Distribution of Vortices in Rotating Helium II

DIETRICH STAUFFER
Max Planck-Institut für Physik und Astrophysik, Munich, Germany

AND

ALEXANDER L. FETTER*
Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California

(Received 16 October 1967)

The distribution of vortices in a rotating cylinder or annulus filled with helium II is studied by two distinct methods. In the continuum approximation, a vortex-free strip of width $\approx 1.4n^{-1/2}$ occurs near each wall, where n is the vortex density. Exact computer calculations show that the vortices in a cylinder tend to form concentric circles about the origin. Only some of these patterns display triangular symmetry. In consequence, any experiment relying on the spatial periodicity of the vortex array is unlikely to give consistent results.

I. INTRODUCTION

THE Onsager-Feynman[1] model of rotating helium II predicts that the superfluid contains an array of rectilinear vortices, each with circulation $\kappa=h/m \approx 10^{-3}$ cm^2 sec^{-1}. In the continuum approximation, the vortices are uniformly distributed[2] with a density $n=2\Omega/\kappa$, where Ω is the angular velocity of the container. Extensions of the continuum approximation have suggested the possibility of vortex-free regions, both near the walls[3–6] and about the central vortex.[7] Previous studies have neglected the discrete nature of the vortex array, except for infinite systems, where the vortices form a triangular lattice.[8–12] The present paper reports a two-dimensional calculation for finite systems, including the effect of image vortices. Section II treats the continuum approximation, while the lattice structure is discussed in Sec. III.

II. CONTINUUM APPROXIMATION

An equilibrium distribution of vortices in a rotating container necessarily represents a stationary value of the free energy $F=E-M\Omega$,[13] where E and M are the energy and angular momentum of the liquid. Since the final expressions are independent of the fluid density ρ, we shall set $\rho_s=\rho[T=0]$ in describing helium II. Furthermore, all calculations refer to a unit length along the axis of rotation. In the limit of many vortices, the free energy is given approximately as

$$F=F_{\mathrm{cl}}+An(\rho\kappa^2/4\pi)\ln(b/a)\,, \tag{1}$$

where A is the area occupied by the vortices with density n, a is the effective core radius, and b is a length comparable with the mean vortex spacing $s\equiv n^{-1/2}$. The first term F_{cl} is the free energy of a classical liquid whose velocity field is equal to the mean velocity $\bar{v}$ of the vortex system averaged over regions containing many vortices. The second term represents the self-energy of the vortices and vanishes in the classical limit ($\kappa\rightarrow 0$, $n\kappa=$const). In previous papers,[2,14] one of us has criticized the form of Eq. (1) as unproved, but Tkachenko[8] has constructed a rigorous derivation for an infinite triangular lattice, where $b=0.27s$ [because $\ln(0.27)=-4.15/\pi$]. Furthermore, b seems to be very insensitive to the precise lattice structure, so that Eq. (1) is presumably generally valid in the limit of high vortex density.

As a first application of this continuum approximation, consider the vortex-free region near the wall of a rotating cylinder of radius R. We assume that the vortices are confined to a circle of radius $R_a\leq R$. Since the vortices must move with the velocity $\mathbf{\Omega}\times\mathbf{r}$,[13] the vortex density is $n=2\Omega/\kappa$ for $r\leq R_a$. It follows that the mean velocity is $\bar{v}(r)=\Omega r$ for $r\leq R_a$, and $\bar{v}(r)=\Omega R_a^2/r$ for $R_a\leq r\leq R$. The calculation of F is straightforward, and the equilibrium condition $dF/dR_a=0$ yields

$$\begin{aligned}(R-R_a)/s&=[(2\pi)^{-1}\ln(b/a)]^{1/2}+O(s/R)\\&\approx 1.4.\end{aligned} \tag{2}$$

In obtaining Eq. (2), we assume $s\ll R$ and use the expansion $\ln(1+x)\approx x-\frac{1}{2}x^2$. If N denotes the actual

* Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under AFOSR Contract No. AF49(638)1389.

