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Stable States of a Scalar Particle in Its Own Gravational Field*†

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Structures similar to classical geons are constructed using a matter field described by the Klein-Gordon equation. The matter field creates a gravitational field which interacts with the matter. The coupled system has stable, bound states. A spectrum of states is shown to exist with masses of the order of 10^{-10} kg. A variational principle is derived which is in precise analogy with the Rayleigh-Ritz principle of conventional quantum mechanics.

1. INTRODUCTION

THE geon was developed by Wheeler[1] in order to introduce the concept of body into classical general relativity. In addition to the original (photon) geons, there have been geons constructed out of neutrinos[2] and gravitational[3] radiation. All of these fields may be described geometrically[4,5]; thus the subject of geons may be considered a portion of the subject of geometrodynamics, a term which connotes the philosophy of describing a major portion of physics in terms of geometry. These structures are all very large, and are not fully stable, although the lifetimes may be quite long.

It is possible to create a similar structure which is much smaller in both linear dimensions and mass, and which is completely stable. To do so, it is necessary to invoke a field with nonvanishing mass. This is a rather unsatisfactory feature, as only the massless scalar field has been geometrized to date,[4] thus some elegance is lost, at least temporarily. However, one can obtain a spectrum of bound states which may be interpreted as a set of physical masses obtained from a single (unobservable) mass. This would provide a model which contains a mass renormalization as an inherent part.

The physical model we construct is that of a field which satisfies the Klein-Gordon equation generalized to curved space-time, which produces a gravitational field. This field in turn affects the dynamics of the Klein-Gordon field. The coupled system of equations is solved by numerical techniques, and normalized, stationary eigenstates which are localized in space are sought.

In order to avoid the necessity of neglecting direct interactions between particles which are very close to each other, solutions are only sought for systems which contain exactly one particle.

2. PARTICLE STATE FUNCTION

In a curved space, a Klein-Gordon field of mass m may be described in terms of the Lagrangian density

$$\mathcal{L}=(1/2m)(g^{\alpha\beta}\psi_{;\alpha}{}^{*}\psi_{;\beta}+m^2\psi^*\psi)\,, \tag{1}$$

where units are chosen so that $\hbar=c=1$. The variational principle

$$\delta\int\mathcal{L}(-g)^{1/2}dx^1dx^2dx^3dx^4=0 \tag{2}$$

yields the equation of motion for the field:

$$g^{\mu\nu}\psi_{;\mu\nu}-m^2\psi=0\,. \tag{3}$$

We take coordinates (r,α,β,t) which reduce to spherical coordinates in regions sufficiently distant from the structure that space-time is essentially flat, but not distant enough to make over-all curvature effects of the universe significant. This study is restricted to structures of spherical symmetry, which is here taken to mean that the field amplitude is independent of α and β. We seek solutions that are harmonic in the time variable in order to represent a stable configuration. Then

$$\psi=\varphi(r)e^{-iEt}\,. \tag{4}$$

This gives a metric tensor similar to that of Schwarzschild: $g_{\mu\nu}=\mathrm{diag}(e^\lambda,\, r^2,\, r^2\sin^2\alpha,\, -e^\nu)$. The functions $\lambda(r)$ and $\nu(r)$ are yet to be determined. With this form for the metric, and setting

$$k=(m^2-E^2)^{1/2}\,,\quad \rho=kr\,,\quad \chi=E^2/m^2\,, \tag{5}$$

(3) simplifies to

$$\ddot{\varphi}+\left(\frac{\dot{\nu}-\dot{\lambda}}{2}+\frac{2}{\rho}\right)\dot{\varphi}+e^\lambda\varphi\frac{e^{-\nu}\chi-1}{1-\chi}=0\,. \tag{6}$$

Dots denote differentiation with respect to ρ.

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[1] J. A. Wheeler, Phys. Rev. **97**, 511 (1955).
[2] D. Brill and J. A. Wheeler, Rev. Mod. Phys. **29**, 465 (1957).
[3] D. Brill and J. Hartle, Phys. Rev. **135**, B271 (1964).
[4] A review article which includes the geometrizing of the electromagnetic and massless Klein-Gordon fields is D. Brill, Nuovo Cimento Suppl. **2**, 1 (1964).
[5] A. Inomata and W. A. McKinley, Phys. Rev. **140**, B1467 (1965).

