Relativistic Classical Mechanics with Time as a Dynamical Variable

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A relativistically invariant classical Hamiltonian mechanics is presented, in which each particle is described by the eight dynamical variables of position, time, momentum, and energy. The two-particle scattering problem which consists of both inelastic scattering and elastic scattering is completely solved and reduced to quadratures. Special attention is given to elastic scattering, and it is shown that the two particles participating in elastic scattering remain at a spacelike separation with respect to each other throughout their trajectories. It is also shown that this theory is capable of describing the decay of one particle into two particles.

I. INTRODUCTION

A CLASSICAL mechanics is essentially a rule for constructing an ensemble of trajectories. For a relativistically invariant mechanics, the rule will be such that when each trajectory in reference frame A is Lorentz transformed into a trajectory in reference frame B, the *ensemble* of possible trajectories will "look the same" from reference frame B as it looked from reference frame A (i.e., if the coordinates of reference frame B are substituted for the coordinates of reference frame A in the equations for the trajectories in reference frame A, it will be found that these new equations describe the actual trajectories in reference frame B).

Recently there have appeared some new formulations of relativistically invariant classical mechanics. Apart from the intrinsic interest in such a classical mechanics, there is the hope that it could lead to the construction of a satisfactory relativistically invariant quantum mechanics.[1]

In the mechanics of Van Dam and Wigner,[2] the trajectories are solutions of a set of integro-differential equations. Roughly speaking, the force felt by particle 1 at time t is exerted by particle 2 not only at time t, but over the whole time interval during which the trajectory of particle 2 is spacelike with respect to the space-time location of particle 1. This theory is not expressed in Hamiltonian form, so it cannot be used to formulate a quantum mechanics.

In the mechanics of Currie[3] and Hill,[4] the trajectories are solutions of the usual set of second-order differential equations. The forces however, must be specially restricted functions of position and velocity, in order that the equations of motion be Lorentz invariant. Kerner and Hill[5] have pointed out that any set of second-order differential equations can be cast in Hamiltonian form by any one of a number of suitable changes of variables, and Hill[4] has shown how to apply this procedure to the relativistically invariant differential equations. However, although the equations are expressed in Hamiltonian form, the canonical variables are not position and momenta. This means that the usual replacement of canonical coordinates by operators, which is used in constructing quantum mechanics from Hamiltonian mechanics, may not produce a physically meaningful set of quantum-mechanical equations.

In this paper we have decided to investigate a relativistically invariant classical mechanics which satisfies four useful requirements: (1) The equations of motion are expressed in Hamiltonian form. (2) The particle's positions and momenta are (at least some of the) dynamical variables. (3) The theory is "manifestly" Lorentz invariant. (4) The results reduce to those of the usual Galilean-invariant classical mechanics in the limit of small particle velocities.

The first requirement of course means that we do not need to search for a set of differential equations, but only need to find a single function of the dynamical variables. This Hamiltonian function will not be the energy of the system of particles (although it will be a constant of the motion): it is merely a function through whose agency one can obtain a set of equations of motion.

The third requirement is interpreted to mean that the Hamiltonian must be a function of Lorentz-invariant scalar products of four-vectors. To satisfy the second requirement, we shall introduce a time variable and an energy variable for *each* particle as *independent dynamical variables*. This completes the position-time and momentum-energy four-vectors out of which the scalar products will be formed. In the next section we shall discuss some of the unfamiliar features of this mechanics, as compared with the usual mechanics where there is a single time for all particles (which is not a dynamical variable, but simply a parameter) and a single energy (which is not an *independent* dynamical variable).

[1] An interesting relativistically invariant quantum mechanics is described by B. Bakamjian and L. H. Thomas, Phys. Rev. **92**, 1300 (1953); L. L. Foldy, *ibid.* **122**, 275 (1961); R. Fong and J. Sucher, J. Math. Phys. **5**, 956 (1964). However, the analogous classical mechanical equations are not relativistically invariant, for the position variables do not transform properly under Lorentz transformations; see also, D. G. Currie, T. F. Jordan, and E. C. Sudarshan Rev. Mod. Phys. **35**, 350 (1963).

[2] H. Van Dam and E. P. Wigner, Phys. Rev. **138**, 1576 (1965); 142, 838 (1966).

[3] D. G. Curie, Phys. Rev. **142**, 817 (1966).

[4] R. N. Hill, J. Math. Phys. **8**, 201 (1967); **8**, 1756 (1967).

[5] R. N. Hill and E. H. Kerner, Phys. Rev. Letters **17**, 1156 (1966).

One of the results of this paper will show that the fourth requirement is satisfied if the Lorentz-invariant Hamiltonian is constructed from the Hamiltonian of the Galilean-invariant theory by replacing the rotationally invariant scalar products of three-vectors with Lorentz-invariant scalar products of four-vectors.

These four requirements were stipulated in the hope that the classical theory which satisfied them would be the basis for a useful relativistically invariant quantum theory. The classical theory presented here does not fulfill that hope; the reasons for this will be indicated at the conclusion of this paper, in Sec. X. However, it still may be possible to construct a theory possessing some of the features of the theory presented here, which would fulfill that hope.

II. SOME UNUSUAL FEATURES

The position-time and momentum-energy four-vectors for the ith particle will be denoted by x_i^μ and p_i^μ, respectively. The Poisson bracket of two variables A and B is defined as

$$[A,B]\equiv\sum_i g^{\mu\nu}\left[\frac{\partial A}{\partial x_i^\mu}\frac{\partial B}{\partial p_i^\nu}-\frac{\partial A}{\partial p_i^\mu}\frac{\partial B}{\partial x_i^\nu}\right],$$

where $-g^{00}=g^{11}=g^{22}=g^{33}=1$, and for $i\neq j$, $g^{ij}=0$; repeated Greek indices are to be summed over. In particular, the Poisson bracket relations $[x_i^\mu,p_i^\nu]=g^{\mu\nu}$ show that $-p_i^0$ is the variable canonically conjugate to $t_i(t_i\equiv x_i^0)$.

We next parametrize the trajectories: All the dynamical variables are functions of a single "evolution parameter" s. The equation of motion of a variable A is then $dA/ds=[A,H]$. In particular, the equations of motion for x_i^μ and p_i^μ are $dx_i^\mu/ds=g^{\mu\nu}\partial H/\partial p_i^\nu$, $dp_i^\mu/ds=-g^{\mu\nu}\partial H/\partial x_i^\nu$. These equations will be form-invariant under a Lorentz transformation (and therefore the trajectory solutions will satisfy the criteria for a relativistically invariant mechanics mentioned in the first paragraph of the previous section) if s is a scalar invariant under a Lorentz transformation.

These procedures are well known.[6] To elucidate the first unusual feature of this theory, we turn to a system of two particles moving in one spatial dimension. Consider the two world lines as they are traced out with increasing s, on a graph whose vertical axis is labeled by both t_1 and t_2, and whose horizontal axis is labeled by both x_1 and x_2. One may imagine a vector, drawn between the two points representing the space-time locations of the two particles at an "instant" of s. This vector will usually not be horizontal [since usually $t_1(s)\neq t_2(s)$], and its orientation will change as s increases. In other words, the force exerted by particle 1 at time t_1 is felt by particle 2 at time t_2, and vice versa. This is in contrast to the usual Galilean-invariant theory, where this vector would always be horizontal. We shall call this vector the "relative event" vector. The solution of the two-particle problem will amount to the determination of the dynamics of the relative event vector, just as in the Galilean-invariant theory the crucial dynamics is that of the relative position vector.

The most unusual features of this theory stem from the fact that it is necessary to specify eight initial conditions for each particle instead of the usual six initial conditions (for particles moving in three spatial dimensions). That is, once we have specified the position and momentum of each particle at $s=s_a$, we still have to specify the initial time and energy coordinates; different choices of these coordinates will result in different particle trajectories. Let us examine these initial conditions in the case of a scattering problem (the bound-state problem, which is more complex, will be discussed in Sec. VIII). Initially, we start out with two widely separated particles, of masses m_1 and m_2 and momenta $\mathbf{p}_1(s_a)$ and $\mathbf{p}_2(s_a)$. The statement that a particle has mass m_i already implies the initial condition for p_i^0, the energy of the ith particle: At the "instant" s_a, we must have $p_i^0(s_a)=[\mathbf{p}_i^2(s_a)+m_i^2]^{1/2}$. As s increases, the particles move toward each other and (it will be seen that) p_i^0 and $\mathbf{p}_i$ do not change until the particles interact. Long after the particles separate, say at $s=s_b$, p_i^0 and $\mathbf{p}_i$ are once more constant. We can then calculate $-p_i^2(s_b)\equiv p_i^{02}(s_b)-\mathbf{p}_i^2(s_b)$ which may or may not equal m_i^2. This is where the choice of the initial time coordinates comes in. It will be possible, in a typical case, to choose the initial time coordinates so that $-p_i^2(s_b)=m_i^2$, i.e., the scattering is elastic. Other choices of these coordinates will lead to inelastic scattering $[0\leq -p_i^2(s_b)\neq m_i^2]$ or even nonphysical scattering $[0>-p_i^2(s_b)]$.

It is perhaps preferable to consider that, instead of specifying the two initial conditions of time and energy, we are specifying one initial condition and one final condition: These are the values of $-p_i^2$ before and after the collision.[6a] In this theory, the particle's mass $\sqrt{(-p_i^2)}$ is a dynamical variable, instead of being just a parameter. Since it is a Lorentz-invariant dynamical variable, the choice of initial and final values of $\sqrt{(-p_i^2)}$ selects a subset (of the full set) of trajectories in an invariant way. The subset still satisfies the criteria for

[6] C. Lanczos, *The Variational Principles of Mechanics* (University of Toronto Press, Toronto, 1964); H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959).

[6a] *Note added in proof*. For some potentials, there is a range of initial conditions (particle masses, positions and momenta) for which it is impossible to choose the initial time coordinates (or alternatively, the final values of $-p_i^2$) so that the scattering is elastic. It should be emphasized that in this paper we do not discuss such existence questions as whether a class of potentials exists for which one can obtain elastic scattering trajectories for *all* choices of initial masses, positions and momenta. We conjecture, however, that a large class of such potentials does exist, including those potentials which have "hard cores," i.e., are infinite for some spacelike or lightlike value of their argument (see Sec. V), as in the example presented in Appendix A.

a relativistically invariant set of trajectories, which was stated in the first paragraph of Sec. I.

To recapitulate: In the classical mechanics with which we are familiar, all solutions of the pertinent equations are possible trajectories for the particles, and the particles do not change their masses during their motion. This is not the case in the present theory, where the equations must be supplemented by asymptotic conditions, in order to select out the physical trajectories of interest.

There are some other aspects of this theory that are not found in the usual Galilean-invariant theories, but they will be discussed in later sections. We shall now proceed to examine the one-particle problem (Sec. III), and then we will introduce a useful set of coordinates with which to discuss the two-particle problem (Sec. IV). The two-particle scattering problem in one spatial dimension (Secs. V and VI) and in three spatial dimensions (Sec. VII) will follow: these form the bulk of this paper. Miscellaneous topics including the bound-state problem and the nonrelativistic limit appear in Sec. VIII. A decay problem will be treated in Sec. IX.

III. NONINTERACTING PARTICLES

The Hamiltonian for n noninteracting particles will be taken as

$$H\equiv\sum_{i=1}^{n} p_i^2/2m_i=\sum_{i=1}^{n} \mathbf{p}_i^2/2m_i-p_i^{02}/2m_i \tag{1}$$

(we have set $c=1$), following the rule given in Sec. I for constructing the Lorentz-invariant Hamiltonian from the Hamiltonian of the Galilean-invariant theory. It is worth noticing that the mass m_i appears in two places in this theory: in the asymptotic condition on $-p_i^2$ and also as a parameter in the Hamiltonian.