[1] L. Onsager, Nuovo Cimento **6**, Suppl. 2, 249 (1949); R. P. Feynman, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Co., Amsterdam, 1955), Vol. I, p. 17.
[2] A. L. Fetter, Phys. Rev. **152**, 183 (1966).
[3] H. E. Hall, Advan. Phys. **9**, 89 (1960); see p. 102.
[4] P. J. Bendt and T. A. Oliphant, Phys. Rev. Letters **6**, 213 (1961).
[5] M. P. Kemoklidze and I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. **46**, 1677 (1964) [English transl.: Soviet Phys.—JETP **19**, 1134 (1964)].
[6] I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (W. A. Benjamin, Inc., New York, 1965), p. 97.
[7] M. P. Kemoklidze and Yu. G. Mamaladze, Zh. Eksperim. i Teor. Fiz. **46**, 165 (1964) [English transl.: Soviet Phys.—JETP **19**, 118 (1964)].
[8] V. K. Tkachenko, Zh. Eksperim. i Teor. Fiz. **49**, 1875 (1965) [English transl.: Soviet Phys.—JETP **22**, 1282 (1966)].
[9] V. K. Tkachenko, Zh. Eksperim. i Teor. Fiz. **50**, 1573 (1966) [English transl.: Soviet Phys.—JETP **23**, 1049 (1966)].
[10] A. L. Fetter, Phys. Rev. **162**, 143 (1967).
[11] L. Reatto, Phys. Rev. (to be published).
[12] D. Stauffer, Phys. Letters **24A**, 72 (1967); **25A**, 540 (1967); note the footnote in the second paper.
[13] G. B. Hess, Phys. Rev. **161**, 189 (1967).
[14] A. L. Fetter, Phys. Rev. **153**, 285 (1967).

number of vortices and N_0 $(=2\pi R^2\Omega/\kappa)$ the number that would fill the cylinder at the same density, Eq. (2) predicts

$$\begin{aligned} N &= N_0\{1-[2\ln(b/a)]^{1/2}N_0^{-1/2}\} \\ &\approx N_0\{1-5N_0^{-1/2}\} \end{aligned} \tag{3}$$

as $N_0\to\infty$, which agrees with Hall's[3] expression, but differs from Khalatnikov's[6] by a numerical factor.

Equation (1) also applies to a large number of vortices in a rotating annulus (inner radius R_1, outer radius R_2, $s\ll R_1$, $s\ll d\equiv R_2-R_1$). We assume a circulation Γ about the inner cylinder and a vortex-free region $R_1\le r\le R_i$, surrounded by vortices at the equilibrium density $n=2\Omega/\kappa$. The mean velocity is $\bar{v}(r)=\Gamma/2\pi r$ for $R_1\le r\le R_i$, and $\bar{v}(r)=(\Gamma/2\pi r)+\Omega(r^2-R_i^2)/r$ for $R_i\le r\le R_2$. Equation (1) must now be minimized with respect to both parameters Γ and R_i. The condition $\partial F/\partial R_i=0$ implies

$$(R_i/s)^2=\pi^{-1}\{(\Gamma/\kappa)+\tfrac{1}{2}\ln(b/a)[\ln(R_2/R_i)]^{-1}\}, \tag{4a}$$

while $\partial F/\partial\Gamma=0$ implies

$$\Gamma=2\pi[\ln(R_2/R_1)]^{-1} \times\{\Omega R_i^2[\tfrac{1}{2}+\ln(R_2/R_i)]-\tfrac{1}{2}\Omega R_1^2\}. \tag{4b}$$

A combination of Eqs. (4a) and (4b) with the condition $s\ll R_1$ eventually yields

$$(R_i-R_1)/s=[(2\pi)^{-1}\ln(b/a)]^{1/2}+O(s/R_1), \tag{4c}$$

$$\Gamma=2\pi\Omega R_i^2\{1+O(s^2/R_1^2)\}. \tag{4d}$$

Equation (4d) shows that the mean velocity $\bar{v}(r)$ is equal to Ωr for $r\ge R_i$, as it must be to avoid frictional drag from the normal component. Our results agree with calculation 1 of Ref. 4, but disagree with calculation 2 of the same reference and with Ref. 5.

Near the outer wall of the annulus, the vortex-free region must be the same as that in a cylinder. Comparison of Eqs. (2) and (4c) then shows an identical vortex-free region at each side of the annulus and at the edge of the cylinder.

If Γ is set equal to κ, Eq. (4a) may also be used to study the vortex-free region around the central vortex in a cylinder. In this case, we find $R_i\approx 0.9s$, which is comparable with the nearest-neighbor distance $1.1s$ in a triangular lattice. This result means that there is no vortex-free region, in conflict with the conclusion of Ref. 7.

The source of the disagreement with previous work[4–7] is easily found. In these calculations,[4] the mean velocity was allowed to be discontinuous at $r=R_a$ or $r=R_i$. Such a velocity pattern can be produced only by an additional ring of vortices situated at the discontinuity, but the energy of these vortices [$(\rho\kappa^2/4\pi)\ln(b/a)$ per vortex] was apparently omitted.[5] Note that our calculations increase the already considerable discrepancy between theory and Tsakadze's experiment.[15]

[15] D. S. Tsakadze, Zh. Eksperim. i Teor. Fiz. **46**, 505 (1964) [English transl.: Soviet Phys.—JETP **19**, 343 (1964)].