3. EQUATIONS FOR THE METRIC

Using a prescription due to Rosenfeld,[6]

$$T_\nu{}^\mu = \frac{\partial \mathcal{L}}{\partial \psi_{;\mu}{}^*}\psi_{;\nu}{}^* + \frac{\partial \mathcal{L}}{\partial \psi_{;\mu}}\psi_{;\nu} - \mathcal{L}\delta_\nu{}^\mu , \tag{7}$$

we have, for the energy-momentum tensor given by the Lagrangian (1)

$$T_\nu{}^\mu = (1/2m)[g^{\mu\alpha}(\psi_{;\alpha}\psi_{;\nu}{}^* + \psi_{;\alpha}{}^*\psi_{;\nu}) - (g^{\alpha\beta}\psi_{;\alpha}{}^*\psi_{;\beta} + m^2\psi^*\psi)\delta_\nu{}^\mu]. \tag{8}$$

This gives

$$T_1{}^1 = \frac{1}{2m}\left[e^{-\lambda}\left(\frac{d\varphi}{dr}\right)^2 - (m^2 - e^{-\nu}E^2)\varphi^2\right] \tag{9}$$

and

$$T_4{}^4 = -\frac{1}{2m}\left[e^{-\lambda}\left(\frac{d\varphi}{dr}\right)^2 + (m^2 + e^{-\nu}E^2)\varphi^2\right]. \tag{10}$$

These are the only important components since $T_2{}^2 = T_3{}^3$, as required for compatibility with the structure of the Einstein tensor, the off-diagonal elements vanish, and the Bianchi identities yield one equation relating the nonzero components.

The corresponding components of the Einstein tensor for a metric of the type under consideration are[7]

$$G_1{}^1 = -\frac{1}{r}\frac{d\nu}{dr}e^{-\lambda} + \frac{1}{r^2}(1 - e^{-\lambda}) \tag{11}$$

and

$$G_4{}^4 = -\frac{1}{r}\frac{d\lambda}{dr}e^{-\lambda} + \frac{1}{r^2}(1 - e^{-\lambda}). \tag{12}$$

Equating the Einstein and energy-momentum tensors, making the substitutions (5), defining

$$e^{-\lambda} = 1 - \sigma(\rho)/\rho \tag{13}$$

and

$$\tau(\rho) = e^{\lambda - \nu}, \tag{14}$$

one finds

$$\dot\sigma = \tfrac{1}{2}\kappa\rho(\rho - \sigma)\left\{\dot\varphi^2 + \left[\left(\frac{m}{k}\right)^2 \frac{\rho}{\rho - \sigma} + \left(\frac{E}{k}\right)^2 \tau\right]\varphi^2\right\} \tag{15}$$

and

$$\dot\tau = \frac{\tau}{\rho - \sigma}\left[\kappa\rho^2\left(\frac{m}{k}\right)^2\varphi^2 - \frac{2\sigma}{\rho}\right]. \tag{16}$$

In these equations, $\kappa = 8\pi G/m$. In all the numerical work, G/m has been set equal to unity. The matter field satisfies, in terms of the new variables, the following equation:

$$\ddot\varphi + \left[\frac{1}{\rho} + \frac{1}{\rho - \sigma} - \tfrac{1}{2}\kappa\left(\frac{m}{k}\right)^2 \frac{\rho^2\varphi^2}{\rho - \sigma}\right]\dot\varphi + \left(\tau\chi - \frac{\rho}{\rho - \sigma}\right)\frac{\varphi}{1 - \chi} = 0. \tag{17}$$

[6] L. Rosenfeld, Mem. Acad. Roy. Belgique Cl. Sci. **18**, No. 6 (1940).

[7] See, e.g., C. Møller, *The Theory of Relativity* (Oxford University Press, New York, 1952), Chap. XI, Secs. 121–123.

4. DISCUSSION OF THE EQUATIONS

In order to solve the system of equations (15)–(17), we must determine a set of boundary conditions.

We first require that the state functions must be normalizable. This implies that the solution which approaches the zeroth-order solution in the gravitational field, $\varphi = Ae^{-\rho}/\rho$ (for large ρ), must be taken.