Hamilton's equations of motion for the ith particle are

$$\begin{aligned} &d\mathbf{x}_i/ds=\mathbf{p}_i/m_i, \quad dt_i/ds=p_i^0/m_i, \\ &d\mathbf{p}_i/ds=0, \quad dp_i^0/ds=0. \end{aligned} \tag{2}$$

The solutions of these equations are straight world lines:

$$\begin{aligned} &\mathbf{p}_i=\text{const vector}, \quad p_i^0=\text{const}, \\ &\mathbf{x}_i=(s-s_{ia})\mathbf{p}_i/m_i+\mathbf{x}_{ia}, \quad t_i=(s-s_{ia})p_i^0/m_i+t_{ia}, \end{aligned} \tag{3}$$

and we may eliminate the parameter s, obtaining

$$\mathbf{x}_i=\mathbf{x}_{ia}+(t_i-t_{ia})\mathbf{p}_i/p_i^0. \tag{4}$$

Equation (4) is the proper relativistic expression for a particle of mass m_i moving with constant speed, provided we give p_i^0 the initial value $(\mathbf{p}_i^2+m_i^2)^{1/2}$. Since $-p_i^2$ is a constant throughout the trajectory, the initial value t_{ia} is of no consequence in determining the final mass value. This is a degenerate case of the interacting-particle problem in that there is an overabundance of initial conditions which produce the same world lines.

The masses m_i which appear in the Hamiltonian have disappeared from Eq. (4), so that the mass of a free particle is solely determined by the initial value of $-p_i^2$. It would therefore be possible to have a different mass in the ith term of the Hamiltonian than the initial value given to $\sqrt{(-p_i^2)}$, without affecting the shape of the ith trajectory. Yet the *rate* at which this world line is traversed as s increases does depend upon the mass in the Hamiltonian, as can be seen from Eqs. (2) or (3). For example, if we do initially set $-p_i^2=m_i^2$, we find that the increment of proper time for the ith particle satisfies the simple relationship $d\tau_i^2 \equiv -g^{\mu\nu}dx_i^\mu dx_i^\nu=(-p_i^2/m_i^2)ds^2=ds^2$, so that all particles have the same increment of proper time over the interval ds.

IV. COORDINATES FOR TWO PARTICLES

Before proceeding to the interacting two-particle problem, we will discuss a useful change of coordinates in the context of the noninteracting two-particle problem. The new set of canonical coordinates are related to the old set of coordinates by

$$\begin{aligned} &P^\nu\equiv p_1^\nu+p_2^\nu, \quad X^\nu\equiv(m_1x_1^\nu+m_2x_2^\nu)/M, \\ &p^\nu\equiv\mu(p_1^\nu/m_1-p_2^\nu/m_2), \quad x^\nu\equiv x_1^\nu-x_2^\nu, \end{aligned} \tag{5}$$

[with $M\equiv m_1+m_2$, $\mu\equiv m_1m_2/(m_1+m_2)$]. The inverse of these equations is

$$\begin{aligned} &p_1^\nu=m_1P^\nu/M+p^\nu, \quad x_1^\nu=X^\nu+\mu x^\nu/m_1, \\ &p_2^\nu=m_2P^\nu/M-p^\nu, \quad x_2^\nu=X^\nu-\mu x^\nu/m_2. \end{aligned} \tag{6}$$

Of course $\mathbf{P}$ and P^0 are the total momentum and energy, respectively, while $\mathbf{x}$ and t $(\equiv x^0)$ are the relative space and time coordinates, respectively.

$\mathbf{X}$ and T $(\equiv X^0)$ are the coordinates of a vector (in the world line space) whose tip lies on the line connecting the events $(\mathbf{x}_1,t_1)$ and $(\mathbf{x}_2,t_2)$. We shall call $\mathbf{X}$ and T the "center-of-mass" coordinates. It is well to emphasize here the difference between $\mathbf{X}$ and the usual relativistic center-of-energy (c.e.) coordinate. When there is no interaction, $\mathbf{X}$ and $\mathbf{X}_{\text{c.e.}}$ both are coordinates of points at rest in the center-of-mass reference frame (a Lorentz frame moving with velocity $\mathbf{P}/P^0$, in which the total momentum vanishes), but they are not necessarily coordinates of the same point. The usual c.e. coordinate $\mathbf{X}_{\text{c.e.}}(T_L)\equiv[p_1^0\mathbf{x}_1(T_L)+p_2^0\mathbf{x}_2(T_L)]/P^0$ [we have set $p_i^0=(\mathbf{p}_i^2+m_i^2)^{1/2}$] is computed when $\mathbf{x}_1$ and $\mathbf{x}_2$ have as arguments the same instant of time in the laboratory reference frame. The expressions for $\mathbf{x}_1$ and $\mathbf{x}_2$ given in Eq. (4), with $t_1=t_2\equiv T_L$, yield

$$\begin{aligned} \mathbf{X}_{\text{c.e.}}(t)=&(p_1^0\mathbf{x}_{1a}+p_2^0\mathbf{x}_{2a})/P^0 \\ &-(\mathbf{p}_1t_{1a}+\mathbf{p}_2t_{2a})/P^0+T_L\mathbf{P}/P^0. \end{aligned} \tag{7}$$

On the other hand, $\mathbf{X}(s)\equiv[m_1\mathbf{x}_1(s)+m_2\mathbf{x}_2(s)]/M$ is computed when $\mathbf{x}_1$ and $\mathbf{x}_2$ have as arguments the same "instant" of s. Substituting Eqs. (3) (with $s_{1a}=s_{2a}\equiv s_a$)

into the definitions of $\mathbf{X}$ and T yields

$$\mathbf{X}(s)=(m_1\mathbf{x}_{1a}+m_2\mathbf{x}_{2a})/M+(s-s_a)\mathbf{P}/M\,,\quad T(s)=(m_1t_{1a}+m_2t_{2a})/M+(s-s_a)P^0/M\,. \tag{8}$$

Since T is in one-to-one correspondence with s, we may write $\mathbf{X}$ in terms of the time T:

$$\mathbf{X}(T)=(m_1\mathbf{x}_{1a}+m_2\mathbf{x}_{2a})/M-(m_1t_{1a}+m_2t_{2a})\mathbf{P}/MP^0+T\mathbf{P}/P^0. \tag{9}$$

From Eqs. (7) and (9) we see that $\mathbf{X}_{\text{c.e.}}$ computed at time T_L and $\mathbf{X}$ computed at time T are points moving with velocity $\mathbf{P}/P^0$, but they are only the same point for special values of the initial coordinates ($X_{\text{c.e.}}$ and $\mathbf{X}$ can also be brought into coincidence simply by computing $\mathbf{X}$ at time $T+\tau$, where τ is an appropriate constant).

An important difference between $\mathbf{X}$ and $\mathbf{X}_{\text{c.e.}}$ arises when we introduce an interaction. It will later become obvious that even in this case $d\mathbf{X}/dT=\mathbf{P}/P^0$, where $\mathbf{P}$, P^0 are constants of the motion. On the other hand, $\mathbf{X}_{\text{c.e.}}$ does not move with constant speed during the interaction, (although before and after the interaction, $\mathbf{X}_{\text{c.e.}}$ does move with constant speed $\mathbf{P}/P^0$). Similarly, when each momentum is evaluated at a common time T_L and not at a common value of s, the sum of the momenta is not constant during the interaction (although before and after the interaction, the sum remains at the constant value $\mathbf{P}$). It is therefore the center-of-mass Lorentz frame, moving with constant speed $\mathbf{P}/P^0$, that is useful in this theory. In this Lorentz frame, the sum of momenta evaluated at the instant s vanishes. It is only before and after the interaction, that the center-of-mass reference frame coincides with the c.e. reference frame.

Much of the algebraic simplicity of this theory occurs because the useful coordinates (5) in which the equations of motion simplify are related to the individual particle coordinates by essentially the linear relationships of the Galilean-invariant theory, even though this is a Lorentz-invariant theory (compare $\mathbf{X}$ for example, with the nonlinear expression for $\mathbf{X}_{\text{c.e.}}$). This is possible because the particles are not forced to interact at the same time.

To see what interpretation to give $\mathbf{p},p^0$, we write the two-particle Hamiltonian [Eq. (1)] in terms of the new variables [using Eqs. (6)]

$$H=(\mathbf{P}^2-P^{02})/2M+(\mathbf{p}^2-p^{02})/2\mu\,. \tag{10}$$

Hamilton's equations of motion are then

$$d\mathbf{X}/ds=\mathbf{P}/M\,,\quad dT/ds=P^0/M\,,\quad d\mathbf{x}/ds=\mathbf{p}/\mu\,,\quad dt/ds=p^0/\mu\,, \tag{11}$$

while $d\mathbf{P}/ds=dP^0/ds=d\mathbf{p}/ds=dp^0/ds=0$. Thus p^ν/μ is the "velocity" (quotation marks because the derivatives are taken with respect to s) of the tip of the relative event vector x^ν in the $\mathbf{x}$-t plane.

We cannot regard x^ν as the position-time coordinates and p^ν as the momentum-energy variables of a fictitious particle of mass μ, for three important reasons. The first reason is simply that $\mathbf{p}^2-p^{02}\neq-\mu^2$. In fact, if $p_i^0=(\mathbf{p}_i^2+m_i^2)^{1/2}$, we find by using Eq. (5) that

$$\mathbf{p}^2-p^{02}=2\frac{\mu}{M}[(\mathbf{p}_1^2+m_1^2)^{1/2}(\mathbf{p}_2^2+m_2^2)^{1/2}-(\mathbf{p}_1\cdot\mathbf{p}_2+m_1m_2)]\geq0 \tag{12}$$

[the inequality follows from the Schwarz inequality $|a|\,|b|\geq a\cdot b$ applied to the four-dimensional vectors $(\mathbf{p}_1,m_1)$ and $(\mathbf{p}_2,m_2)$]. Thus the second reason is that when particles obey the correct energy-momentum-mass relationship, p^ν is a spacelike vector [assuming $(\mathbf{p}_1,m_1)\neq(\mathbf{p}_2,m_2)$]. The actual velocity of the fictitious particle would therefore be greater than the speed of light ($|d\mathbf{x}/dt|=|\mathbf{p}/p^0|>c$), so we cannot even regard p^ν as the momentum-energy of a particle whose mass is not μ. Thirdly, the sign of p^0 need not be positive.

The solutions of Eqs. (11) are Eqs. (8) and

$$\mathbf{x}=\mathbf{x}_a+(s-s_a)\mathbf{p}/\mu\,,\quad t=t_a+(s-s_a)p^0/\mu\,. \tag{13}$$

If p^0 is negative, Eq. (13) tells us that t will become increasingly negative as s increases. This is perfectly proper: It does not mean that "time is running backwards" or anything of the sort, since the particle's time coordinates are not t, they are t_i.[7] Equation (13) also tells us that the two particles asymptotically have a spacelike separation, since as $|s|\to\infty$, we have $\mathbf{x}^2-t^2\to s^2(\mathbf{p}^2-p^{02})/\mu^2\to+\infty$.

The considerations of the last paragraphs are relevant to the scattering problem, because it is asymptotically a free particle problem. *In the scattering problem, the particles start out and end up at a spacelike separation.* Actually, we have not quite shown that the particles end up at a spacelike separation when there is inelastic scattering [$\sqrt{(-p_i^2)}=m_i'\neq m_1$ as $s\to+\infty$], since in Eq. (12) we have chosen $\sqrt{(-p_i^2)}=m_i$. However, we shall shortly see that as a result of the dynamics, $\mathbf{p}^2-p^{02}$ is the same finally $(s\to+\infty)$ as initially $(s\to-\infty)$. Therefore the inequality (12) holds for outgoing particles since it holds for incoming particles, from which it follows that the outgoing particles achieve a spacelike separation as $s\to+\infty$.

As a matter of convention, we shall always choose $p^0\geq0$ as $s\to-\infty$ (if p^0 is negative, we need only exchange particles 1 and 2 to make it positive) so that

[7] The equation of motion $dt_i/ds=p_i^0/m_i$ [Eq. (2)], which also holds when there is an interaction, assures that t_i will increase with s provided the particle's energy p_i^0 is positive. If there is a particularly strong interaction, however, p_i^0 may become negative during part of some trajectories. In these cases, x_i will be three-valued over an interval of t_i. This "doubling back" of the relative event world line may be interpreted as particle-antiparticle creation, following Wheeler and Feynman [R. P. Feynman, Phys. Today **19**, 31 (1966)]. This is not "permanent" particle creation because the final asymptotic world line will still be single-valued.

for large negative s, t either increases or remains constant as s increases.

