-He inelastic cross sections.

Note added in proof. After this paper was prepared, it was brought to the authors attention that a result equivalent to the first-order correction in Eq. (27) has been obtained previously by H. J. Schnitzer, Nuovo Cimento **28**, 752 (1963). Although Schnitzer's result is defined in terms of an infinite limit, it can be seen to be equivalent to Eq. (27) by using our Eq. (A1).

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APPENDIX A

We shall show that for large P

$$2\int_0^P x j_l^2 dx = \ln 2P - \eta_l' + O(1/P) . \quad (A1)$$

[9] M. M. Block and D. Koetke (private communication).

First, note that from Eq. (10)

$$\eta_l' = \Gamma'(l+1)/\Gamma(l+1) = -\gamma + 1 + \tfrac{1}{2} + \cdots + 1/l, \quad \text{(A2)}$$

where γ is Euler's constant. It is easy to integrate j_0 and j_1 directly to obtain this result for $l=0$ and $l=1$. We shall therefore derive a recurrence formula for higher l values.

Using the standard recursion relation for j_l

$$j_l(x) = [(2l-1)/x] j_{l-1}(x) - j_{l-2}, \quad \text{(A3)}$$

we obtain

$$2\int_0^P xj_l^2(x)dx = 2\int_0^P xj_{l-2}^2(x)dx + 2(2l-1)\int_0^P [j_l(x) - j_{l-2}(x)] j_{l-1}(x)dx. \quad \text{(A4)}$$

The second integral may be evaluated directly by using the indefinite integral

$$\int j_p(x) j_q(x) dx = \frac{x^2[j_{p-1}(x)j_q(x) - j_p(x)j_{q-1}(x)]}{p(p+1) - q(q+1)} - \frac{xj_p(x)j_q(x)}{p+q+1}. \quad \text{(A5)}$$

Equation (A4) becomes

$$2\int_0^P xj_l^2(x)dx = 2\int_0^P xj_{l-2}^2(x) - 1/(l-1) - 1/l, \quad \text{(A6)}$$

which is the desired recursion relation.

APPENDIX B

By expanding Eq. (14) to second order in ξ^2 and using this in Eq. (15), we obtain directly [by comparing with Eq. (17)]

$$\begin{aligned}(c - \tfrac{2}{3}a^3) = {} & 8\left[\int_0^{kR} xj_l^2(x)dx\right]^3 - 8\int_0^{kR} xj_l^2(x)dx \int_x^{kR} yn_l^2(y)dy \int_0^y zj_l^2(z)dz \\ & - 8\int_0^{kR} xj_l^2(x)dx \int_x^{kR} yj_l(y)n_l(y)dy \int_y^{kR} zj_l(z)n_l(z)dz - 8\int_0^{kR} xj_l(x)n_l(x)dx \int_0^x yj_l^2(y)dy \int_0^y zj_l(z)n_l(z)dz \\ & - 8\int_0^{kR} xj_l(x)n_l(x)dx \int_0^x yj_l(y)n_l(y)dy \int_0^y zj_l^2(z)dz. \quad \text{(B1)}\end{aligned}$$

[Note that the term $(ia\xi^2)$ in Eq. (17) appears trivially in our expansion, and thus we do not discuss it here.] By inspection, the last three terms in Eq. (B1) are finite as kR gets large, but the first two diverge. By interchanging orders of integration, the sum of the last three terms can be written

$$-16\int_0^{kR} xj_l^2(x)dx\left[\int_x^{kR} yj_l(y)n_l(y)dy\right]^2, \quad \text{(B2)}$$

and the second term as

$$-8\int_0^{kR} xn_l^2(x)dx\left[\int_0^x yj_l^2(y)dy\right]^2. \quad \text{(B3)}$$

The first term in Eq. (B1) is simply $(-a^3)$, since (see Appendix A)

$$\int_0^{kR} xj_l^2(x)dx = -\tfrac{1}{2}(\eta_l' - \ln 2kR) + O(1/kR). \quad \text{(B4)}$$

Thus, by using Eqs. (B2) and (B3) in comparison with Eq. (B1), we find

$$c = -8\left\{\int_0^{kR} xn_l^2(x)dx\left[\int_0^x yj_l^2(y)dy\right]^2 - \tfrac{1}{3}\left[\int_0^{kR} xj_l^2(x)dx\right]^3\right\} - 16\int_0^\infty xj^2(x)dx \times \left[\int_x^\infty yj_l(y)n_l(y)dy\right]^2. \quad \text{(B5)}$$

The limit $kR \to \infty$ has been taken in the second term. The first term is also easily seen to be finite in this limit by considering its derivative with respect to (kR),

$$-8[kRn_l^2(kR) - kRj_l^2(kR)]\left[\int_0^{kR} xj_l^2(x)dx\right]^2.$$

For large kR this becomes proportional to

$$\frac{\cos kR \ln^2 kR}{kR} + \cdots.$$

Since this is integrable for large (kR), we see that the expression for c as given in Eq. (B5) is finite as required.