The metric at large distances should approach that of Schwarzschild; this leads to the requirement that $\sigma \to 2Ek$ and $\tau \to (1 - \sigma/\rho)^{-2}$ in our coordinates as $\rho \to \infty$.

The constant A is not trivial since this system of equations is nonlinear and it must be determined with E to get the desired stationary states. To do so we fix a norm for the state function to be unity. The choice of norm requires some argumentation; this will be done in Sec. 5.

It is not clear *a priori* that the coordinate system chosen will be usable everywhere. If well-behaved solutions to the equations exist everywhere, the metric is well behaved, and the coordinate system is satisfactory. This situation will occur unless $\sigma = \rho$, which will produce a pseudosingularity of the type found in the Schwarzschild solution, indicating the presence of a bridge or wormhole. No proof of the lack of this behavior was constructed, but the form of the equations makes it appear unlikely. In the computed solutions no indication of singular behavior exists. This suggests that the ostensibly static solutions sought will be truly static, in contrast to the Schwarzschild-Kruskal solution.[8]

The energy spectrum is determined by insisting on smooth behavior of the state function at the origin. It is of interest to note that the energy appears four times in the equations: twice (in the form of its square) in the variable χ in Eq. (17), once (again squared) in Eq. (15), and once (to the first power) in the initial value of σ.

It can be shown that, except for eigenstates, no power-series solution exists for the system in a neighborhood of the origin. The only simple solution to the equations in that region is the set

$$\dot\varphi = e/\rho, \qquad \varphi = e\ln\rho + b, \tag{18}$$

$$\dot\tau = 2\tau/\rho, \qquad \tau = c\rho^2, \tag{19}$$

$$\dot\sigma = -\tfrac{1}{2}\kappa e\sigma/\rho, \qquad \sigma = f\rho^{-(1/2)\kappa e}, \tag{20}$$

where b, c, e, and f are constants, as yet undetermined.

The eigenvalue criterion is that the constant e vanish. The constant f is negative except for eigenvalues. At an eigenvalue, a solution with $f = 0$ could exist; however, it is not clear whether such must be the case.

[8] The latter solution is described in M. Kruskal, Phys. Rev. **119**, 1743 (1960).

5. NORMALIZATION

The normalization which we use in this paper is

$$\int e^{-\nu/2}\varphi^2({}_3g)^{1/2}d\rho d\alpha d\beta=1. \tag{21}$$

Except for the first factor, everything seems obvious: There is the usual square of the space part of the state function, multiplied by the invariant three-volume element, and of course everything is integrated over the entire three volume. But why the first factor? One answer is given in the presence for this norm, and only for this norm, of the energy minimum principle of the next section.

A stronger justification is found in the consideration of energy. For a stationary state, the difficulties in the definition of an "energy density" caused by the difference between the covariant divergence and the ordinary one do not appear and we have a conserved quantity: the energy-momentum vector[9]

$$J^\mu=-\int T_4{}^\mu(-g)^{1/2}dx^1dx^2dx^3. \tag{22}$$

Requiring that J^4, the Hamiltonian, be E^2/m where the symbol E must have the same meaning as it does in Eq. (4) yields Eq. (21) as the normalization condition.

This discussion raises another interesting point. In quantum mechanics, a "probability" current density j^μ is defined. For a well-defined problem, j^μ should be related in a natural way to J^μ, but such a relation is not immediately evident. Nevertheless it exists. To see this, note that for a self-consistent solution, the static gravitational field assumed in Eq. (22) is equivalent to the separability of the state function into the product of a spatial part and a harmonic time dependence.

Direct calculation then yields

$$\begin{aligned}J^\mu&=E\int \frac{i}{2m}g^{\mu\alpha}(\psi_{;\alpha}{}^*\psi-\psi^*\psi_{;\alpha})(-g)^{1/2}dx^1dx^2dx^3\\&=E\int j^\mu(-g)^{1/2}dx^1dx^2dx^3,\end{aligned} \tag{23}$$

which may be verified in any standard quantum-mechanics text.[10] Comparing Eq. (23) with Eq. (22), one has the relation

$$J^\mu=Ej^\mu, \tag{24}$$

which is as natural as could be desired. This result holds in every coordinate system for which the system is static. A standard argument[11] shows that the right side of Eq. (23) and the expression which reduces to the right side of Eq. (22) for static fields each transform like a vector under Lorentz transformations. Hence Eq. (24) holds for all those cases where it is clear that it must.