For our last topic dealing with noninteracting particles, we examine the nonrelativistic limit $|\mathbf{v}_i|/c\approx 0$ (we shall not set $c=1$ for this discussion). Inserting the definitions of $\mathbf{p}$ and p^0 [Eq. (5)] into Eq. (11), expanding in powers of $\mathbf{v}_i\equiv \mathbf{p}_i/m_i$ and retaining only the leading terms gives us

$$d\mathbf{x}/ds\approx \mathbf{v}_1-\mathbf{v}_2, \quad dt/ds\approx \tfrac{1}{2}(\mathbf{v}_1^2/c^2-\mathbf{v}_2^2/c^2). \tag{14}$$

We see that the temporal separation changes much more slowly than the spatial separation, in the nonrelativistic limit. When c is set equal to infinity, *the temporal separation becomes a constant of the motion.* If the initial condition then specifies that the temporal separation equals zero, we obtain the Galilean-invariant theory. It will be shown in Sec. VIII that for any problem, the Galilean-invariant theory is obtained in the limit as $c\to\infty$ and $c(t_i-t_j)\to 0$.

V. SCATTERING OF TWO PARTICLES IN ONE SPATIAL DIMENSION

The one-dimensional case shall be treated first because of its relative mathematical simplicity as compared to the three-dimensional case, and because it is easier to visulize a relative world line in the x-t plane than in the four-dimensional $\mathbf{x}$-t space. However, since the one-dimensional trajectories correspond to those three-dimensional trajectories which have zero angular momentum, the one-dimensional results have a somewhat limited generalization to three dimensions.

The Hamiltonian for two interacting particles is

$$H\equiv (p_1^2-p_1^{0\,2})/2m_1+(p_2^2-p_2^{0\,2})/2m_2 + V_+([(x_1-x_2)^2-(t_1-t_2)^2]^{1/2}), \tag{15}$$

[our notation is $x_i^\nu\equiv(x_i,t_i)$, $p_i^\nu\equiv(p_i,p_i^0)$], which is constructed from the Galilean-invariant theory with potential $V_+(|x_1-x_2|)$. Immediately the following worry appears: What if the dynamics is such that the argument of V_+ becomes imaginary? If V_+ depends upon an odd power of its argument, then V_+ and the dynamical variables will become complex, which is physically meaningless. To insure that this does not happen, we simply define a new potential function that is always real and which equals V_+ for $\rho^2\equiv(x_1-x_2)^2-(t_1-t_2)^2\geq 0$, e.g.,

$$V\equiv\theta(\rho^2)V_+(\sqrt{\rho^2})+\theta(-\rho^2)V_-(\sqrt{-\rho^2})$$

[where $\theta(z)=1$ for $z\geq 0$, $\theta(z)=0$ for $z<0$]. Of course the Galilean theory furnishes no guidelines as to what the function V_- may be. We shall pay special attention to those trajectories for which the two particles remain at spacelike (or lightlike) separation for all s, since for these trajectories, the choice of V_- is immaterial. In particular, we shall concentrate on showing that the important *elastic scattering trajectories are all "spacelike-separated" trajectories* [$\rho^2(s)\geq 0$ for $\infty>s>-\infty$].

Our aim in this section will be to find the general solution of the scattering problem, in the form of quadratures. First, we write the Hamiltonian [(Eq. (15)] in terms of center-of-mass and relative-event coordinates:

$$H=P\cdot P/2M+p\cdot p/2\mu+V(\rho).$$

[The notation is: $x^\nu\equiv(x,t)$, $p^\nu\equiv(p,p^0)$, $X^\nu\equiv(X,T)$, $P^\nu\equiv(P,P^0)$; $\rho^2\equiv x^2-t^2$; given a two-vector a^ν, we write $a\cdot a\equiv(a^1)^2-(a^0)^2$ instead of using our previous notation $a^2\equiv(a^1)^2-(a^0)^2$, in order to avoid writing, e.g., $p^2=p^2-p^{0\,2}$.] The first term in the Hamiltonian gives rise to a center-of-mass motion indentical to that discussed in the previous section. If the initial conditions are that at $s=s_a$, particles 1 and 2 have momenta p_{1a}, p_{2a} and energies $(p_{1a}^2+m_1^2)^{1/2}$, $(p_{2a}^2+m_2^2)^{1/2}$, then $P=p_{1a}+p_{2a}$, $P^0=(p_{1a}^2+m_1^2)^{1/2}+(p_{2a}+m_2^2)^{1/2}$ are constant for all s.

The dynamics is contained in the second and third terms of the Hamiltonian

$$h\equiv(p^2-p^{0\,2})/2\mu+V((x^2-t^2)^{1/2}). \tag{16}$$

The Hamiltonian equations of motion are

$$\begin{aligned} dx/ds&=p/\mu, \quad dp/ds=-[dV(\rho)/d\rho]x/\rho,\\ dt/ds&=p^0/\mu, \quad dp^0/ds=-[dV(\rho)/d\rho]t/\rho, \end{aligned}\tag{17}$$

or more simply, $d^2x^\nu/ds^2=-V'(\rho)x^\nu/\rho$. We seek those solutions $x(s)$, $t(s)$ of the differential equations which satisfy these initial conditions at $s=s_a$:

$$\begin{aligned} x(s_a)&=x_a, \quad dx/ds|_{s=s_a}=p_a/\mu\equiv p_{1a}/m_1-p_{2a}/m_2;\\ t(s_a)&=t_a, \quad dt/ds|_{s=s_a}=p_a^0/\mu\equiv p_{1a}^0/m_1-p_{2a}^0/m_2. \end{aligned}\tag{18}$$

We shall assume that V vanishes (or is negligibly small) for a sufficiently large value of its argument: say for $\rho>\rho_0$, where ρ_0, the range of the potential, is some positive number.[8] The choice of initial conditions will be such that $x_a^2-t_a^2\gg\rho_0^2$, so that the relative event world line will be initially a straight line.

There are two important constants of the motion. Of course, one is h. It follows from the vanishing of V at $s=s_a$ and the two-dimensional version of the inequality (12) that *the value of $h=p_a\cdot p_a/2\mu$ is positive.* The other constant of the motion is

$$N\equiv xp^0-pt. \tag{19}$$

It is easy to show directly that $dN/ds=0$, using Eqs. (17). Alternatively, N is the generating function of Lorentz transformations. That is, if $\bar{x}=x\cosh w+t\times\sinh w$, $\bar{t}=t\cosh w+x\sinh w$, $\bar{p}=p\cosh w+p^0\sinh w$,

[8] We shall also assume that $V(\rho)$ vanishes (or is negative valued) for sufficiently large negative ρ^2: say for $\rho^2<-\bar{\rho}_0^2$, where $\bar{\rho}_0^2$ is some positive number. This will ensure that the relative event vector becomes spacelike and not timelike as $s\to\infty$, as consideration of the "energy diagram" associated with Eq. (20) (*et seq.*) will make apparent.

$\bar{p}^0 = p^0 \cosh w + p \sinh w$, and $F(x,t,p,p^0)$ is any function, then

$$F(\bar{x},\bar{t},\bar{p},\bar{p}^0) = F(x,t,p,p^0) + w[N,F] + (w^2/2!)[N,[N,F]] + \cdots.$$

(The brackets are the Poisson brackets defined in Sec. II.) If h is substituted for F in the above equation, we see that the manifest invariance of h implies that $[N,h]=0$, and therefore $dN/ds=0$. Actually, it will be most useful to think of N as "angular momentum" in the x-t plane (since $\mu^{-1}N = x dt/ds - t dx/ds$), rather than as a generating function.

Since there are four dynamical variables and we know of only two constants of the motion, we must find two quadratures in order to solve the problem. The first quadrature, obtained by expressing Eq. (16) in the form of a differential equation, will enable us to find $\rho(s)$. In order to do this, we must write $p\cdot p$ in terms of $\rho(s)$. The square of Eq. (19) is

$$N^2 = (x\cdot p)^2 - \rho^2 p\cdot p,$$

and using $d\rho^2/ds = 2x\cdot p/\mu$ [Eq. (17)] we have

$$p\cdot p = (\mu^2/4\rho^2)(d\rho^2/ds) - N^2/\rho^2$$

which, inserted into Eq. (16), yields

$$\tfrac{1}{8}\mu(d\rho^2/ds)^2 + [V(\rho)-h]\rho^2 = N^2/2\mu. \quad (20)$$

This is a most useful form because of its similarity to the one-dimensional energy-conservation equation of ordinary Galilean mechanics [the equivalent form $h = \frac{1}{2}\mu(d\rho/ds)^2 + V(\rho) - N^2/2\mu\rho^2$ is not as useful because $d\rho/ds$ becomes imaginary for $\rho^2<0$, so that $(d\rho/ds)^2$ is not positive definite].

Visualize an "energy" diagram of Eq. (20), where the "effective potential" $[V(\rho)-h]\rho^2$ is graphed versus ρ^2. The "interaction term" $V(\rho)\rho^2$ is superimposed upon the "constant force" term $-h\rho^2$, which tends to push the particle in the direction of increasing ρ^2 (since h is positive). If $V(\rho)\rho^2 \to 0$ as $\rho^2 \to 0$ (which we shall assume, for simplicity), the effective potential vanishes at $\rho^2=0$. This has the consequence that, for all potentials satisfying $[V(\rho)-h]\rho^2 \leq 0$ for $\rho^2 \geq 0$ (this includes all attractive potentials), the only trajectories which do not become timelike-separated for some range of s are those for which the "effective energy" $N^2/2\mu$ vanishes (i.e., $N=0$). For a strong repulsive effective potential, where $[V(\rho)-h]\rho^2>0$ for $\rho^2>0$, the trajectories are spacelike-separated for all $|N|$ between zero and some maximum value. For $|N|$ above this maximum value, the trajectories become timelike-separated over some range of s.[8]

It is also useful to visualize the relative world line on an x-t diagram. The spacelike-separated region lies outside the light cone. In this region lie the two hyperbolas $x^2-t^2=\rho_0^2$, beyond which the potential vanishes. We may visualize a relative world line starting as a straight line of slope p^0/p $(1>p^0/p>0)$ in the negative-x–negative-t quadrant. As s increases, the relative world line enters the left-hand hyperbolic potential "boundary." It continues through the potential region, and may emerge via the right-hand boundary. This "transmissive" scattering world line must obviously cross inside the light cone, where ρ^2 is timelike, unless it passes through the origin (in which case $N=0$). The line may also veer inside the potential region, and exit from the left-hand potential boundary through which it entered. This is "reflective" scattering, for which the relative world line may or may not cross inside the light cone.

The first quadrature we seek is the integral of Eq. (20):

$$s - \text{const} = \pm\tfrac{1}{2}\mu \int^{\rho^2} \frac{d\rho^2}{\{N^2 + 2\mu\rho^2[h - V(\rho)]\}^{1/2}}. \quad (21)$$

The negative sign is used when $d\rho^2/ds \leq 0$, in which case the constant is s_a and the lower limit of the integral is $\rho^2(s_a)$. The positive sign is used when $d\rho^2/ds \geq 0$, in which case the lower limit of the integral is the solution of $N^2 + 2\mu\rho^2[h-V(\rho)]=0$, while the constant is the value of s corresponding to that value of ρ^2.

The second quadrature is most simply expressed in terms of a new variable. We define

$$\begin{aligned} x &\equiv \rho\cosh\alpha, & t &\equiv \rho\sinh\alpha & \text{for } x^2-t^2 &\geq 0; \\ x &\equiv \bar{\rho}\sinh\bar{\alpha}, & t &\equiv \bar{\rho}\cosh\bar{\alpha} & \text{for } x^2-t^2 &\leq 0. \end{aligned} \quad (22)$$

Curves of constant ρ $(\bar{\rho})$ are hyperbolas outside (inside) the light cone; curves of constant α $(\bar{\alpha})$ are straight lines of constant slope outside (inside) the light cone. We find that

$$d\alpha/ds = d(\tanh^{-1}(t/x))/ds = N/\mu\rho^2, \quad (23)$$

which follows from the equations of motion [Eq. (17)] and the definition of N [Eq. (19)]. Similarly, $d\bar{\alpha}/ds = -N/\mu\rho^2 = N/\mu\bar{\rho}^2$ (since in the timelike region, $\rho^2 = x^2-t^2 = -\bar{\rho}^2 < 0$). Upon integration we have

$$\alpha(s) - \alpha(s_a) = (N/\mu)\int_{s_a}^{s} ds'/\rho^2(s'). \quad (24)$$

Equation (24) is all that is required for spacelike-separated trajectories. In this case, we see that $\alpha(s)$ monotonically increases (if $N>0$), decreases (if $N<0$) or remains constant $(N=0)$. If the trajectories become timelike-separated, the equation

$$\bar{\alpha}(s) = (N/\mu)\int^{s} ds'/\rho^2(s') + \text{const}$$

must be used as well.

