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Statistical Mechanics of Rotating Quantum Liquids

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The statistical physics of rotating fluids is discussed with particular emphasis on the rules for taking the thermodynamic limit of infinite size. The thermodynamics of a rotating ideal Bose liquid is explicitly worked out. It is found that the rotating ideal Bose liquid undergoes a Bose-Einstein condensation; however, the first-order phase transitions predicted by Blatt and Butler are not present when the thermodynamic limit is properly taken.

1. INTRODUCTION

IN recent years, there has been a great deal of experimental and theoretical work done on rotating quantum liquids.[1] The simplest model available is the ideal quantum liquid, where the interaction between particles vanishes, but where quantum statistics are fully taken into account. In spite of its simplicity, there is in the literature some confusion about the properties of this model because the rules for taking the thermodynamic limit of infinite size have been incorrectly stated.[2] The purpose of this paper is (i) to establish the rules for taking the thermodynamic limit of infinite size in rotating systems, and (ii) to explicitly calculate the thermodynamic properties of the rotating ideal Bose liquid.

2. STATISTICAL THERMODYNAMICS

Consider a system of N particles in a cylindrical container of volume V and radius R. From symmetry considerations it is evident that the angular momentum about the cylinder axis is a good quantum number. If Γ is the degeneracy of eigenstates with sharp values for the energy E and angular momentum L, then the entropy associated with an equilibrium state of rotation is given by

$$S=k\ln\Gamma(E,L,V,N,R)\,, \tag{1}$$

where k is Boltzman's constant. In principle, Eq. (1) contains complete information about the thermodynamic properties of the system. The differential of Eq. (1) with R held constant is the fundamental thermodynamic relation

$$dE=TdS+\omega dL-PdV+\mu dN\,. \tag{2}$$

For a large class of systems (e.g., classical fluids and rigid bodies), the thermal parameter ω defined in Eq. (2) represents the angular velocity of rigid-body rotation. For quantum fluids, this is not necessarily so. A discussion of this important point may be found in Appendix A. It is physically evident that the parameters S, L, V, N, and E scale extensively if the radius R is fixed. We can therefore supplement the thermodynamic relation (2) with a scaling law

$$\alpha E=E(\alpha S,\alpha L,\alpha V,\alpha N,R) \tag{3}$$

and its associated Euler equation

$$E=TS+\omega L-PV+\mu N\,. \tag{4}$$

The differential of Eq. (4) can be combined with Eq. (2) to yield the Gibbs-Duhem equation for rotating systems

$$d\mu=-sdT+vdP-ld\omega\,, \tag{5}$$

where s, v, and l represent, respectively, the entropy, volume, and angular momentum per particle. The Gibbs-Duhem equation is a convenient starting point for thermodynamic considerations (e.g., the thermodynamics of phase transitions).

Note that rotating fluids have inhomogeneous equilibrium states. For example, the thermodynamic density N/V is the spatial average of an inhomogeneous density function $n(\mathbf{r})$ which varies with the distance from the cylinder axis. Similarly, the pressure P defined in Eq. (2) is the surface average of the stresses on the ends of the cylinder. It is a property of inhomogeneous systems that the thermodynamic equations of state will often depend on the geometric shape of the system. For example, the angular momentum of a fluid in a cylindrical container will differ from the angular momentum of a fluid in a spherical container even if N, V, T, and ω are identical for both systems. If $\omega=0$, then the system is homogeneous, and the thermodynamic parameters depend on N, V, and T, but not on the details of the geometric shape of the container. If $\omega\neq 0$, then the system is inhomogeneous, and each geometry is a separate problem. In this paper, the cylindrical geometry is chosen for two reasons: (i) It is the geometry commonly used by experimentalists; and (ii) the cylindrical geometry has a simple scaling law [Eq. (3)] and a correspondingly simple associated Gibbs-Duhem equation [Eq. (5)], which cannot be easily generalized to other geometries, and which simplify the thermodynamics of rotating systems.