6. ENERGY MINIMUM PRINCIPLE

It would be interesting if an energy minimum could be shown to exist for the states sought. This is only possible for the normalization of Eq. (21), as it is only for this choice that the stationary property of the Lagrangian afforded by the variational principle of general relativity leads to a stationary property for the Hamiltonian (i.e., $-T_4{}^4$).

We want to demonstrate a minimum principle for E^2 as defined in Eq. (22) subject to the constraint of Eq. (21), and to do so we show that

$$W=-4\pi\rho^2e^{(\lambda+\nu)/2}T_4{}^4+\frac{E^2}{m}\left[1-\int e^{-\nu/2}\varphi^2({}_3g)^{1/2}d\rho d\alpha d\beta\right]f(\rho) \tag{25}$$

has the desired property, $\int Wd\rho=E^2/m$, whenever the constraint is satisfied. The function $f(\rho)$ is arbitrary except that $\int fd\rho=1$.

We shall use a theorem given by Bliss[12] after some necessary preliminaries: Define

$$I\equiv\int Wd\rho;$$

then

$$\delta I=\int (W_\varphi\eta+W_{\dot\varphi}\dot\eta)d\rho \tag{26}$$

and

$$\delta^2 I=\int 2w(\rho,\eta,\dot\eta)d\rho, \tag{27}$$

where

$$\begin{aligned}\eta&=\delta\varphi,\\2w&=W_{\varphi\varphi}\eta^2+2W_{\varphi\dot\varphi}\eta\dot\eta+W_{\dot\varphi\dot\varphi}\dot\eta^2,\end{aligned} \tag{28}$$

and subscripts indicate partial differentiation. The integrand function (W) must have continuous derivatives through the fourth order on some region $R(\rho,\varphi,\dot\varphi)$. An arc is defined as a specific functional dependence of the dependent variable (φ) on the independent variable (ρ), e.g., that illustrated in Fig. 1. The arc in question must contain only points which are interior to R or, under some conditions, on the boundary. In our case, the origin is such a boundary point.[13] The desired theorem states that W is minimized by an arc that satisfies four conditions:

Condition I is essentially the requirement that the

[9] Reference 7, p. 299.

[10] See, e.g., L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Co., New York, 1955), 2nd ed., Eq. 42.8.

[11] Reference 7, pp. 339–340.

[12] G. A. Bliss, *Lectures on the Calculus of Variations* (University of Chicago Press, Chicago, 1946), pp. 82, 83.

[13] This causes no difficulty since the origin may be eliminated by excluding a neighborhood of radius ϵ of the origin from the range of integration. The theorem is certainly true with this restriction. Then pass to the limit as $\epsilon\rightarrow 0$.

first variation of W be zero. This satisfied, since

$$\int W d\rho = \int \mathfrak{L}(-g)^{1/2} d\rho d\alpha d\beta + E^2/m, \qquad (29)$$

and we insist that the first variation of

$$\int \left[\int \mathfrak{L}(-g)^{1/2} d\rho d\alpha d\beta \right] dt$$

vanish for arbitrary time extent.

Condition II requires that along the arc

$$\mathcal{E}(\rho,\varphi,\dot{\varphi},\dot{\Phi}) \equiv W(\rho,\varphi,\dot{\Phi}) - W(\rho,\varphi,\dot{\varphi}) - (\dot{\Phi}-\dot{\varphi})^2 W_{\dot{\varphi}}(\rho,\varphi,\dot{\varphi}) > 0 \qquad (30)$$

for every admissible set $(\rho,\varphi,\dot{\Phi})$ different from $(\rho,\varphi,\dot{\varphi})$. In this problem one finds

$$\mathcal{E} = 2\pi\rho^2 e^{(\nu-\lambda)/2} (\dot{\Phi}-\dot{\varphi})^2 > 0 \quad (\dot{\Phi} \neq \dot{\varphi}). \qquad (31)$$