This completes our determination of the general solution of Eqs. (17) with initial conditions (18). Once $\rho^2(s)$ and $\alpha(s)$ $[\bar{\alpha}(s)]$ are known, one can determine $x(s)$ and $t(s)$ using Eqs. (22), while $p(s)$ and $p^0(s)$ can be found by differentiating x and t [Eqs. (17)]. The four constants h, N, $\rho^2(s_a)$, $\alpha(s_a)$, which appear in the solutions (21), (24), can be evaluated in terms of the four initial conditions x_a, t_a, p_a, and $p_a{}^0$.

VI. ELASTIC SCATTERING IN ONE SPATIAL DIMENSION

The masses of the particles after the scattering is completed will depend upon the values of p, p^0 for large s. If we affix subscript b's to momenta, energies, and masses to denote their final values, it follows from the definition $m_{ib}{}^2 \equiv -p_{ib}\cdot p_{ib}$, from Eqs. (6), from the constancy of $P^\nu = P_b{}^\nu$, and from the asymptotic h-conservation law $p_b \cdot p_b = p_a \cdot p_a$ that

$$m_{1b}{}^2 = -(m_1P/M + p_b)\cdot(m_1P/M+p_b) = m_1{}^2 - (2m_1/M)P\cdot(p_b - p_a)$$

and (25)

$$m_{2b}{}^2 = -(m_2P/M - p_b)\cdot(m_2P/M-p_b) = m_2{}^2 + (2m_2/M)P\cdot(p_b - p_a).$$

By eliminating $P\cdot(p_b-p_a)$ from Eqs. (25) (or alternatively, by using asymptotic H-conservation $p_{1b}{}^2/2m_1 + p_{2b}{}^2/2m_2 = p_{1a}{}^2/2m_1 + p_{2a}{}^2/2m_2$), we find that the final masses squared obey the linear relation

$$m_{1b}{}^2/m_1 + m_{2b}{}^2/m_2 = m_1 + m_2. \qquad (26)$$

Any two final masses consistent with Eq. (26) can arise from suitably chosen trajectories in a suitably chosen potential. Even an imaginary mass is possible.

According to Eq. (25), elastic scattering occurs when $P(p_b-p_a) - P^0(p_b{}^0 - p_a) = 0$. In the center-of-mass reference frame, where $P=0$, this elastic scattering condition is $p_b{}^0 = p_a{}^0$. It then follows from Eqs. (6) that $p_{1a}{}^0 = p_{1b}{}^0$ and $p_{2a}{}^0 = p_{2b}{}^0$, as expected. From asymptotic h conservation $p_b{}^2 - (p_b{}^0)^2 = p_a{}^2 - (p_a{}^0)^2$ we conclude that $|p_b| = |p_a|$. When $p_b = -p_a$, the scattering is reflective (notice that if $P=0$, then $p_1 = -p_2 = p$) while the scattering is transmissive when $p_b = +p_a$.

We will now elucidate some interesting properties of *reflective* elastic scattering in the center-of-mass reference frame.

1. $t_b = -t_a$

The initial conditions at $s=s_a$ were defined in Eq. (18). We now define some useful "final" conditions. The relative world line crosses the point (x_a, t_a) at s_a, continues through the interaction region, and finally returns to the spatial coordinate x_a at some larger value of s, which we shall call s_b. When $s=s_b$, the considerations of the previous paragraph tell us that $p_b = -p_a$ and $p_b{}^0 = p_a{}^0$. In order to relate t_b to t_a, we employ these relations and the constancy of N [Eq. (19)]:

$$x_a p_a{}^0 - p_a t_a = x_b p_b{}^0 - p_b t_b = x_a p_a{}^0 - (-p_a) t_b,$$

so that $t_b = -t_a$. We summarize the "final" conditions

$$x(s_b) = x_a, \quad dx/ds|_{s_b} = -p_a/\mu,$$
$$t(s_b) = -t_a, \quad dt/ds|_{s_b} = p_a{}^0/u. \qquad (27)$$

2. $x(s) = x(s_b+s_a-s)$, $t(s) = -t(s_b+s_a-s)$

Now we can show that a relative world line for reflective elastic scattering is unchanged if it is reflected across the x axis and run backwards. We define

$$\bar{x}(s) \equiv x(s_b+s_a-s), \quad \bar{t}(s) \equiv -t(s_b+s_a-s). \qquad (28)$$

One can easily see that $\bar{x}(s)$, $\bar{t}(s)$ satisfy the same second-order differential equations that $x(s)$ and $t(s)$ satisfy [Eqs. (17)]. Furthermore, the *final* conditions on x^ν [Eq. (27)] determine the *initial* conditions on $\bar{x}^\nu$:

$$\bar{x}(s_a) = x(s_b+s_a-s_a) = x_a,$$
$$d\bar{x}/ds|_{s_a} = -dx/ds|_{s_b} = p_a/\mu,$$
$$\bar{t}(s_a) = -t(s_b+s_a-s_a) = t_a, \qquad (29)$$
$$d\bar{t}/ds|_{s_a} = +dt/ds|_{s_b} = p_a{}^0/\mu,$$

which are identical to the intial conditions on x^ν [Eqs. (18)]. Since $\bar{x}^\nu(s)$ and $x^\nu(s)$ satisfy the same equations and initial conditions, they are identically the same functions.

3. $d\rho^2/ds = 0$ when $t=0$

From the equation $t(s) = -t(s_b+s_a-s)$ computed at $s=\frac{1}{2}(s_b+s_a)$, we find that $t(\frac{1}{2}(s_b+s_a))=0$.

By taking the derivative of $x(s)=x(s_b+s_a-s)$ with respect to s, and evaluating the resulting equation at $s=\frac{1}{2}(s_b+s_a)$, we find that $dx(s)/ds|_{\frac{1}{2}(s_b+s_a)}=0$.

Now we see that $d\rho^2/ds|_{(s_b+s_a)/2}=0$, since $d\rho^2/ds = 2x dx/ds - 2t dt/ds$ and $dx/ds|_{(s_b+s_a)/2} = t(\frac{1}{2}(s_b+s_a))=0$.

4. $\rho^2(s) \geq 0$ for all s

Finally, we are in a position to prove that all reflective elastic scattering trajectories are spacelike-separated. Consideration of the differential equation (20) for $\rho^2(s)$ and it associated "energy" diagram shows that in a scattering problem there is only one minimum of the function $\rho^2(s)$ and no maximum. But we have just found that $\rho^2(s)$ is stationary at $s=\frac{1}{2}(s_b+s_a)$. Therefore, it is at this value of s that the unique minimum occurs, and so

$$\rho^2(s) \geq x^2(\tfrac{1}{2}(s_b+s_a)) - t^2(\tfrac{1}{2}(s_b+s_a)) = x^2(\tfrac{1}{2}(s_b+s_a)) - 0 \geq 0.$$

This treatment of reflective elastic scattering is quite similar to the discussion of three-dimensional elastic scattering (next section). On the other hand, transmissive elastic scattering is somewhat unusual. To begin with, we can prove that all transmissive trajectories satisfying $N=0$ are elastic scattering trajectories. Since

$$\mu^{-1}N = dxt/ds - tdx/ds = x^2 d(t/x)/ds = 0$$

and $N=xp^0-pt=0$ imply that $t/x=p^0/p=\text{const}$ (we need not consider the possibility that $x=0$), *these relative world lines are straight.* Although $p_0(s)$ and $p(s)$ change during the interaction, they maintain their ratio, so that $p_a{}^0/p_a=p_b{}^0/p_b$. This equality, together with asymptotic h conservation $p_a{}^2-(p_a{}^0)^2=p_b{}^2-(p_b{}^0)^2$ implies that either $p_b{}^0=-p_a{}^0$ and $p_b=-p_a$ or else that $p_b{}^0=p_a{}^0$, $p_b=p_a$. The latter corresponds to elastic transmissive scattering (the former corresponds to an inelastic reflective scattering).

The $N=0$ elastic transmissive scattering has many of the properties of elastic reflective scattering:

(1) $t_b=-t_a$. The useful final conditions for transmissive elastic scattering are that the relative world line travels from the point (x_a,t_a) at s_a through the scattering region and eventually reaches the spatial coordinate $-x_a$ at some larger value of s, which we shall call s_b. If we define $t(s_b)\equiv t_b$, the constancy of the ratio t/x requires that $t_b=-t_a$.

(2) $x(s)=-x(s_b+s_a-s)$, $t(s)=-t(s_b+s_a-s)$. It is readily verified that

$$\bar{x}(s)\equiv -x(s_b+s_a-s)\,,\quad \bar{t}(s)\equiv -t(s_b+s_a-s)$$

satisfy the same differential equations and initial conditions that $x(s)$ and $t(s)$ satisfy, so that they describe the same relative world line. Note that $\bar{x}(s)$, $\bar{t}(s)$ describe the world line reflected through the origin (this differs from the reflection symmetry of reflective elastic scattering) and run backwards.

(3) $d\rho^2/ds=0$ *when* $t=0$. Since the relative world line passes through $(x,t)=(0,0)$, both $\rho^2=0$ and $d\rho^2/ds=0$ when $t=0$.

(4) $\rho^2(s)\geq 0$ *for all s*. Since the relative world line is a straight line which has a slope p^0/p of magnitude less than 1, and it passes through the origin, it lies wholly in the spacelike region.

The question arises as to whether there are any transmissive elastic scattering trajectories other than those which satisfy $N=0$. We will just state, without proof, that for some (not all) potentials it is possible to specify initial conditions of some transmissive elastic scattering trajectories which satisfy neither $N=0$, nor $\rho^2(s)\geq 0$ for all s. We shall regard these trajectories as anomalous: They can be eliminated by a suitable choice of V for $\rho^2<0$.

The hallmark of the elastic scattering relative world lines (when $P=0$) is their reflection symmetry in the $x-t$ plane. Crudely speaking, the particles leave the interaction region on trajectories which are the "reverse" of the trajectories on which they entered. The other interesting property of the elastic scattering trajectories is that they are spacelike-separated (of course, there are inelastic scattering trajectories which are also spacelike-separated). This property is also possessed by the trajectories of the Galilean-invariant theory. In both theories, the elastic scattering trajectories are determined by the same potential function of positive argument.

VII. SCATTERING IN THREE SPATIAL DIMENSIONS

This section will contain essentially a duplication of the discussion that appeared in Secs. V and VI. To find the quadratures of the three-dimensional scattering problem, we consider the Hamiltonian

$$H=p_1{}^2/2m_1+p_2{}^2/2m_2+V(\rho)=P^2/2m+p^2/2\mu+V(\rho)$$

($\rho^2\equiv\mathbf{x}^2-t^2$; we have returned to the three-dimensional notation of Sec. IV, e.g., $p^2=\mathbf{p}^2-p^{0 2}$). The center-of-mass motion caused by the term $P^2/2M$ has been discussed. The interaction part of the Hamiltonian

$$h=\tfrac{1}{2}[\mathbf{p}^2-(p^0)^2]+V((\mathbf{x}^2-t^2)^{1/2}) \tag{30}$$

gives rise to the equations of motion:

$$\begin{aligned} d\mathbf{x}/ds&=\mathbf{p}/\mu\,,\quad d\mathbf{p}/ds=-[dV(\rho)/d\rho]\mathbf{x}/\rho\,,\\ dt/ds&=p^0/\mu\,,\quad dp^0/ds=-[dV(\rho)/d\rho]t/\rho\,. \end{aligned} \tag{31}$$

The interesting new feature is that there are six constants of the motion. In addition to h [which is positive because of the inequality (12)], there is the well-known generator of homogeneous Lorentz transformations

$$N^{\nu\mu}\equiv x^\mu p^\nu-x^\nu p^\mu\,, \tag{32}$$

which satisfies $dN^{\mu\nu}/ds=0$, as may be verified by direct differentiation and use of Eqs. (31). $N^{\mu\nu}$ can be written in terms of two three-vectors

$$\mathbf{j}\equiv\mathbf{x}\times\mathbf{p}\quad\text{and}\quad\mathbf{n}\equiv\mathbf{x}p^0-\mathbf{p}t\,. \tag{33}$$

It might be concluded that Eqs. (33) contain six independent constants of the motion but there are only five because

$$\mathbf{j}\cdot\mathbf{n}=0\,. \tag{34}$$

Of course $\mathbf{j}$ is the angular momentum, while $\mathbf{n}$ might be thought of as components of "angular momentum" in each space-time plane. However, we find it preferable to think of $\mathbf{n}$ simply as a vector in three-dimensional $\mathbf{x}$ space. It lies in the plane of the scattering, since it is orthogonal to the angular momentum [Eq. (34)]. We will shortly see that in the center-of-mass reference frame, the elastic scattering trajectories traced out by the relative position vector $\mathbf{x}(s)$ are unchanged by reflection across the vector $\mathbf{n}$.