This discussion has been based on the microcanonical ensemble where the energy and the angular momentum have sharp values. Concrete calculations are more commonly carried out in the canonical ensemble where the energy and the angular momentum are exponentially distributed. The rules of statistical physics for changing

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[1] A comprehensive review of the literature on this subject may be found in E. L. Andronikashvili and Yu. G. Mamaladze, Rev. Mod. Phys. **38**, 567 (1967).

[2] J. M. Blatt and S. T. Butler, Phys. Rev. **100**, 476 (1955).

ensembles yield the canonical density matrix[3]

$$\rho=Q_N^{-1}\exp[-(H-\omega L)/kT], \tag{6}$$

where the partition function Q_N is determined by normalizing the trace of the density matrix to unity,

$$Q_N=\mathrm{tr}\exp[-\beta(H-\omega L)], \quad \beta=(kT)^{-1}. \tag{7}$$

Once the partition function is known, the free energy can be calculated from

$$F=-kT\ln Q_N=E-TS-\omega L, \tag{8}$$

and the fundamental thermodynamic relation [Eq. (2)] becomes

$$dF=-SdT-Ld\omega-PdV+\mu dN. \tag{9}$$

It is well known that the various ensembles are equivalent only in the limit of large systems. Mathematically, we let $V\to\infty$ and $N\to\infty$, with N/V remaining finite. A superficial examination of the scaling law [Eq. (3)] might lead one to believe that the radius R should also remain finite as $V\to\infty$. This "long-tube" thermodynamic limit has been suggested by Blatt and Butler.[2] This limiting procedure is certainly incorrect, because the ratio of the surface area to the volume remains finite as $V\to\infty$. The reason why this is unacceptable (for the special case $\omega=0$) has been discussed by Fisher.[4] In particular, the ideal Bose liquid will not undergo a Bose-Einstein condensation in the "long-tube" thermodynamic limit even when $\omega=0$,[5] whereas it is well known that Bose-Einstein condensation does take place when the thermodynamic limit is properly taken.[6] The proper procedure is to allow $R\to\infty$, with the ratio R^3/V remaining finite. Having established that R must grow ever larger, a new problem presents itself. For finite ω, the particles near the surface of the container will have a velocity of order ωR.[7] The kinetic energy per particle will therefore diverge unless $\omega\to 0$ in such a way that ωR is kept finite. This added condition on ω completes the rules for taking the thermodynamic limit of infinite size for rotating systems in the canonical ensemble.[8]

3. IDEAL BOSE LIQUID

For an ideal Bose liquid, the operator $(H-\omega L)$ can be written in the second quantized form

$$H-\omega L=\int\psi^\dagger(\mathbf{r})(h-\omega l)\psi(\mathbf{r})d^3r, \tag{10}$$

where h and l are the differential operators

$$h=-(\hbar^2/2m)\nabla^2 \tag{11}$$

and

$$l=-i\hbar\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right). \tag{12}$$

The eigenfunctions

$$(h-\omega l)\psi_j(\mathbf{r})=E_j\psi_j(\mathbf{r}) \tag{13}$$

are a convenient set of basis functions for the occupation-number representation. It follows from Eqs. (10) and (13) that

$$H-\omega L=\sum_j a_j^\dagger a_j E_j, \tag{14}$$

where $a_j^\dagger$ and a_j are, respectively, the creation and destruction operators for the normalized one-particle state $\psi_j(\mathbf{r})$. The partition function will then have the form

$$Q_N=\sum_{\{n_j\}}{}'\exp(-\beta\sum_j n_jE_j), \tag{15}$$

where the prime on the summation restricts the total occupation number $\sum n_j$ to the value N. It is convenient to sum over the occupation number n_0 of the lowest eigenvalue E_0 after summing over all of the n_j, with $j\neq 0$. Therefore we write

$$Q_N=\sum_{n_0}K(N-n_0)\exp(-\beta n_0E_0), \tag{16}$$

with

$$K(N)=\sum_{\{n_j\}}{}'\exp(-\beta\sum_{j\neq 0}n_jE_j). \tag{17}$$

The generating function for $K(N)$ is defined by

$$\Psi(z)=\sum_N K(N)z^N. \tag{18}$$

If $\Psi(z)$ is known, then the coefficient $K(N)$ may be found by doing a contour integral