Condition III requires that $W_{\dot{\varphi}\dot{\varphi}} > 0$ everywhere on the arc. In our case we have

$$W_{\dot{\varphi}\dot{\varphi}} = \tfrac{1}{2}\rho^2 e^{-\lambda/2} > 0. \qquad (32)$$

Condition IV requires that there be no solution to the problem

$$\delta \int 2w d\rho = 0, \qquad (33)$$

which is zero at the initial point of the arc and at another point, but not identically zero. We regard $\rho = \infty$ as the initial point. Evaluating w, we find that

$$w = \int \mathfrak{L}(\eta,\dot{\eta})(-g)^{1/2} d\alpha d\beta. \qquad (34)$$

Thus, the Euler equation for this problem is the same as Eq. (6), except that φ is replaced by η. If there are acceptable solutions with no nodes in φ and with $\varphi(\infty)=0$, condition IV will be satisfied. This occurs since the metric is kept fixed during the variation, hence λ and ν are the same for the problem (33) as for the original one, and Eq. (6) is linear in φ.[14] Numerical calculation shows that the ground state has the required properties.

On the other hand, examination of the necessity statement of condition IV only permits the possibility of a node at the final point, $\rho=0$. Thus, any state with a node at a finite value of ρ cannot even be a relative minimum. Such states include the higher-energy states.

With the normalization of (24) the theorem yields the following result: The ground state is a minimum energy configuration under sufficiently small variations of φ, and arbitrary variations of $\dot{\varphi}$, while the excited states are not minimum energy configurations. It is interesting to note that these are the same results one has in elementary quantum mechanics.

[14] Linearity is required since the solution for η is proportional to that for φ, but not necessarily equal to it, since there is no normalization requirement on the η solution.

7. SOLUTION OF THE EQUATIONS

The solution of the nonlinear system of equations (15)–(17) with two eigenvalues (E^2 and A) which enter nonlinearly has been done by numerical methods.

The basic-solution method is to choose a value of ρ large enough that asymptotic solutions of the system are valid, and use these to start a step-by-step numerical integration toward the origin. When a sufficiently small value of ρ is achieved that the approximate solutions (18), (19), and (20) are valid, the integrations are terminated and the quantities $e=\rho\dot{\varphi}$ and d, defined as a measure of the deviation of the normalization integral from unity, are calculated.

We then assume that we have to solve simultaneously the transcendental equations

$$e(E^2,A)=0, \qquad (35)$$

$$d(E^2,A)=0. \qquad (36)$$

At this point, the normalization becomes very significant. For reasons detailed below, the norm of Eq. (21) makes the location of normalized eigenstates very difficult. The first goal was thus to obtain eigenstates which were not normalized. Figure 1 depicts the ground state so obtained. Other states were checked, and it was found that, in analogy with usual quantum-mechanical results, the ground state had no nodes while the nth excited state possessed n nodes. This result is independent of the satisfaction of Eq. (21); hence it is possible to apply the results of Sec. 6 to assert that the ground state is a minimum energy configuration with respect to variations in the state function (for fixed metric) and that the higher states are not. Note that in flat-space quantum mechanics the last point is almost trivial since there are perturbations in each excited state which mix some of the ground state into the wave function, and this must lower the expectation energy. In the present case this argument fails due to lack of superposition; nevertheless the result still holds.

The results of the analysis of the entire system with m^2 set arbitrarily to 0.1 are shown in Fig. 2. An unexpected difficulty appeared. The region in which the family of eigenfunctions meet the curve of normalized functions occurs at values of E^2 and A so small that the equations become quite ill-behaved. It was possible to find solutions where σ/ρ reaches 3×10^{-6} of unity. This caused τ to become large ($\sim10^{12}$), and shortly thereafter σ would reach such large negative values ($\sim-10^{300}$) that there was acute danger of machine overflow. At least the start of this phenomenon is not due to the exceeding of the capabilities of the numerical procedures involved. These procedures involved a fairly sophisticated self-checking for error, and cut the