We have eight dynamical variables and six constants of the motion, so again we need two quadratures. The quadrature for $\rho(s)$ is once more obtained by eliminating p^2 from the expression for h. We calculate $N^2\equiv\frac{1}{2}N^{\mu\nu}N^{\lambda\kappa}g_{\mu\kappa}g_{\nu\lambda}$:

$$N^2=(x\cdot p)^2-\rho^2p^2=\mathbf{n}^2-\mathbf{j}^2\,. \tag{35}$$

Using $d\rho^2/ds=2x\cdot p/\mu$ [Eqs. (31)], we again obtain $p^2=(\mu^2/\rho^2)(d\rho^2/ds)-N^2/\rho^2$, and upon insertion into

Eq. (30),

$$(\mu/8)(d\rho^2/ds)^2+[V(\rho)-h]\rho^2=(\mathbf{n}^2-\mathbf{j}^2)/2\mu. \quad (36)$$

The only difference between Eq. (36) and its one-dimensional counterpart (20) is that in three dimensions the constant N^2 can take on negative as well as positive values, because the angular momentum $\mathbf{j}$ need not be zero. Upon consideration of the energy diagram associated with Eq. (36) we see that, since we assume the effective potential $[V(\rho)-h]\rho^2$ vanishes as $\rho\to 0$, all trajectories for which the effective energy $N^2/2\mu$ is negative (or zero) are spacelike-separated trajectories. When $N^2>0$, the trajectories may or may not be spacelike-separated, depending upon the position and height of the maximum value of the potential, just as in the one-dimensional case.

The first quadrature, the integral of Eq. (36), is identical to Eq. (21). The second quadrature is most simply expressed in terms of new variables α (or $\bar{\alpha}$):

$$\begin{aligned} |\mathbf{x}|&\equiv\rho\cosh\alpha, \quad t\equiv\rho\sinh\alpha, \quad \mathbf{x}^2-t^2\geq 0;\\ |\mathbf{x}|&\equiv\bar{\rho}\sinh\bar{\alpha}, \quad t\equiv\bar{\rho}\cosh\bar{\alpha}, \quad \mathbf{x}^2-t^2\leq 0. \end{aligned} \quad (37)$$

Equations (37) are identical to their one-dimensional counterparts (22), except that here we are dealing with the $|\mathbf{x}|-t$ *half-plane* instead of the $x-t$ plane, so that the ranges of the variables must be suitably restricted (i.e., $\rho\geq 0$, and $\bar{\rho}\geq 0, \bar{\alpha}\geq 0$ or $\bar{\rho}\leq 0, \bar{\alpha}\leq 0$). We compute $d\sinh\alpha/ds$,

$$d\sinh\alpha/ds=(\rho^3\mu)^{-1}(\mathbf{x}^2p^0-t\mathbf{x}\cdot\mathbf{p})=(\rho^3\mu)^{-1}\mathbf{n}\cdot\mathbf{x}, \quad (38)$$

by using Eqs. (31) and the definition (33) of $\mathbf{n}$. In order to obtain $\mathbf{n}\cdot\mathbf{x}$ in terms of α and ρ, we first note from Eqs. (33) that $\mathbf{n}\times\mathbf{x}=\mathbf{j}t$. This expression, together with the identity $(\mathbf{n}\times\mathbf{x})^2=\mathbf{n}^2\mathbf{x}^2-(\mathbf{n}\cdot\mathbf{x})^2$ and the expression in (35), $N^2=\mathbf{n}^2-\mathbf{j}^2$ yields

$$(\mathbf{n}\cdot\mathbf{x})^2=\mathbf{n}^2\rho^2+N^2t^2=\rho^2(\mathbf{n}^2+N^2\sinh^2\alpha). \quad (39)$$

Putting Eq. (39) into Eq. (38), we obtain,

$$(\mathbf{n}^2+N^2\sinh^2\alpha)^{-1/2}d\sinh\alpha/ds=\pm(\rho^2\mu)^{-1}. \quad (40)$$

The integral of Eq. (40) depends upon whether N^2 is is positive or negative. After a little manipulation, the results are

$$\sinh\alpha=\pm\left(\frac{\mathbf{n}^2}{-N^2}\right)^{1/2}\sin\left[\left(\frac{-N^2}{\mu^2}\right)^{1/2}\times\int_{s_a}^{s}\frac{ds'}{\rho^2(s')}+\text{const}\right], \quad N^2<0 \quad (41a)$$

$$\sinh\alpha=\pm\left(\frac{\mathbf{n}^2}{N^2}\right)^{1/2}\sinh\left[\left(\frac{N^2}{\mu^2}\right)^{1/2}\times\int^{s}\frac{ds'}{\rho^2(s')}+\text{const}\right], \quad N^2>0. \quad (41b)$$

These equations are a bit more complicated than the one-dimensional version (24) [which may be obtained by letting $N^2=\mathbf{n}^2$ in Eq. (41b)]. The constants and signs in these equations may be obtained from the initial conditions of the problem. We can set the lower limit of the integral in Eq. (41a) equal to s_a [and therefore the constant$=\pm\sin^{-1}((-N^2/\mathbf{n}^2)^{1/2}\sinh\alpha_a)$] because the equation is valid for all s (the trajectories never become timelike-separated when $N^2<0$). We observe that Eq. (41a) restricts the magnitude of $\sinh^2\alpha$ according to $\sinh^2\alpha\leq\mathbf{n}^2/(-N^2)$. This restriction is also apparent from Eq. (39), where the right-hand side is forced to be greater than or equal to zero since the left-hand side is the square of a number.

When $N^2>0$, the trajectories may become timelike-separated and Eq. (41b) must be supplemented by a quadrature for $\bar{\alpha}$. The analog of Eq. (38) is

$$d\cosh\bar{\alpha}/ds=-(\bar{\rho}^3\mu)^{-1}\mathbf{n}\cdot\mathbf{x}.$$

Equation (39) becomes $(\mathbf{n}\cdot\mathbf{x})^2=\bar{\rho}^2(N^2\cosh^2\bar{\alpha}-\mathbf{n}^2)$, from which we see that $\cosh^2\bar{\alpha}\geq\mathbf{n}^2/N^2$. The differential equation

$$(N^2\cosh^2\bar{\alpha}-\mathbf{n}^2)^{-1/2}d\cosh\bar{\alpha}/ds=\pm(\rho^2\mu)^{-1}$$

(using $\bar{\rho}^2=-\rho^2$) has the solution

$$\cosh\bar{\alpha}=\left(\frac{\mathbf{n}^2}{N^2}\right)^{1/2}\cosh\left[\left(\frac{N^2}{\mu^2}\right)^{1/2}\int\frac{ds'}{\rho^2(s')}+\text{const}\right]. \quad (41c)$$

This completes the solution. Once $\rho^2(s)$ and $\alpha(s)$ $[\bar{\alpha}(s)]$ are known, one can determine $|\mathbf{x}(s)|$ and $t(s)$ from Eqs. (37). Six other dynamical variables (e.g., $\mathbf{p}$, p^0, y, z) can be written in terms of $|\mathbf{x}(s)|$ and $t(s)$ and the eight initial conditions x_a^ν, p_a^ν, using the expressions for the six constants of the motion h, $N^{\mu\nu}$.

We proceed to investigate the properties of the elastic scattering trajectories in the center-of-mass frame (where $\mathbf{P}=0$). According to Eq. (25), elastic scattering occurs when $p_b^0=p_a^0$. This result, together with asymptotic h-conservation $p_b^2=p_a^2$, tells us that $|\mathbf{p}_b|=|\mathbf{p}_a|$.

Four properties of elastic scattering trajectories, similar or identical to those that appeared in the discussion of one-dimensional scattering, will now be presented. In the following discussion, we will assume that $\mathbf{n}\neq 0$ and $\mathbf{j}\neq 0$. The special cases of $\mathbf{n}=0$ or $\mathbf{j}=0$ will be taken up separately.

1. $t_b=-t_a$

The relative world line at $s=s_a$ passes through the point $(\mathbf{x}_a,t_a)$, proceeds through the interaction region and emerges. Eventually, at some larger value of s which we shall call s_b, the magnitude of the radius vector again equals $|\mathbf{x}_a|$. The time coordinate $t(s_b)\equiv t_b$ will now be related to t_a by use of the final conditions $\mathbf{x}_b^2=\mathbf{x}_a^2$, $\mathbf{p}_b^2=\mathbf{p}_a^2$, $p_b^0=p_a^0$, and the constancy of $\mathbf{n}$, $\mathbf{j}$.

We first need to relate the angles between $\mathbf{x}_a$, $\mathbf{n}$, and $\mathbf{p}_a$ to the angles between $\mathbf{x}_b$, $\mathbf{n}$, and $\mathbf{p}_b$. By equating the value of $\mathbf{j}^2=\mathbf{x}^2\mathbf{p}^2-(\mathbf{x}\cdot\mathbf{p})^2$ at $s=s_a$ to its value at $s=s_b$, we find that $|\mathbf{x}_a\cdot\mathbf{p}_a|=|\mathbf{x}_b\cdot\mathbf{p}_b|$. In fact, because $\mathbf{p}_a$ is an incoming vector while $\mathbf{p}_b$ is an outgoing vector,

$$\mathbf{x}_a\cdot\mathbf{p}_a+\mathbf{x}_b\cdot\mathbf{p}_b=0. \tag{42a}$$

Next, by evaluating $\mathbf{n}\times\mathbf{p}=\mathbf{j}p^0$ [Eq. (33)] at $s=s_a$ and at $s=s_b$, we find that $\mathbf{n}\times\mathbf{p}_a=\mathbf{n}\times\mathbf{p}_b$. This implies that either

$$\mathbf{n}\cdot\mathbf{p}_a+\mathbf{n}\cdot\mathbf{p}_b=0 \tag{42b}$$

or $\mathbf{p}_a=\mathbf{p}_b$. We will exclude this last possibility from discussion.[9]

If we now evaluate $\mathbf{n}\cdot\mathbf{p}=\mathbf{x}\cdot\mathbf{p}\;p^0-\mathbf{p}^2t$ at $s=s_a$ and $s=s_b$ and add the two equations together, we obtain

$$(t_b+t_a)\mathbf{p}_a{}^2=p_a{}^0(\mathbf{x}_a\cdot\mathbf{p}_a+\mathbf{x}_b\cdot\mathbf{p}_b)-(\mathbf{n}\cdot\mathbf{p}_a+\mathbf{n}\cdot\mathbf{p}_b).$$

The expressions in parentheses on the right-hand side of this equation vanish on account of Eqs. (42), which gives the result $t_b=-t_a$ that we seek.