The derivation of Eq. (4) assumes both $s\ll R_1$ and $s\ll d\equiv R_2-R_1$, but the second restriction does not appear essential. If the circulation Γ achieves its equilibrium value, the critical angular velocity Ω_0 for the appearance of the first vortex in a narrow annulus[14] is very nearly equal to the angular velocity Ω_1, at which a row of vortices with spacing s appears in the center of the channel. Since there is a vortex-free region of width $\frac{1}{2}d$ on each side of the row, Ω_1 may be estimated from Eqs. (2) or (4c) to be

$$\Omega_1=(\kappa/\pi d^2)\ln(b/a). \tag{5}$$

Equation (5) reproduces the exact formula [Eq. (54) of Ref. 14] with logarithmic accuracy, so that the continuum approximation correctly describes both a single row of vortices and many rows. Thus it probably also remains valid for all intermediate vortex densities ($s\lesssim\frac{1}{2}d\ll R_1$). In particular, if Ω_2 denotes the critical angular velocity for the appearance of two rows of vortices a distance s apart, an elementary calculation yields

$$\Omega_2/\Omega_1\approx(2\times1.4+1)^2(2\times1.4)^{-2}\approx1.85.$$

This estimate agrees well with the recent experimental value of $\Omega''/\Omega'\approx1.9$ of Bendt and Donnelly.[16]

In contrast to the discussion given by Hall,[3] Hess,[13] and Andronikashvili and Mamaladze,[17] we do not ascribe the vortex-free regions to the presence of images, which were not explicitly included in any of the above calculations. Instead, the following explanation appears more reasonable: The equilibrium distribution of vortices represents a compromise between two opposing effects. In an attempt to minimize its free energy, the rotating helium II imitates solid-body rotation as closely as possible; however, the formation of each vortex requires a certain minimum energy. The omission of vortices near the center of a cylinder greatly alters the mean velocity, which renders such a configuration unfavorable. On the other hand, a vortex-free region near the outer wall reduces the free energy, while the solid-body rotation is essentially unaffected. In an annulus, vortices may also be omitted near the inner cylinder, provided that the change in the mean velocity is compensated by an increased circulation Γ. These arguments are independent of the shape of the boundaries, and we therefore postulate the following: In rotating helium II, a vortex-free strip of width $[(2\pi)^{-1}\ln(b/a)]^{1/2}s$ is formed near all boundaries whose radius of curvature is large compared with the mean vortex spacing s; the equilibrium circulation about each inner boundary is $2\Omega A$, where A is the sum of the area enclosed by the boundary and the area of the vortex-free region. Throughout the occupied region, the vortex density is

[16] P. J. Bendt and R. J. Donnelly, Phys. Rev. Letters **19**, 214 (1967).

[17] E. L. Andronikashvili and Yu. G. Mamaladze, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Co., Amsterdam, 1967), Vol. V, pp. 104–105.

$n=2\Omega/\kappa$. This postulate contains the lowest-order quantum corrections to the classical results of Ref. 2.

III. VORTEX LATTICES

The precise arrangement of vortices in the occupied regions cannot be determined in the continuum approximation. Since this question is of interest, we have computed numerically the free energy of various equilibrium states in a rotating cylinder. Let $\mathbf{r}_j$ be the position of the jth vortex. In equilibrium, the fluid velocity at $\mathbf{r}_j$ produced by the other vortices must be $\mathbf{\Omega}\times\mathbf{r}_j$,[13] because this is the self-consistent velocity of the jth vortex. Such an equilibrium configuration may be obtained from a suitable initial distribution of vortices by an iteration procedure analogous to the Gauss-Seidel single-step algorithm for linear problems:

$$x_j=v_{yj}/\Omega\,,\qquad y_j=-v_{xj}/\Omega\,. \tag{6}$$

Here $\mathbf{r}_j=(x_j,y_j)$ is the new site of the jth vortex, and $\mathbf{v}_j=(v_{xj},v_{yj})$ is the velocity produced at the old site by all the other vortices. The kth vortex contributes a velocity field $\mathbf{v}(\mathbf{r})=\boldsymbol{\kappa}\times(\mathbf{r}-\mathbf{r}_k)/2\pi|\mathbf{r}-\mathbf{r}_k|^2$. For each vortex, the effect of the solid boundary at $r=R$ is incorporated by introducing an image vortex with circulation $-\boldsymbol{\kappa}$ outside the cylinder at the position $\mathbf{r}_j(R/r_j)^2$. The velocity $\mathbf{v}_j$ is calculated by summing the velocity due to all other real vortices and all image vortices. In most cases, this iteration process converged, but the convergence was very slow for large vortex numbers.