$$K(N)=(2\pi i)^{-1}\oint\Psi(z)z^{-(N+1)}dz \tag{19}$$

along a closed curve surrounding the origin. It is easily seen from Eqs. (17) and (18) that

$$\ln\Psi(z)=-\sum_{j\neq 0}\ln(1-ze^{-\beta E_j}), \tag{20}$$

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[3] It is often convenient to use $H'=H-\omega L$ as an effective Hamiltonian in the rotational canonical ensemble, just as it is often convenient to use $H'=H-\mu N$ as an effective Hamiltonian in the grand canonical ensemble. There is no deep physical significance in this fact. In particular, in deriving the rotational canonical ensemble, no effective Hamiltonians for rotating coordinate systems ever need be mentioned.

[4] M. E. Fisher, Arch. Ratl. Mech. Anal. **17**, 377 (1964).

[5] The reason that the "long-tube" thermodynamic limit does not produce a Bose-Einstein condensation is mathematically similar to the reason that a one-dimensional ideal Bose liquid does not produce a Bose-Einstein condensation.

[6] Most statistical-mechanics textbooks discuss Bose-Einstein condensation. A particularly clear discussion may be found in K. Huang, *Statistical Mechanics* (John Wiley & Sons, Inc., New York, 1963).

[7] These considerations will be especially clear if the special example of rigid-body rotation is kept in mind.

[8] If an ensemble where L is microcanonically distributed is used, then the rule becomes $L\to\infty$, with $(NR)^{-1}L$ kept finite.

which may be expanded in powers of z:

$$\ln\Psi(z)=\sum_{s=1} s^{-1}z^s a(s\beta)\,, \tag{21}$$

where

$$a(t)=\sum_{j\neq 0} \exp(-E_j t)\,. \tag{22}$$

The crux of the problem is to evaluate the sum on the right-hand side of Eq. (22) in the thermodynamic limit. This can be carried out by the method of Green's functions.

The retarded Green's function for the operator $(h-\omega l)$ may be defined as

$$g(\mathbf{r},\mathbf{r}',t)=\theta(t)\sum_j \exp(-E_j t)\psi_j(\mathbf{r})\psi_j^*(\mathbf{r}')\,, \tag{23}$$

where $\theta(t)$ is unity if $t>0$, and zero if $t<0$. Once g is known, then $a(t)$ can be found by integration:

$$a(t)=\int g(\mathbf{r},\mathbf{r},t)d^3r-\exp(-E_0 t)\,. \tag{24}$$

It follows from the completeness of the eigenfunctions [Eq. (13)] and the definition [Eq. (23)] that the retarded Green's function satisfies

$$[(\partial/\partial t)+h-\omega l]g(\mathbf{r},\mathbf{r}',t)=\delta(t)\delta(\mathbf{r}-\mathbf{r}')\,. \tag{25}$$

In order to solve this equation directly, we look for a solution of the form

$$g(\mathbf{r},\mathbf{r}',t)=\exp(\omega t l)g_0(\mathbf{r},\mathbf{r}',t)\,. \tag{26}$$

Since $[h,l]=0$, g_0 is the retarded Green's function of the diffusion equation

$$[(\partial/\partial t)+h]g_0(\mathbf{r},\mathbf{r}',t)=\delta(t)\delta(\mathbf{r}-\mathbf{r}')\,. \tag{27}$$

In the limit of large volume,

$$G_0(\mathbf{r},\mathbf{r}',t)=\lim_{V\to\infty} g_0(\mathbf{r},\mathbf{r}',t)\,, \tag{28}$$

it is well known[9] that the solution of Eq. (27) is

$$G_0(\mathbf{r},\mathbf{r}',t)=\theta(t)\left(\frac{m}{2\pi\hbar^2 t}\right)^{3/2}\exp\left[\frac{-m(\mathbf{r}-\mathbf{r}')^2}{2\hbar^2 t}\right]. \tag{29}$$