A. $x_b=2n\,n\cdot x_a/n^2-x_a$, $p_b=p_a-2n\,n\cdot p_a/n^2$

All the final conditions have not yet been related to initial conditions; i.e., expressions for $\mathbf{x}_b$ in terms of $\mathbf{x}_a$ and $\mathbf{p}_b$ in terms of $\mathbf{p}_a$ are still lacking. We now remedy this deficiency. As a preliminary, we need to compliment Eqs. (42) by a relation between $\mathbf{n}\cdot\mathbf{x}_a$ and $\mathbf{n}\cdot\mathbf{x}_b$. By computing $\mathbf{n}\cdot\mathbf{x}=\mathbf{x}^2\mathbf{p}^0-\mathbf{x}\cdot\mathbf{p}t$ at $s=s_a$ and at $s=s_b$ and comparing these two expressions, it follows from relations already derived that $\mathbf{n}$ bisects the angle between $\mathbf{x}_a$ and $\mathbf{x}_b$.

$$\mathbf{n}\cdot\mathbf{x}_a=\mathbf{n}\cdot\mathbf{x}_b. \tag{42c}$$

Next we will show that $\mathbf{x}_a+\mathbf{x}_b$ and $\mathbf{p}_a-\mathbf{p}_b$ are parallel to $\mathbf{n}$. Indeed, $\mathbf{j}$, $\mathbf{n}$, and $\mathbf{j}\times\mathbf{n}$ form an orthogonal set of vectors, of which $\mathbf{n}$ and $\mathbf{j}\times\mathbf{n}$ lie in the plane of the motion. Taking the scalar product of $\mathbf{j}$ with the relationship $\mathbf{n}\times\mathbf{p}=\mathbf{j}p^0$, and evaluating the result at $s=s_a$ and at $s=s_b$, we see that $\mathbf{j}\cdot\mathbf{n}\times\mathbf{p}_a=\mathbf{j}\cdot\mathbf{n}\times\mathbf{p}_b$. A permutation of the variables in the triple product leads to $(\mathbf{p}_b-\mathbf{p}_a)\cdot\mathbf{j}\times\mathbf{n}=0$. Similarly, by taking the scalar product of $\mathbf{j}$ with the relationship $\mathbf{n}\times\mathbf{x}=\mathbf{j}t$, and evaluating the result at $s=s_a$ and at $s=s_b$, we find that $(\mathbf{x}_a+\mathbf{x}_b)\cdot\mathbf{j}\times\mathbf{n}=0$. Therefore, we may write $\mathbf{x}_a+\mathbf{x}_b=A\mathbf{n}$ and $\mathbf{p}_b-\mathbf{p}_a=B\mathbf{n}$. The constants A and B are easily evaluated by taking the scalar product of these equations with $\mathbf{n}$ and employing Eqs. (42b) and (42c). The whole set of relations may then be tabulated:

$$\begin{aligned}&\mathbf{x}_a=2\mathbf{n}\;\mathbf{n}\cdot\mathbf{x}_b/\mathbf{n}^2-\mathbf{x}_b\,,\quad \mathbf{p}_a=-2\mathbf{n}\;\mathbf{n}\cdot\mathbf{p}_b/\mathbf{n}^2+\mathbf{p}_b\,,\\ &t_a=-t_b\,,\quad p_a{}^0=p_b{}^0.\end{aligned} \tag{43}$$

$\mathbf{x}_a$ is the reflection of the vector $\mathbf{x}_b$ across the vector $\mathbf{n}$, $\mathbf{p}_a$ is the reflection of the vector $\mathbf{p}_b$ across the vector $\mathbf{j}\times\mathbf{n}$.

2. $x(s)=2n\,n\cdot x(s_b+s_a-s)/n^2-x(s_b+s_a-s)$, $t(s)=-t(s_b+s_a-s)$

Now we are able to show that the relative world line is unchanged if $\mathbf{x}$ is reflected across the vector $\mathbf{n}$, the sign of t is reversed, and the whole trajectory is run backwards. We define

$$\begin{aligned}&\bar{\mathbf{x}}(s)\equiv 2\mathbf{n}\;\mathbf{n}\cdot\mathbf{x}(s_b+s_a-s)/\mathbf{n}^2-\mathbf{x}(s_b+s_a-s)\,,\\ &\bar{t}(s)\equiv -t(s_b+s_a-s)\,.\end{aligned} \tag{44}$$

It is straightforward to show that

$$\bar{\mathbf{x}}(s)^2-\bar{t}(s)^2=x^2(s_b+s_a-s)-t^2(s_b+s_a-s)\,,$$

and then to show that

$$\frac{d^2\bar{x}^\nu(s)}{ds^2}=-\frac{V'((\bar{\mathbf{x}}^2-\bar{t}^2)^{1/2})\bar{x}^\nu}{(\bar{\mathbf{x}}^2-\bar{t}^2)^{1/2}}$$

follows from the equations of motion (31) for x^ν. The initial conditions for $\bar{x}^\nu$ at $s=s_a$ are related to the final conditions for x^ν at $s=s_b$, and when Eqs. (43) are employed, we discover that $\bar{x}^\nu(s_a)=x^\nu(s_a)$, $d\bar{x}^\nu/ds|_{s_a}=dx^\nu/ds|_{s_a}$. Since $\bar{x}^\nu(s)$ and $x^\nu(s)$ satisfy the same equations and initial conditions, they are identically the same functions.

3. $d\varrho^2/ds=0$ when $t=0$

By evaluating the equation $t(s)=-t(s_b+s_a-s)$ at $s=\frac{1}{2}(s_b+s_a)$, we conclude that $t(\frac{1}{2}(s_b+s_a))=0$.

If we compute $\mathbf{n}=\mathbf{x}p^0-\mathbf{p}t$ at $s=\frac{1}{2}(s_b+s_a)$, we find that $\mathbf{n}=\mathbf{x}p^0|_{(s_b+s_a)/2}-0$, *so that* $\mathbf{x}$ *is parallel to* $\mathbf{n}$ *when* $t=0$.

Upon taking the derivative of Eq. (44) with respect to s [after setting $\bar{\mathbf{x}}(s)=\mathbf{x}(s)$], and evaluating the result

$$\mathbf{p}(s)=-2\mathbf{n}\;\mathbf{n}\cdot\mathbf{p}(s_b+s_a-s)/\mathbf{n}^2+\mathbf{p}(s_b+s_a-s)$$

at $s=\frac{1}{2}(s_b+s_a)$, we discover that $\mathbf{n}\cdot\mathbf{p}(\frac{1}{2}(s_b+s_a))=0$. Since $\mathbf{p}$ is perpendicular to $\mathbf{n}$, it is also perpendicular to $\mathbf{x}$ at $s=\frac{1}{2}(s_b+s_a)$.

It is now clear that

$$d\rho^2/ds|_{(s_b+s_a)/2}=2\mu^{-1}(\mathbf{x}\cdot\mathbf{p}-tp^0)|_{(s_b+s_a)/2}=0\,,$$

since $\mathbf{x}\cdot\mathbf{p}$ and t both vanish at $s=\frac{1}{2}(s_b+s_a)$.

4. $\varrho^2(s)\geq 0$ for all s

The proof of this statement is identical to that employed in the case of one-dimensional reflective

[9] The two cases where $\mathbf{p}_a=\mathbf{p}_b$ are the cases of no interaction, and forward scattering. When there is no interaction, Eq. (42b) *is* satisfied, for in this case $\mathbf{n}\cdot\mathbf{p}_a=\mathbf{n}\cdot\mathbf{p}_b=0$ (see the discussion in Sec. VIII). There also exist forward-scattering trajectories satisfying Eq. (42b), when the potential is attractive. For certain potentials, it is possible to find elastic scattering trajectories satisfying $\mathbf{p}_a=\mathbf{p}_b$ but not satisfying Eq. (42b). This represents forward scattering of an anomalous nature, just as do the one-dimensional transmissive scattering trajectories for which $N\neq 0$. Such trajectories may be eliminated by suitably altering the potential function for $\rho^2<0$.

elastic scattering, and the reader is referred back to that paragraph.

In the above discussion, we have assumed that $\mathbf{j}\neq 0$ and $\mathbf{n}\neq 0$. We shall now examine the elastic scattering trajectories which satisfy $\mathbf{j}=0$ and/or $\mathbf{n}=0$, and demonstrate that they possess the four properties itemized above.

When $\mathbf{j}=0$ and $\mathbf{n}\neq 0$, the three-dimensional problem essentially reduces to the one-dimensional problem. Since $\mathbf{j}=0$ implies that $\mathbf{p}$ and $\mathbf{x}$ are parallel or antiparallel, we may pick our three-dimensional spatial coordinate system so that $\mathbf{x}$ and $\mathbf{p}$ lie along the x axis during all of the motion. The vector $\mathbf{n}$ thus points along the x axis and is of magnitude xp^0-p_xt. It then can be seen that all of the three-dimensional results are identical to the one-dimensional results. For example, the quadrature equation (41b) for α reduces to Eq. (24) when N^2 is set equal to $\mathbf{n}^2$, while the differential equation (36) for ρ^2 reduces to Eq. (29). The four elastic scattering properties are also identical: The invariance of the world line with respect to reflection across the x axis (Sec. VI, property **2** for reflective scattering) is identical to the invariance of the world line with respect to the combined operations of reflection across the $\mathbf{n}$ vector (which has no effect at all) and reversal of the sign of t (Sec. VII, property **2**).

When $\mathbf{n}=0$, it follows from $\mu^{-1}\mathbf{n}=\mathbf{x}dt/ds-td\mathbf{x}/ds=-t^2d(\mathbf{x}/t)/ds=0$ that there are two possibilities: either $\mathbf{x}=\text{const}\times t$ (the constant is $\mathbf{p}/p^0$) or else $t=0$. The former possibility requires that $\mathbf{x}\times\mathbf{p}=\mathbf{j}=0$, and the scattering must be elastic (from the constancy of $\mathbf{p}/p^0$ and asymptotic h conservation). Since $\mathbf{j}=0$, all the comments of the previous paragraph apply, except the statements about the direction of $\mathbf{n}$, which is a zero vector. Actually, since $\mathbf{n}$ bisects trajectories, in the limit as $\mathbf{p}_b\rightarrow\mathbf{p}_a$, the direction of $\mathbf{n}$ becomes *perpendicular* to the trajectory (although $\mathbf{n}\rightarrow 0$). The invariance of the world line with respect to reflection through the origin (Sec. VI, property **2** for transmissive scattering) is identical to the invariance of the world line with respect to the combined operations of reflection across a vector perpendicular to the trajectory and reversal of the sign of t (Sec. VII, property **2**).

The trajectories for which $t=0$ are especially interesting. In fact, when $t(s)=0$, the equations of motion (31) become $p^0=\mu dt/ds=0$, $dp^0/ds=-V't/\rho=0$, and (since $\rho=|\mathbf{x}|$)

$$d\mathbf{x}/ds=\mathbf{p}/\mu\,,\quad d\mathbf{p}/ds=-V'(|\mathbf{x}|)\mathbf{x}/|\mathbf{x}|\,. \qquad (45)$$

These are just the equations of relative motion of the Galilean theory. In the present Lorentz-invariant theory they describe the scattering of particles whose initial and final masses and momenta are such that $p_a^0=p_b^0=0$. These are not necessarily elastic scattering trajectories, even though all solutions of Eqs. (45) satisfy the relation $\mathbf{p}_b^2=\mathbf{p}_a^2$. Elastic scattering only occurs if $P\cdot(p_b-p_a)=\mathbf{P}\cdot(\mathbf{p}_b-\mathbf{p}_a)-0=0$ [Eq. (25)], and $\mathbf{P}$ may not be such as to satisfy this condition. However, in the center-of-mass frame where we have been analyzing the properties of elastic scattering, $\mathbf{P}=0$, and Eqs. (45) together with the relations $t=0$, $p^0=0$ do describe elastic scattering. It is the elastic scattering of equal-mass particles [since $p_a^0=\mu((\mathbf{p}_a^2+m_1^2)^{1/2}/m_1-(\mathbf{p}_a^2+m_2^2)^{1/2}m_2)$ only vanishes if $m_1=m_2$].

These equal-mass elastic scattering trajectories may easily be shown to possess the last four properties that the other elastic scattering trajectories possess. For example, although $\mathbf{n}$ vanishes, its direction (in the limit as $t\rightarrow 0$, $p^0\rightarrow 0$) approaches that of the well-known bisector of the scattering trajectory described by Eqs. (45). Reflection across this bisector and reversal of the sign of t (which has no effect at all) does take the world line into itself.

Thus it can finally be concluded that, with the possible exception of some forward-scattering trajectories, all the elastic scattering trajectories satisfy the same four properties.

VIII. NONRELATIVISTIC LIMIT; NO INTERACTION LIMIT, BOUND STATES; ADVANCED ACTIONS

(A) Let us imagine that we have solved the equations of motion (31) for $\mathbf{x}(s)$, $t(s)$ with appropriate initial and final conditions. We wish to find the limits $\mathbf{x}_{\text{n.r.}}(s)\equiv\lim\mathbf{x}(s)$, $t_{\text{n.r.}}(s)\equiv\lim t(s)$ as $c\rightarrow\infty$, assuming that these limits exist.

It may first be observed that all trajectories become elastic scattering trajectories in the limit as $c\rightarrow\infty$. The leading terms in Eq. (25) are

$$m_{1,2b}^2=m_{1,2}^2 \mp(2m_{1,2}/M)[\mathbf{P}\cdot(\mathbf{p}_b-\mathbf{p}_a)-M(p_b^0-p_a^0)]/c^2$$

and the second term on the right-hand side of this equation vanishes as $c\rightarrow\infty$ [we assume that the initial particle velocities are not expressed as fractions of c, so that each $\mathbf{p}/c\rightarrow 0$ as $c\rightarrow\infty$; the leading terms in each p^0 go as $\mathbf{v}_1^2-\mathbf{v}_2^2$].