These calculations have been carried out with up to 61 vortices. The free energy is calculated by numerical summation instead of from Eq. (1); the kinetic energy E is taken from Eq. (29) of Ref. 14, with $R/a=10^8$, while the angular momentum of the jth vortex is $\frac{1}{2}\rho\kappa(R^2-r_j^2)$.[13] For the small number of vortices considered here, the images scarcely affect the kinetic energy. The most striking result is that the vortices tend to form concentric circles about the center of the cylinder. Two typical equilibrium configurations are shown in Fig. 1, one with nearly perfect triangular symmetry and the other with no particular symmetry. In many cases, as in Fig. 1, a given number of vortices has more than one equilibrium configuration for a fixed angular velocity. Although it is hard to make a reliable estimate of the energy barrier separating these states, the difference between their free energy is small compared with the energy $\ln(b/a)$ (in units of $\rho\kappa^2/4\pi$) necessary to create a single vortex line. This means that the vortex array in helium should not contain large holes, in contrast to the observed flux-line lattice in type-II superconductors.[18] We infer from our calculations that a vortex array in rotating helium II can occur in a variety of states, some displaying triangular structure, others containing lattice defects that destroy the periodicity. For this reason, any experimental search for vortices in rotating helium relying on the presence of a perfect lattice (for example, Bragg scattering) is unlikely to succeed.

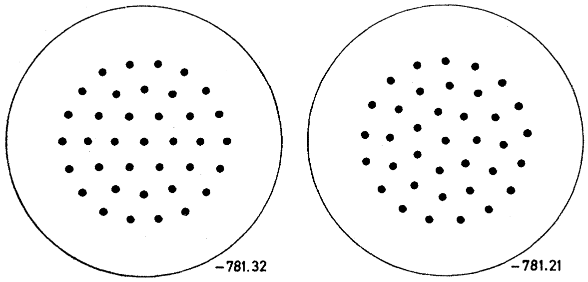

FIG. 1. Two stable configurations of 37 vortices with $s/R=0.2$. The left (right) arrangement was obtained by starting from a triangular (square) lattice and has a free energy of -781.32 (-781.21) in units of $\rho\kappa^2/4\pi$.

These numerical calculations provide a check on the accuracy of the vortex-free regions predicted in the continuum approximation ($s/R\to 0$). If R_a is taken as the distance of the outermost vortex from the center, we find

$$(R-R_a)/s=2.1 \quad \text{for} \quad s/R=0.2\,,$$
$$(R-R_a)/s=1.8 \quad \text{for} \quad s/R=0.1\,,$$

in qualitative agreement with Eq. (2). Furthermore, Fig. 1 is also typical in exhibiting a sharp boundary at R_a. Thus the first derivative of the mean velocity can be discontinuous, which contradicts an assertion in Refs. 5 and 6.

It is now interesting to examine the stability of these configurations with respect to small two-dimensional perturbations $\mathbf{r}_j\to\mathbf{r}_j+\mathbf{u}_j$. Since the quantum-mechanical and thermal oscillations are small,[10] we may use the harmonic approximation and neglect higher-order terms in $\mathbf{u}$. For N vortices, this procedure yields a real eigenvalue problem of dimension $2N$. The finite geometry precludes the use of plane waves to decouple the equations of motion; in the special case of sixfold rotational symmetry, however, introduction of Fourier components in the polar angle[19] leads to a considerable simplification. If the small displacement is written as $\mathbf{u}=\alpha\hat{r}+\beta\hat{\theta}$, where $\hat{r}$ and $\hat{\theta}$ are unit vectors in polar coordinates, then the substitution

$$\begin{Bmatrix}\alpha(r,\theta)\\ \beta(r,\theta)\end{Bmatrix}=\begin{Bmatrix}\alpha_k(r)\\ \beta_k(r)\end{Bmatrix}e^{ik\theta}\quad(k=0,1,\cdots,5) \tag{7}$$

reduces the eigenvalue problem to one involving a complex non-Hermitian matrix of dimension $\frac{1}{3}N$.[19] The matrix elements were calculated analytically, and the eigenvalues were obtained by a computer algorithm. For the favored configurations, the results show that

[18] U. Essmann and H. Träuble, Phys. Letters **24A**, 526 (1967); N. V. Sarma, *ibid.* **25A**, 315 (1967).