Furthermore, if we let

$$G(\mathbf{r},\mathbf{r}',t)=\lim_{V\to\infty} g(\mathbf{r},\mathbf{r}',t)\,, \tag{30}$$

then

$$G(\mathbf{r},\mathbf{r}',t)=\exp(\omega t l)G_0(\mathbf{r},\mathbf{r}',t)\,. \tag{31}$$

The exponential operator in the above equation is most easily examined if the position $\mathbf{r}$ is expressed in cylindrical coordinates (ρ,φ,z). The operator $\exp(\omega t l)$ replaces the real angle φ with the complex angle $\varphi-i\hbar\omega t$. In cylindrical coordinates,

$$(\mathbf{r}-\mathbf{r}')^2=(z-z')^2+\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')\,. \tag{32}$$

Therefore it follows from Eqs. (29) and (31) that

$$G=G_0\exp\gamma\,, \tag{33}$$

where

$$\gamma=(m\rho\rho'/\hbar^2 t)[\cos(\varphi-\varphi'-i\hbar\omega t)-\cos(\varphi-\varphi')]\,. \tag{34}$$

In particular,

$$G(\mathbf{r},\mathbf{r},t)=\theta(t)(m/2\pi\hbar^2 t)^{3/2}\exp(\sigma\rho^2/R^2)\,, \tag{35}$$

$$\sigma=(mR^2/\hbar^2 t)[\cosh(\hbar\omega t)-1]\,. \tag{36}$$

Combining Eqs. (24), (30), and (35), we find that as $V\longrightarrow\infty$,

$$V^{-1}a(t)\longrightarrow(m/2\pi\hbar^2 t)^{3/2}(1/\sigma)(e^{\sigma}-1)-V^{-1}e^{-E_0 t}\,. \tag{37}$$

At this point, it is crucial that $\omega\longrightarrow 0$ and $R\longrightarrow\infty$, with ωR finite. It is shown in Appendix B that E_0 will remain finite in this limit; therefore

$$\lim\sigma=\tfrac{1}{2}m\omega^2R^2t\,, \tag{38}$$

and

$$\lim V^{-1}a(t)=(m/2\pi\hbar^2 t)^{3/2}(2/m\omega^2R^2t)(e^{m\omega^2R^2t/2}-1)\,. \tag{39}$$

If we let

$$\chi(z)=\lim V^{-1}\ln\Psi(z)\,, \tag{40}$$

then it follows from Eqs. (21) and (39) that

$$\chi(z)=(\lambda^3\xi)^{-1}\sum_{s=1}^{\infty} s^{-(7/2)}(e^{s\xi}-1)z^s\,, \tag{41}$$

where

$$\lambda=(2\pi\hbar^2/mkT)^{1/2} \tag{42}$$

is the thermal wavelength, and

$$\xi=\tfrac{1}{2}\beta m\omega^2R^2\,. \tag{43}$$

The explicit expression for $\chi(z)$ may be used in the usual formalism of statistical physics to deduce the thermodynamic equations of state.

4. THERMODYNAMIC EQUATIONS OF STATE

The free energy per particle

$$f=-kT\lim N^{-1}\ln Q_N \tag{44}$$

as a function of n, ω, and T contains complete information about the thermodynamic properties of the system. If $Q(N,n_0)$ is defined as

$$Q(N,n_0)=K(N-n_0)\exp(-\beta n_0 E_0)\,, \tag{45}$$

then it follows from Eq. (16) that

$$\ln\max_{n_0} Q(N,n_0)\leq\ln Q_N\,, \tag{46a}$$

$$\ln Q_N\leq\ln N\max_{n_0} Q(N,n_0)\,. \tag{46b}$$

The above inequalities enable one to approximate the

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[9] R. Kubo, *Statistical Mechanics* (North-Holland Publishing Co., Amsterdam, 1965), p. 178.