The equation of motion $dct(s)/ds=p^0/\mu c$ reduces to the equation $dct_{\text{n.r.}}(s)/ds=0$ in the limit as $c\rightarrow\infty$, so $ct_{\text{n.r.}}$ is a constant. From the equation of motion $c^{-1}dp^0(s)/ds=-V'(\rho)ct/\rho$ taken in the limit as $c\rightarrow\infty$, we see that the constant value of $ct_{\text{n.r.}}$ is in fact zero [assuming $V'(\rho)\neq 0$: see Eq. (14) *et seq.* for the case in which $V(\rho)=0$]. Now if $ct_{\text{n.r.}}=0$, the other two equations of motion (31) become

$$d\mathbf{x}_{\text{n.r.}}/ds=\mathbf{p}_{\text{n.r.}}/\mu\,,\ d\mathbf{p}_{\text{n.r.}}/ds=-V'(|\mathbf{x}_{\text{n.r.}}|)\mathbf{x}_{\text{n.r.}}/|\mathbf{x}_{\text{n.r.}}|\,,$$

which are the equations of the Galilean-invariant theory. While $\mathbf{x}_{\text{n.r.}}$ and a solution $\mathbf{x}$ of Eqs. (45) are identical functions of s, we emphasize that they are interpreted differently. $\mathbf{x}$ is a solution of the Lorentz-invariant equations for certain special initial and final conditions, while $\mathbf{x}_{\text{n.r.}}$ is the $c\to\infty$ limit of *any* solution of the Lorentz-invariant equations.

(B) Although no-interaction trajectories are not really scattering trajectories, they obviously do satisfy the elastic scattering asymptotic conditions, and it is of some interest to exhibit the way in which they can be made to possess the four properties of elastic scattering trajectories. The free-particle world line solutions are $x_i^\nu(s)=(s-s_{ia})p_i^\nu/m_i+x_{ia}^\nu$ [Eq. (3)]. Without affecting the world lines for particles 1 and 2, we may choose s_{1a} and s_{2a} so that for some value of s—say $s=0$—the trajectories satisfy the two conditions $t=0$ and $d\rho^2/ds=2p\cdot x/\mu=0$ (these two conditions lead to two linear equations in s_{1a} and s_{2a} which can be solved). Then the expression for $x^\nu=x_1^\nu-x_2^\nu$ becomes

$$\mathbf{x}(s)=s\mathbf{p}/\mu+\mathbf{x}_0,\quad t=sp^0/\mu\quad(\text{where }\mathbf{p}\cdot\mathbf{x}_0=0).\qquad(46)$$

A relative world line (46) is the limit of an elastic scattering relative world line as $V\to 0$. The other world lines [Eqs. (3)] for which s_{1a} and s_{2a} have not been so chosen are the $V\to 0$ limit of inelastic scattering world lines.

It is straightforward to verify that Eqs. (46) possess all the properties of elastic scattering trajectories. The vector $\mathbf{n}=\mathbf{x}_0p^0$ satisfies $\mathbf{n}\cdot\mathbf{p}=0$, and bisects the trajectory. Likewise $\rho^2=s^2p^2/\mu^2+\mathbf{x}_0^2\geq 0$, the equality occurring only if the two particle world lines intersect at $s=0$ (i.e. $\mathbf{x}_0=0$).

(C) The difficulty in the two-particle bound-state problem is that there is no relationship which plays the role that the asymptotic energy-momentum–mass relationship plays for the scattering problem. From the point of view of consistency of the theory, the most satisfactory way to obtain a bound state would be to consider a four-particle scattering problem. For example, suppose particles 1 and 2 interact with each other (they are the particles which will become bound) while particles 3 and 4 each only interact with particle 1. Starting out with all particles far apart, but particle 4 much farther away from the others, we arrange the initial conditions so that particle 3 "hits" particle 1 when it is in the vicinity of particle 2, taking away some energy and leaving it bound to particle 2. At a later value of s, particle 4 arrives and breaks up the bound state. The particle trajectories are then completely determined by the initial positions and momenta and the initial and final masses of all particles.

When we are given only two particles, however, the problem is underdetermined. It is possible to have two *different* bound-state trajectories for which the particles in both bound states have *identical* positions and momenta at some "instant" of s. In order to uniquely describe a two-particle bound state, it is necessary to also specify the inital time and energy coordinates, and it does not seem that there is a physically justifiable way of doing this, other than the method of the previous paragraph. We conclude that it is not very meaningful to talk about the bound state of two particles outside of the context of the creation and destruction of the bound state. The two-particle bound-state problem is thus intimately bound up with the many-particle problem, and we shall not discuss it further.

(D) Because the scalar parameter s correlates the particle motions at different values of their respective time coordinates, the forces may be regarded as being both advanced and retarded instead of being instantaneous as in the Galilean-invariant theory. When the particles are spacelike-separated, the interaction may be regarded as proceeding faster than the speed of light. This means that a retarded force viewed from one Lorentz frame will appear as an advanced force in other Lorentz frames.

Some unusual features of advanced interactions in a classical theory have been carefully discussed by Wheeler and Feynman.[10] One occurs in the present theory, for example, if particles 1 and 2 are interacting with $t_1(s)>t_2(s)$, and particle 3 (which only interacts with particle 1) comes along and hits particle 1. What we observe is a kink in the trajectory of particle 2 at a time before the collision of particles 1 and 3 occurred. It appears that particle 2 has anticipated the collision, and undergone a violent disturbance without prior cause.

There is another type of advanced interaction which is peculiar to this theory. Suppose we are considering the elastic scattering of particles 1 and 2 with specified initial positions and momenta, and we calculate the trajectories of the two particles. Let the interaction of particles 1 and 2 be fairly long range.

Now we consider a new problem, by introducing particle 3 which has a short-range interaction with particle 1 alone. We wish to calculate the three-particle elastic scattering trajectories, when particles 1 and 2 have the same initial positions and momenta as previously. We arrange for particle 3 to hit particle 1, say at $s=0$, when particle 2 is fairly close to particle 1. What we will find is that in order to have all three particles come out of the scattering having the same masses with which they entered, it is necessary to have the initial time coordinates of particles 1 and 2 be different for the three-particle problem than for the two-particle problem. This means that the trajectories of particles 1 and 2 will be different in the three-particle problem from what they were in the two-particle problem, *even earlier than the times for which $s=0$*. In other words, to make the scattering elastic we must anticipate the interaction with particle 3 and adjust the initial time conditions. In consequence, a comparison of the trajectories in the two problems makes it appear that both particles 1 and 2 have anticipated the collision, and altered their trajectories long before particle 3 has actually collided with particle 1.

[10] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **21**, 425 (1949).

IX. PARTICLE CREATION: DYNAMICS OF TWO-PARTICLE DECAY

We have already commented[7] that a strong enough potential might cause a particle's world line to have a number of position coordinates for a range of its time variable in the region of interaction, and this might be interpreted as particle creation. In such a case the number of final asymptotic trajectories will still be the same as the number of initial asymptotic trajectories, so that it is not possible to "permanently create" particles in this manner.

It is possible to permanently create particles in a slightly artificial but interesting way. We will illustrate the concepts involved by describing a particle which decays into two particles. The major idea is that the equations of motion allow a meaningful dynamics of two world lines, one of which extends from $t_1=-\infty$ to $t_1=+\infty$, while the other one extends only from $t_2=0$ to $t_2=+\infty$. Particle 1 is both the decaying particle (as $t_1\rightarrow-\infty$) and a decay product (as $t_1\rightarrow+\infty$) while particle 2 is a truly "created" particle since it has no world line for $t_2<0$.

For simplicity we shall discuss the decay in the center-of-mass reference frame. The initial conditions are of course quite different from those in the scattering problem. We will set the initial momentum and energy of particle 2 equal to zero at $s=s_a$, and there will be no loss in generality if we place it at the origin of coordinates: thus $\mathbf{x}_{2a}=t_{2a}=\mathbf{p}_{2a}=p_{2a}{}^0=0$. Since this is the center-of-mass reference frame, the decaying particle must be at rest, so $\mathbf{p}_{1a}=0$, $p_1{}^0=\bar{m}_1$. We shall give particle 1 an initial position coordinate $\mathbf{x}_{1a}$, which is not too far from the origin, but which is otherwise an adjustable parameter, and we shall make t_{1a} a very large negative number. This means that $-\rho^2(s_a)=t_{1a}{}^2-\mathbf{x}_{1a}{}^2\gg0$: *The particles start off with a timelike separation.*

What we wish to see happen is particle 2 remaining at rest at the origin of coordinates while particle 1 traces out a straight world line parallel to the t axis and approaching the origin of coordinates. When particle 1's world line nears the origin (say when $t_a{}^2-\mathbf{x}_a{}^2=\bar{\rho}_0{}^2$) then both particles interact, and particle 2 starts to move. As $s\rightarrow\infty$ we wish to see particles 1 and 2 at a spacelike separation, tracing out straight world lines commensurate with new mass values m_1, m_2 ($m_1+m_2\leq\bar{m}_1$) and the conservation laws.

The Hamiltonian we use to describe the system is as usual $H=p_1{}^2/2m_1+p_2{}^2/2m_2+V(\rho)$. However, the masses m_1 and m_2 in the Hamiltonian are the final asymptotic particle masses, and not the initial masses as they were for the scattering problem. Moreover, we must carefully select the functional form of $V(\rho)$. In order that particle 2 not change its space and time coordinates while particle 1 is approaching it, the force which is proportional to $V'(\rho)$ must vanish for $-\infty\leq\rho^2\leq-\bar{\rho}_0{}^2$. We therefore choose $V(\rho)=V_0$ in this region, where V_0 is a constant to be determined. When ρ^2 lies between the timelike "range" and the spacelike "range" $(-\bar{\rho}_0{}^2<\rho^2<\rho_0{}^2)$, $V(\rho)$ decreases from V_0 to 0. The form of $V(\rho)$ in this region will help determine whether or not a decay with satisfactory final asymptotic conditions is possible, as will shortly be explained.

We may "milk" the conservation laws in order to obtain the information they contain about the final asymptotic coordinates at $s=s_b$. The consequence of momentum conservation is $\mathbf{p}_{1b}=-\mathbf{p}_{2b}=\mathbf{p}_b$, and energy conservation yields

$$\bar{m}_1=p_{1b}{}^0+p_{2b}{}^0. \tag{47}$$

Asymptotic H conservation guarantees that $-\bar{m}_1{}^2/2m_1+V_0=p_{1b}{}^2/2m_1+p_{2b}{}^2/2m_2$. We desire $p_{1b}{}^2=-m_1{}^2$, $p_{2b}{}^2=-m_2{}^2$, so that if we choose

$$V_0=\bar{m}_1{}^2/2m_1-m_1/2-m_2/2\,, \tag{48}$$

we have at least assured ourselves that

$$p_{1b}{}^{02}/m_1+p_{2b}{}^{02}/m_2 =(\mathbf{p}_b{}^2+m_1{}^2)/m_1+(\mathbf{p}_b{}^2+m_2{}^2)/m_2. \tag{49}$$

Equations (47) and (49) may be solved for $p_{1b}{}^0$ and $p_{2b}{}^0$, which are then expressed in terms of m_1, m_2, $\bar{m}_1$, and $\mathbf{p}_b{}^2$. These relations will not have the desired forms $p_{1b}{}^0=(\mathbf{p}_b{}^2+m_1{}^2)^{1/2}$, $\mathbf{p}_{2b}{}^0=(\mathbf{p}_b{}^2+m_2{}^2)^{1/2}$, unless the initial coordinate $\mathbf{x}_{1a}$ and the potential function are so chosen that the dynamics produces a final value of $\mathbf{p}_b{}^2$ satisfying $\bar{m}_1=(\mathbf{p}_b{}^2+m_1)^{1/2}+(\mathbf{p}_b{}^2+m_2{}^2)^{1/2}$, i.e.,

$$\mathbf{p}_b{}^2=[\bar{m}_1{}^2-(m_1+m_2)^2][\bar{m}_1{}^2-(m_1-m_2)^2]/4m_1{}^2. \tag{50}$$

We emphasize that the final value of $\mathbf{p}_b{}^2$ depends upon the dynamics, and not upon any conservation law. It is always possible to "sculpt" a shape for $V(\rho)$ and to choose an $\mathbf{x}_{1a}$ so that the relative world line ends up with the asymptotic value (50) for $\mathbf{p}_b{}^2$.