[19] T. H. Havelock, Phil. Mag. **11**, 617 (1931). In Eq. (25), a term $4n(p^n-1)^{-1}$ should be added to the expression for Q.

TABLE I. The logical relation between a local minimum of the free energy $E-M\Omega$ and the stability of a vortex system against small perturbations. The influence of mutual friction with the normal component is calculated as in Refs. 12 and 13. The shorter arrow means that the relation is assumed but unproved.

Without image vortices
(Energy minimum) $\rightleftarrows$ (Stability with friction) $\rightleftarrows$ (Stability without friction)
With image vortices
(Energy minimum) $\rightleftarrows$ (Stability with friction) $\longrightarrow$ (Stability without friction)

the image vortices have only a small influence on the eigenfrequencies. In practice, it is difficult to obtain a clear distinction between stable and unstable configurations. Some of the eigenvalues are very sensitive to unavoidable inaccuracies in the vortex positions, and even stable arrangements exhibit small imaginary (unstable) frequencies.

Fortunately, these vortex systems can be proved stable without recourse to eigenvalue calculations, because the iteration procedure of Eq. (6) necessarily yields a stable configuration. This result is easily proved with Hess's Hamiltonian formalism.[13] It is convenient to employ the rotating frame of reference, and we shall use capital letters to denote the corresponding components of position and velocity. Each vortex satisfies the following Hamiltonian equations:

$$\dot{X}_j=\partial H/\partial Y_j, \quad \dot{Y}_j=-\partial H/\partial X_j, \tag{8}$$

where $H=F/\rho\kappa$. If $X_j^{(n)}$ and $Y_j^{(n)}$ denote the coordinates after n iteration steps, Eq. (6) may be rewritten as

$$\begin{aligned} X_j^{(n+1)}-X_j^{(n)}&=\Omega^{-1}V_{yj}=-\Omega^{-1}\partial H/\partial X_j^{(n)}, \\ Y_j^{(n+1)}-Y_j^{(n)}&=-\Omega^{-1}V_{xj}=-\Omega^{-1}\partial H/\partial Y_j^{(n)}, \end{aligned} \tag{9}$$

which shows that each vortex is shifted in the direction of the negative gradient of free energy. If the iteration converges, the resulting configuration must be in a local minimum of free energy. Such a configuration is guaranteed to be stable: Any small displacement would produce dissipation from mutual friction with the normal component in helium II, but this is impossible at an energy minimum; Eq. (6) therefore leads to a stable configuration.

It is curious that the converse statement is untrue, at least if mutual friction is neglected: As shown below, there exist stable vortex configurations that cannot be reached by the iterative procedure, no matter how the initial configuration is chosen. This result clarifies the relation between a system of vortices and the corresponding Newtonian system in which Newton's second law replaces the "Magnus force" [Eq. (8)].[20] Stability in Newtonian dynamics necessarily implies a local minimum in the free energy, and it is tempting to impose the same condition on vortex dynamics. Indeed, in the absence of boundaries, any stable vortex system (finite or infinite) can be proved to occupy a local minimum. Nevertheless, the inclusion of images renders this assertion incorrect, as shown by the following simple example.[21] Consider three vortices symmetrically arranged on a circle of radius r in a cylinder of radius R. The stability of this configuration can be investigated analytically.[19] Without using a computer, it is easy to demonstrate that both the vortex system and the corresponding Newtonian system are unstable for $r/R=0.57$, both are stable for $r/R=0.55$, but for $r/R=0.56$, the vortex system is stable, while the Newtonian system is unstable. As additional confirmation, the iteration based on Eq. (6) diverged for $r/R=0.56$.

In the absence of mutual friction, we conclude that a stable vortex system need not occupy a local minimum of the free energy. Instead, the system presumably lies at a saddle point, where vortex dynamics confines the motion to stable regions, even though Newtonian dynamics would lead to an instability. Note, however, that an arbitrarily small frictional force probably renders a saddle point unstable in vortex dynamics, although an explicit proof has not been constructed. The relation between the stability of various vortex systems is shown schematically in Table I. For all practical situations (for example, $T>0$), we expect that an energy minimum is necessary as well as sufficient for a stable vortex array.

ACKNOWLEDGMENTS

We thank G. B. Hess for many helpful comments and suggestions. One of us (D.S.) is indebted to the Institut für Plasmaphysik for computing time and to the Max Planck-Institut for a grant.

[20] A. L. Fetter and P. C. Hohenberg, Phys. Rev. **159**, 330 (1967).

[21] A similar situation occurs for a single vortex in the center of a stationary cylinder, but we are interested in systems with many vortices.