partition function by the largest term in the sum [Eq. (16)]. The error made in the free energy per particle will be of order $N^{-1}\ln N$, which vanishes in the limit $N\rightarrow\infty$. Therefore we may write the free energy per particle as

$$f=\min_{0\leq\alpha\leq 1} f(\alpha), \tag{47}$$

where

$$f(\alpha)=-kT\lim N^{-1}\ln Q(N,\alpha N). \tag{48}$$

The behavior of $Q(N,n_0)$ in the thermodynamic limit may be found from Eqs. (19), (40), and (45):

$$Q(N,n_0)=(2\pi i)^{-1}\oint \exp V\Phi(N/V,n_0/N,z)dz, \tag{49}$$

where

$$\Phi(n,\alpha,z)=\chi(z)-n(1-\alpha)\ln z+n\alpha\xi. \tag{50}$$

In deriving Eq. (50) we have used the result (proved in Appendix B) that

$$\lim\beta E_0=-\tfrac{1}{2}\beta m\omega^2R^2\equiv-\xi. \tag{51}$$

Consider the analytic properties of $\Phi(n,\alpha,z)$ as a function of z for fixed values of n and α. Along the positive real z axis, $\chi(z)$ is monotonically increasing, while $-n(1-\alpha)\ln z$ is monotonically decreasing. Therefore we expect $\Phi(n,\alpha,z)$ to pass through a minimum for some value $z=\zeta$ on the real axis. A necessary condition on ζ is that

$$\Phi'(n,\alpha,\zeta)=0, \tag{52}$$

or

$$n(1-\alpha)=\zeta\chi'(\zeta). \tag{53}$$

It can be directly checked from Eq. (41) that the above equation has a solution if $n(1-\alpha)$ is less than the critical density

$$n_c(\omega,T)=(\lambda^3\xi)^{-1}\sum_{s=1}^{\infty}s^{-(5/2)}(1-e^{-s\xi}). \tag{54}$$

Furthermore, $\Phi''(n,\alpha,\zeta)>0$, which shows that ζ is at a minimum of Φ along the real axis. Therefore Eq. (49) may be evaluated by the method of "saddle-point" integration.[10] In particular,

$$\ln Q(N,n_0)=V\Phi(N/V,n_0/N,\zeta)+O(\ln V), \tag{55}$$

which, together with Eqs. (48) and (50), implies that

$$f(\alpha)=kT[(1-\alpha)\ln\zeta-\alpha\xi-n^{-1}\chi(\zeta)]. \tag{56}$$

The parameter ζ may be eliminated via Eq. (53). Physically, $f(\alpha)$ represents the free energy per particle associated with a fraction α of the particles in the Bose condensate $\psi_0(\mathbf{r})$. The actual fraction of particles in the condensate will be, according to Eq. (47), the fraction α_0 which minimizes the free energy per particle. There are two regions (i.e., phases) of interest. (i) If the density is less than the critical density [Eq. (54)], then $\alpha_0=0$, and there is no Bose-Einstein condensation. The free energy per particle is obtained from the parametric equations

$$f=kT\{\ln\zeta-n^{-1}\chi(\zeta)\}, \tag{57a}$$

$$n=\zeta\chi'(\zeta). \tag{57b}$$

(ii) If the density is greater than the critical density, then $\alpha_0=(n-n_c)/n$, and the free energy per particle is given by

$$f=-kT\{\xi+n^{-1}\chi(e^{-\xi})\}. \tag{58}$$

All of the thermodynamic equations of state for the two phases follow from Eqs. (41), (57), and (58). A useful set of functions for describing the thermodynamic properties is

$$g_k(z)=\sum_{s=1}^{\infty}s^{-k}z^s. \tag{59}$$

A partial list of important thermodynamic equations of state will be given below. For each equation, the upper expression refers to the normal phase ($n<n_c$), and the lower one refers to the condensation region ($n>n_c$):

$$\alpha_0=0 \tag{60a}$$

$$=1-(n\lambda^3\xi)^{-1}[g_{5/2}(1)-g_{5/2}(e^{-\xi})], \tag{60b}$$

$$\beta P=(\lambda^3\xi)^{-1}[g_{7/2}(e^{\xi}\zeta)-g_{7/2}(\zeta)] \tag{61a}$$

$$=(\lambda^3\xi)^{-1}[g_{7/2}(1)-g_{7/2}(e^{-\xi})], \tag{61b}$$

$$\mu=kT\ln\zeta \tag{62a}$$

$$=-kT\xi, \tag{62b}$$

$$l=(mR^2\omega/\lambda^3n\xi^2)[g_{5/2}(\zeta e^{\xi})\xi-g_{7/2}(\zeta e^{\xi})+g_{7/2}(\zeta)] \tag{63a}$$