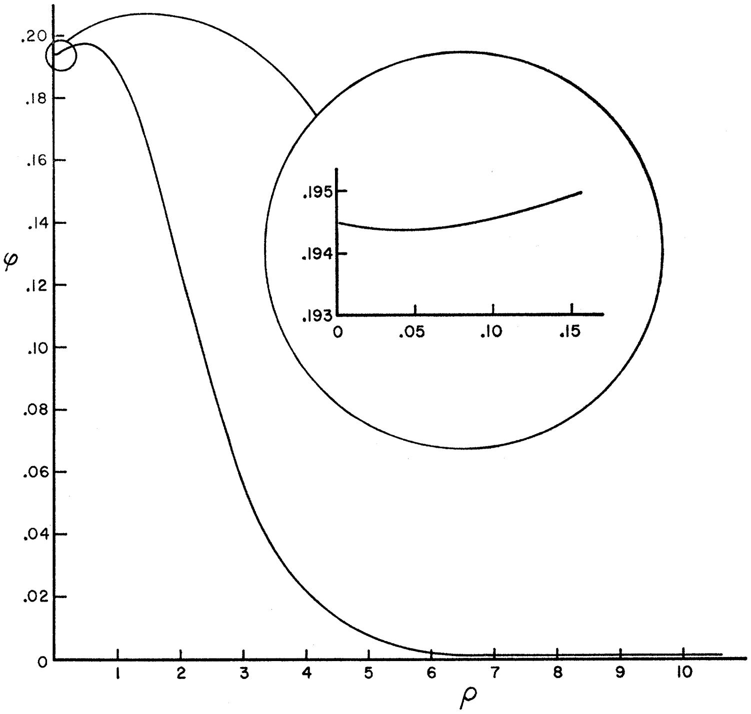

FIG. 1. The state function for an *un*normalized ground state. The small negative slope for $\rho<0.03$ is due to residual errors in locating the eigenvalue. Examination of the function for $\rho<0.002$ shows the presence of a weak divergence ($\rho\dot{\varphi}\approx-10^{-4}$).

step size to keep the single-step error sufficiently small that numerical instability should never have been possible. A rather singular function was integrated by this program with entirely satisfactory results.

In any case, the portion of the curve of normalized functions (CNF) lying in the region $A>\frac{1}{2}E^2$, $E^2<0.05$ was not observed except to place upper bounds on the value of A at each energy. The belief that the CNF actually passes through this region and therefore intersects the curves (35) requires justification. The main evidence is that the portion of the CNF at small values of A and $E^2>0.05$ is well established; thus there are two breaks in the CNF or none. Also, Eqs. (15)–(17) are sufficiently well behaved everywhere in the region $E^2<m^2$, $A\neq0$ that unless $\sigma=\rho$ for some $\rho\neq0$ (never observed), the solutions are continuous functions of their initial values. This implies that the left-hand sides of Eqs. (35) and (36) are continuous, which implies that none of the curves described by these equations can terminate except on the line $A=0$. Upon noting that $E^2=A=0$ is a trivial (and well-behaved) solution to Eqs. (15)–(17) and (35), but not to (36), one becomes convinced that at least the ground-state curve reaches this point and that the CNF lies above it.

8. DISCUSSION

We have shown the existence of a spectrum of bound states for the particular mass chosen. Converting back to conventional units, we find that the mass parameter was selected to be 1.28×10^{-15} kg or 7.71×10^{8} amu. The eigenenergy of the system, or (what is the same thing) the mass of the system as seen by a distant observer, is less than or about $0.07m$, hence $E\lesssim5\times10^{7}$ amu. The diameter of the system is of the order of 4×10^{-32} m. Thus the system analyzed is much heavier and more

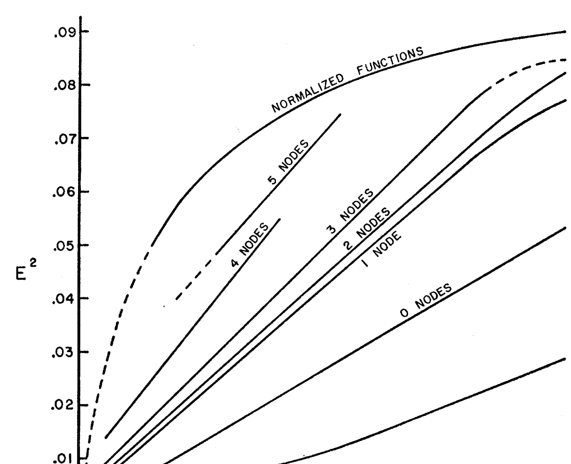

FIG. 2. The curve of normalized functions and the lowest six members of the family of curves satisfying the eigenfunction criterion for the case $m^2=0.1$. The dashed portion of the curve of normalized functions is imprecisely located and may be to the right of its correct position. The ground state is located at $E^2\leq5\times10^{-4}$; $A\leq2\times10^{-4}$.