$$\begin{aligned}&=mR^2\omega\{\alpha_0+(\lambda^3n\xi^2)^{-1}\\&\qquad\times[\xi g_{5/2}(1)-g_{7/2}(1)+g_{7/2}(e^{-\xi})]\}.\end{aligned} \tag{63b}$$

Other thermal equations of state can be derived from Eqs. (61) and (62) by eliminating the parameter ζ and using the Gibbs-Duhem relation [Eq. (5)]. Note that no qualitatively new phase transitions (e.g., the first-order phase transitions of Blatt and Butler[2]) arise when the ideal Bose liquid is set into rotational motion. There will of course be quantitative changes in the thermal properties of the system. For example, if the container has a low angular velocity, then the fraction of particles in the Bose condensate has the temperature dependence

$$\alpha_0(T)=1-(T/T_c)^{3/2}, \quad (n>n_c\ \ \xi\ll 1). \tag{64}$$

In the high angular velocity region, the exponent changes:

$$\alpha_0(T)=1-(T/T_c)^{5/2}, \quad (n>n_c\ \ \xi\gg 1) \tag{65}$$

but the curve is qualitatively similar. Further quantitative details are left for the reader to derive for himself.

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[10] The method of "saddle-point" integration (also known as the method of "steepest descents") is discussed in M. M. Morse and H. Feshbach [*Methods of Mathematical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, p. 440]. The application of this method in statistical physics is discussed in most standard textbooks [e.g., K. Huang (Ref. 6)].

5. CONCLUSIONS

We shall conclude this paper with a brief physical description of the particle and current density that is present when the ideal Bose gas is in rotational thermal equilibrium. The mathematical details are discussed in Appendix C. If the average density of the system is less than the critical density, so that there is no Bose-Einstein condensation, then the particle density increases in a smooth manner as the distance from the cylinder axis increases, and the velocity field is of the rigid-body type (i.e., $\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}$). If the average density is larger than the critical density, so that there is Bose-Einstein condensation, then the particle and current density have two distinct parts: (i) the contribution due to the particles in the condensate and (ii) the contribution due to the particles in the other single-particle states. The density contribution due to the particles in the condensate is nonzero only in a very small region near the walls of the container. The current density due to the particles in the condensate is mathematically described by a single vortex line on the axis of the cylinder; however, this is physically indistinguishable from rigid-body rotation since the particle density of the condensate is so strongly peaked at the cylinder radius R. The contribution to the particle and current density due to the particles which are not in the condensate remains qualitatively unchanged as the fluid passes through the phase transition.

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APPENDIX A

Consider a liquid in a cylindrical rigid-body container. If the container can exchange energy with the liquid, then it follows from thermodynamics that a necessary condition for thermal equilibrium is that the temperature of the liquid be equal to the temperature of the container (i.e., $T_1=T_2$). Similarly, if the container can also exchange angular momentum with the liquid, then a necessary condition for thermal equilibrium is that $\omega_1=\omega_2$. It is easily shown that the thermal parameter ω for the rigid-body container is equal to its angular velocity of rotation. Furthermore, the physical significance of ω for the liquid is simple in all cases where classical statistics is sufficiently accurate. This is deduced from a theorem in statistical physics which is the rotational analog of the Bohr–van Leeuwan theorem[11] of magnetic theory. The distribution function for classical systems in the canonical ensemble is

$$f=(N!)^{-1}(2\pi\hbar)^{-3N}Q_N^{-1}\exp[-\beta(H-\omega L)], \quad (A1)$$

where

$$H-\omega L=(1/2m)\sum_j(\mathbf{P}_j-m\boldsymbol{\omega}\times\mathbf{R}_j)^2 +U-\tfrac{1}{2}m\sum_j(\boldsymbol{\omega}\times\mathbf{R}_j)^2. \quad (A2)$$