compact than a baryon.[15] It would be of major interest to investigate the effects of varying the parameter m, but the numerical difficulties already mentioned interfered too strongly with the cases $m^2=1$ and $m^2=10$ for any information to be obtained that would be precise enough to indicate a trend.

There is an apparent difficulty due to the behavior of σ. It seems to be nonzero at the origin [i.e., f of Eq. (20) seems to be nonzero]. If this result is true, the curvature scalar would diverge indicating that space-time fails to be a differentiable manifold at the origin, although it would still be simply connected. However, there are two reasons for casting doubt on this observation, even though the indications are sufficiently clear that they would usually be taken as conclusive. First, although f is always negative, normally suggesting that it never vanishes, analytic investigation of the system shows that whenever the eigenvalue criterion is not *exactly* satisfied, f must be negative. Nevertheless, for a precise eigensolution, f may vanish. This situation could never be observed numerically, of course. Second, after the completion of this calculation, one of the authors (D.A.F.) learned of a similar calculation which differed from the one described here in that a system of many particles was investigated, and in the direction of the numerical integration.[16] In this calculation the mathematically risky but physically reasonable assumption was made that space-time is differentiable at the origin, and this led to certain initial conditions. At (and only at) eigensolutions, the functions behaved at infinity so as to produce the conditions used in this work for starting the integration. It turned out that the solutions were extraordinarily sensitive to the energy, in the sense that a slight variation from eigenenergy produced an unexpectedly rapid divergence of the solutions. One is inclined to believe that this is a result of the lack of any solution of the type required by the initial conditions except at eigenvalues. It seems most likely that a combination of analytic and numerical characteristics have combined to indicate a spurious difficulty.

[15] Quantum gravitational effects are expected to be of the same order as the classical ones when masses of the order of 2×10^{-8} kg are present in a region of radius of the order of 1.6×10^{-35} m. They are therefore still fairly small in the system investigated here.

[16] R. Ruffini, Ph.D. thesis, University of Rome, 1966 (unpublished), also private communications.

9. INTERPRETATIONS AND EXTENSIONS

One is inclined to wonder whether the structures that have been investigated have any relation to fundamental particles. We now have structures which are some 45-orders of magnitude less in mass than the original ones of Wheeler[1]; however, they are still much more massive than any of the known particles. The successes of the quark model of fundamental particles also argues against the interpretation of these structures as fundamental particles themselves since the group-theoretic considerations which are at the core of the quark model depend on a flat, simply connected space. One may interpret these structures as analogs of quarks in this sense: Two or more of these structures will interact and, because of the attractive nature of gravity, will form bound states. If the state function of one structure has significant amplitude where it is near the center of another, this compound structure may be strongly bound. The nonlinearities of the Einstein equations will enhance this effect. Such a compound structure may have a smaller mass than the individual components, and would have a larger radius. The larger radius results in a smaller average curvature over the complex. When the complex structures become relatively large and light, space-time may be quite flat on the scale in question.[17] It should be mentioned that the above remarks apply to any gravitationally derived structure, not only the particular one investigated in this work.

In order to investigate compound structures, one should know the entire spectrum of states, including states with angular momentum. Such states are under investigation by A. Jones, who has kindly given permission to mention his work. An angular part of this problem can be solved analytically; the remainder is under investigation by numerical techniques.

ACKNOWLEDGMENTS

Preliminary calculations were performed on donated time on machines operated by the General Electric Company and the State University of New York at Buffalo. The many courtesies extended by the personnel at these centers are gratefully acknowledged.

[17] This discussion is due in part to remarks made during the talk "The End of Time" given by J. A. Wheeler during the January 1967 meeting of the American Physical Society.