Here we are treating the thermal parameter ω as a vector along the cylinder axis, and we are assuming that the potential energy U is the potential of velocity-independent forces. The statistical average of a quantity A is given by

$$\langle A\rangle=\int Af\prod_j d^3P_jd^3R_j. \quad (A3)$$

By making the change of integration variables $\mathbf{P}_j\to\mathbf{p}_j=\mathbf{P}_j-m\boldsymbol{\omega}\times\mathbf{R}_j$ and integrating over $\mathbf{p}_j$ first, it is easily seen that

$$\langle(\mathbf{P}_j-m\boldsymbol{\omega}\times\mathbf{R}_j)\delta(\mathbf{r}-\mathbf{R}_j)\rangle=0. \quad (A4)$$

The mass current and mass density associated with an equilibrium rotational state are, respectively,

$$\langle\mathbf{J}(\mathbf{r})\rangle=\sum_j\langle\mathbf{P}_j\delta(\mathbf{r}-\mathbf{R}_j)\rangle \quad (A5)$$

and

$$\langle\rho(\mathbf{r})\rangle=\sum_j m\langle\delta(\mathbf{r}-\mathbf{R}_j)\rangle. \quad (A6)$$

Therefore we have the following theorem[12]:

$$\langle\mathbf{J}(\mathbf{r})\rangle=\langle\rho(\mathbf{r})\rangle(\boldsymbol{\omega}\times\mathbf{r}),\quad\text{(classical statistics)}. \quad (A7)$$

The physical significance of this theorem is apparent. A classical system will rotate as a rigid body with angular velocity $\boldsymbol{\omega}$ when in thermal equilibrium. A simple consequence of this theorem is that the average angular momentum

$$\langle\mathbf{L}\rangle=\int\mathbf{r}\times\langle\mathbf{J}(\mathbf{r})\rangle d^3r \quad (A8)$$

may be written[13]

$$\langle\mathbf{L}\rangle=\boldsymbol{\omega}\int\langle\rho(\mathbf{r})\rangle(x^2+y^2)d^3r\equiv\boldsymbol{\omega}I. \quad (A9)$$

It is impossible to generalize (A7) to quantum liquids, whose equilibrium currents can form much more interesting configurations than equilibrium currents in classical liquids. However, there is a compelling physical argument due to Landau and Lifshitz[14] which states that (A7) would hold for all liquids if the current $\langle\mathbf{J}(\mathbf{r})\rangle$ and the density $\langle\rho(\mathbf{r})\rangle$ were "coarse grained" over small macroscopic regions of space. The size of the spatial cells needed for averaging the current and the density is obviously connected with the coherence

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[11] J. H. Van Vleck, *Theory of Electric and Magnetic Susceptibilities* (Oxford University Press, London, 1932), Chap. 4.

[12] To the author's knowledge, this theorem has not been previously stated in the literature.

[13] In the limit $\omega\to 0$, this was first shown by J. M. Blatt, S. T. Butler, and M. R. Schafroth, Phys. Rev. **100**, 481 (1955).

[14] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press Ltd., London, 1958), pp. 34 and 35.

length introduced by Blatt, Butler, and Schafroth,[15] but it is difficult to make precise statements about this roughly defined physical quantity.

APPENDIX B

The lowest eigenvalue of the operator $(h-\omega l)$ has the form[2]

$$E_0=\min_n[(\hbar^2x_n{}^2/2mR^2)-\hbar\omega n], \tag{B1}$$

where x_n is the smallest root of the nth-order Bessel function

$$J_n(x_n)=0. \tag{B2}$$

Since the root x_n is a monotonically increasing function of the integer index n with the asymptotic form[16]

$$x_n=n+O(n^{1/3}) \tag{B3}$$

as $n\rightarrow\infty$, it is easily seen that in the thermodynamic limit

$$E_0\rightarrow-\tfrac{1}{2}m\omega^2R^2=-kT\xi. \tag{B4}$$

This limiting value is needed in calculating the free energy per particle.

APPENDIX C

The particle and current density can be calculated from the single-particle density matrix $\langle\psi^\dagger(\mathbf{r}')\psi(\mathbf{r})\rangle$. If the eigenfunctions [Eq. (13)] are used to expand the field operators $\psi(\mathbf{r})$ and $\psi^\dagger(\mathbf{r}')$, then it follows from Eqs. (6) and (14) that

$$\langle\psi^\dagger(\mathbf{r}')\psi(\mathbf{r})\rangle=\sum_j\langle a_j{}^\dagger a_j\rangle\psi_j{}^*(\mathbf{r}')\psi_j(\mathbf{r}). \tag{C1}$$

First, let us consider the normal phase (i.e., no Bose-Einstein condensation). The average occupation number is given by

$$\langle a_j{}^\dagger a_j\rangle=f(E_j), \tag{C2}$$

where

$$f(E_j)=[\zeta^{-1}\exp(\beta E_j)-1]^{-1} \tag{C3a}$$

$$=\sum_{s=1}^{\infty}\zeta^s\exp(-s\beta E_j) \tag{C3b}$$

is the Bose distribution function. The above expansion implies that

$$\langle\psi^\dagger(\mathbf{r}')\psi(\mathbf{r})\rangle=\sum_{s=1}^{\infty}\zeta^s g(\mathbf{r},\mathbf{r}',s\beta), \tag{C4}$$

where g is defined by Eq. (23). The single-particle density matrix in the thermodynamic limit

$$D(\mathbf{r},\mathbf{r}')=\lim\langle\psi^\dagger(\mathbf{r}')\psi(\mathbf{r})\rangle \tag{C5}$$

is then given by

$$D(\mathbf{r},\mathbf{r}')=\sum_{s=1}^{\infty}\zeta^s\lim g(\mathbf{r},\mathbf{r}',s\beta). \tag{C6}$$

The right-hand side of the above equation can be evaluated by using Eqs. (30), (33), and (34). First, we have

$$\lim(g)=G_0\exp\Gamma, \tag{C7}$$

where

$$\Gamma=\lim(\gamma); \tag{C8}$$

then it follows from Eq. (34) that

$$\Gamma(\mathbf{r},\mathbf{r}',t)=(m\rho\rho'/\hbar^2t)\times[i\hbar\omega t\sin(\varphi-\varphi')+\tfrac{1}{2}(\hbar\omega t)^2\cos(\varphi-\varphi')]; \tag{C9}$$

finally, we can combine (C6) through (C9) to yield

$$D(\mathbf{r},\mathbf{r}')=\sum_{s=1}^{\infty}\zeta^sG_0(\mathbf{r},\mathbf{r}',s\beta)\exp\Gamma(\mathbf{r},\mathbf{r}',s\beta). \tag{C10}$$

Equations (29), (C9), and (C10) constitute an explicit expression for the single-particle density matrix in the normal phase. The modifications needed to take into account Bose-Einstein condensation are straightforward. A condensate contribution of the form $N\alpha_0\psi_0{}^*(\mathbf{r}')\psi_0(\mathbf{r})$ must be added, and the value of ζ must be equal to $\exp(-\xi)$ [see Eq. (62)]. Therefore, in the region where the condensate is macroscopically occupied, the density matrix has the form

$$D(\mathbf{r},\mathbf{r}')=N\alpha_0\psi_0{}^*(\mathbf{r}')\psi_0(\mathbf{r})+\sum_{s=1}^{\infty}e^{-s\xi}G_0(\mathbf{r},\mathbf{r}',s\beta)\exp\Gamma(\mathbf{r},\mathbf{r}',s\beta). \tag{C11}$$

The nature of the wave function $\psi_0(\mathbf{r})$ has been discussed by Blatt and Butler[2] and will not be reviewed here. The explicit expressions (C10) and (C11) for the single-particle density matrix allow one to calculate the density and current distributions which are qualitatively discussed in this paper.

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[15] J. M. Blatt, S. T. Butler, and M. R. Schafroth, Phys. Rev. **100**, 495 (1955).

[16] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, New York, 1966), 2nd ed., Chap. 2.