We have not yet made use of the conservation of $\mathbf{j}$ and $\mathbf{n}$ [Eqs. (33)]. From the initial conditions for the relative coordinates $\mathbf{p}_a=0$, $p_a{}^0=\mu\bar{m}_1/m_1$, $\mathbf{x}_a=\mathbf{x}_{1a}$, $t_a=t_{1a}$, we see that $\mathbf{j}=0$ and $\mathbf{n}=\mathbf{x}_{1a}\mu\bar{m}_1/m_1$. By equating $\mathbf{n}$ at $s=s_a$ to $\mathbf{n}$ as $s=s_b$,

$$(\mu\bar{m}_1/m_1)\mathbf{x}_{1a}=\mathbf{x}_bp_b{}^0-\mathbf{p}_bt_b\,,$$

and realizing that $\mathbf{j}=0$ implies that $\mathbf{x}_b$ is parallel to $\mathbf{p}_b$, we discover that the decay products are ejected along the vector $\mathbf{x}_{1a}$. If we align our coordinate axes so that the x axis points along $\mathbf{x}_{1a}$, the y and z coordinates of $\mathbf{x}$ remain at zero, and the problem reduces to a one-dimensional decay problem. The quadratures (21), (24) for the scattering problem are the quadratures for this decay problem as well, since the equations of motion for the two problems are identical.

It may be expected that a multiple decay or an inelastic scattering with production may be described by similar methods.

X. CONCLUDING REMARKS

This relativistically invariant theory has a number of useful features, chief among them being that it is algebraically simple and that it is expressed in Hamiltonian form. This latter property is useful both because it is a familiar classical formulation encompassing a large body of thoroughly understood techniques, and because a Hamiltonian formulation of a classical mechanics is a necessary intermediate step in the creation of a quantum mechanics out of the classical mechanics. The question naturally arises as to whether a quantum mechanics can be constructed out of the classical theory presented here. In fact, the present work was undertaken with a view toward investigating whether it is possible to construct a useful, logically consistent quantum mechanics in which time is an operator, and not a scalar parameter.

Suppose one replaces the dynamical variables p_i^μ by operators $(\hbar/i)\partial/\partial x_{i\mu}$ in the expression (15) for the Hamiltonian function H (as well as in the appropriately symmetrized expressions for other dynamical variables). One may solve the "Schrödinger equation" $H\psi=i\partial\psi/\partial s$ to obtain a solution $\psi(\mathbf{x}_1,t_1,\mathbf{x}_2,t_2,s)$ which possesses the property

$$\int d^4x_1\int d^4x_2|\psi|^2=\text{const}.$$

This follows from the conservation of probability in the eight-dimensional configuration space:

$$\frac{1}{i}(\psi^*H\psi-\psi H\psi^*)=-\left(\frac{\partial j_1{}^\mu}{\partial x_1{}^\mu}+\frac{\partial j_2{}^\mu}{\partial x_2{}^\mu}\right)=\frac{\partial\psi^*\psi}{\partial s}.$$

How can we interpret ψ? We cannot regard

$$|\psi(\mathbf{x}_1,t_1,\mathbf{x}_2,t_2,s)|^2d^3x_1d^3x_2$$

as the probability for finding particle 1 in the volume d^3x_1 about the point $\mathbf{x}_1$ at time t_1, and particle 2 in the volume d^3x_2 about the point $\mathbf{x}_2$ at time t_2: One reason is that $\int|\psi|^2d^3x_1d^3x_2$ is not a constant, yet we know that the probability is 1 that particles 1 and 2 will be found somewhere in space at times t_1 and t_2. It is clear that our interpretation of ψ, our use of it to make predictions, and the theory of measurement which must be constructed if we are to have a complete quantum theory will be somewhat different from what is usual in quantum theory. We may tentatively interpret $|\psi|^2$ as a probability density in configuration space, in view of the conservation of probability equation that $|\psi|^2$ satisfies [i.e., at a given instant of s, $|\psi|^2d^4x_1d^4x_2$ is the fraction of systems belonging to an ensemble whose coordinates lie in the volume $d^4x_1d^4x_2$ about the point $(\mathbf{x}_1,t_1,\mathbf{x}_2,t_2)$]. Of course we have not shown that such an interpretation can lead to a consistent theory, nor shall we investigate whether this is possible within the context of the relativistically invariant "quantum theory outlined here, because this quantum theory possesses a serious defect which would obscure such an investigation.

The defect in this quantum theory arises because the particle masses are not conserved quantities. This does not matter in the classical theory because one has 16 initial conditions (for two particles) at one's disposal with which to characterize the particle's trajectories. By judiciously choosing these initial conditions, one can select elastic scattering trajectories or inelastic scattering trajectories.

However, in the quantum theory, one has, so to speak, only eight initial conditions at one's disposal, since a wave function can only be characterized by the eigenvalues of a complete set of commuting variables (for a quantum theory constructed out of a classical theory with n variables, there can be no more than $\frac{1}{2}n$ complete commuting variables: This follows from the similarity of the algebras of commutator brackets and Poisson brackets, and the fact that one can only construct $\frac{1}{2}n$ functions whose Poisson bracket relations with each other vanish). If we choose the eight variables to be the four-momenta of the two particles, and we construct a wave packet in configuration space from a superposition of eigenfunctions of these eight variables whose eigenvalues are narrowly spread, we are left with a probability density function $|\psi|^2$, which occupies a certain volume in configuration space. A classical ensemble of two-particle systems, which has the same probability density function in configuration space and whose four-momenta are all chosen to be equal to $\langle\psi|p_1{}^\mu|\psi\rangle$ and $\langle\psi|p_2{}^\mu|\psi\rangle$, will develop with increasing s into an ensemble of particles with varying masses. It can be shown that the quantum theory wave packet will develop with increasing s so that one predicts the same behavior. It does not seen to be possible to project out of the wave function a piece of it which only describes elastic scattering behavior. This quantum theory therefore describes a universe which consists of particles which scatter into other particles with a continuum of masses (including imaginary masses).

The relativistically invariant classical theory presented here is interesting in its own right, but it does not appear suitable as a jumping-off place for a useful quantum theory. Modifications of this classical theory which have the property that particle masses are conserved (at last asymptotically) are presently under investigation.

APPENDIX A: EXAMPLE: HARD-SPHERE SCATTERING

Because the mathematical expressions in the theory presented here are very similar to those of the usual Galilean-invariant theory, one might expect that most of the analytically soluble problems of the latter theory are also analytically soluble problems of the former theory. It has been the author's experience that this is the case: The additional complications of the relativistic theory do not render such problems as Coulomb scattering intractable. As a simple example, we will consider the scattering of two particles where the interaction potential function $V(\sqrt{x^2})$ is

$$V(\sqrt{x^2})=0 \quad \text{for} \quad x^2>d^2>0, \qquad V(\sqrt{x^2})=\infty \quad \text{for} \quad x^2<d^2. \tag{A1}$$

This is the relativistic analog of the familiar Galilean problem of the scattering of two hard spheres, the sum of whose radii is d. The two particles move along straight world lines with momenta $p_{1a}{}^\nu, p_{2a}{}^\nu$ until their spacelike separation x^2 decreases to d^2, where they collide. We will select the value of s at which they collide to be $s=0$, and define the four-vector d^ν as the value of the relative event vector when the collision takes place; $x^\nu(s=0)=d^\nu$. After the collision, the two particles move off with momenta $p_{1b}{}^\nu$, $p_{2b}{}^\nu$, along straight world lines, and it is these final momenta which must be calculated in order to solve the problem.

Before and after the collision, the solutions of the equations of motion $dx^\nu/ds=p^\nu/\mu$, $dp^\nu/ds=-\partial V/\partial x_\nu$ are

$$x^\nu=p_a{}^\nu s/\mu+d^\nu, \quad s<0 \tag{A2a}$$

$$x^\nu=p_b{}^\nu s/\mu+d^\nu, \quad s>0, \tag{A2b}$$

respectively. The straight world line (A2a) would cut the hypersurface $\mathbf{x}^2-x_0{}^2=d^2$ in two points, when $s=0$ and when $s=-2p_a\cdot d/\mu p_a{}^2$, if this latter value of s is negative. To ensure that this value of s is positive so that the spheres do not collide before $s=0$, d^ν must satisfy the inequality

$$p_a\cdot d<0 \tag{A3}$$

[recall that $p_a{}^2>0$: see Eq. (12)].

Because the potential function is so singular, the equations of motion cannot be solved directly. However, the value of $p_b{}^\nu$ can be found in a straightforward fashion by using the known conservation laws, just as can be done for the Galilean hard-sphere problem. We prefer, however, to present an illustration of a direct solution of the dynamical equations, and so we will consider the hard-sphere problem to be the limiting case of a scattering of two particles whose interaction potential function is

$$V(\sqrt{x^2})=0 \quad \text{for} \quad x^2>d^2, \qquad V(\sqrt{x^2})=V_0(d^2-x^2)/(d^2-c^2) \quad \text{for} \quad d^2>x^2>c^2, \qquad V(\sqrt{x^2})=V_0 \quad \text{for} \quad c^2>x^2, \tag{A4}$$

and where, as in the problem which is its Galilean counterpart, the hard-sphere problem solutions are obtained in the limit $V_0\rightarrow\infty$.

For this problem, the equation (A2a) is also the correct solution of the equations of motion for $s<0$. For $s=0$ and a short interval of s thereafter, x^2 lies in the region where the collision takes place. The equations of motion during this interval are

$$dx^\nu/ds=p^\nu/\mu, \quad dp^\nu/ds=2V_0x^\nu/\mu(d^2-c^2) \tag{A5}$$

and the solution of these equations which satisfies $x^\nu(s=0)=d^\nu$, $p^\nu(s=0)=p_a{}^\nu$ is

$$x^\nu=d^\nu\cosh\lambda s+(p_a{}^\nu/\mu\lambda)\sinh\lambda s \tag{A6}$$

(where $\lambda\equiv[2V_0/\mu(d^2-c^2)]^{1/2}$). The relative world line $x^\nu(s)$ is governed by Eq. (A6) until it once more intersects the hypersurface $x^2=d^2$. The value of s—call it s_b—at which this occurs is found from Eq. (A6) to satisfy

$$\tanh\lambda s_b=\frac{-2p_a\cdot d}{\mu\lambda(d^2-p_a{}^2/\mu^2\lambda^2)}. \tag{A7}$$

After the relative world line emerges from the interaction region, it becomes a straight line:

$$x^\nu(s)=p^\nu(s_b)(s-s_b)/\mu+x^\nu(s_b), \quad s>s_b \tag{A8}$$

where $x^\nu(s_b)$ and $p^\nu(s_b)$ can be found by putting the value of s_b [Eq. (A7)] into Eq. (A6), and into the derivative of Eq. (A6) with respect to s.

Since we are only interested in the $V_0\rightarrow\infty$ limit of this problem, we may let λ become very large, and expand s_b, $x^\nu(s_b)$ and $p^\nu(s_b)$ in inverse powers of V_0. From Eq. (A7) we find that

$$s_b=-p_a\cdot d(1-c^2/d^2)/V_0+\text{ terms of order } V_0{}^{-2}. \tag{A9}$$

When this expression for s_b is inserted into Eq. (A6), we find that

$$x^\nu(s_b)=d^\nu+\text{terms of order } V_0{}^{-1}. \tag{A10}$$

Finally, upon calculating $p^\nu(s_b)$ by taking the derivative of Eq. (A6) and inserting the expression (A9) for S_b, we obtain

$$p^\nu(s_b)=p_a{}^\nu-d^\nu 2p_a\cdot d/d^2+\text{terms of order } V_0{}^{-1}. \tag{A11}$$

As expected, we see that as $V_0\rightarrow\infty$, the interval s_b over which the collision takes place vanishes, the relative vector x^ν between the particles during the collision approaches the constant value d^ν, and the final momentum $p_b{}^\nu$ approaches a value independent of V_0 or c. The solution of the hard-sphere problem for $s>0$ is [from Eqs. (A8)–(A11)]:

$$x^\nu(s)=[p_a{}^\nu-d^\nu 2p_a\cdot d/d^2]s/\mu+d^\nu, \quad s>0. \tag{A12}$$

Now that the problem has been completely solved, we can examine the solution (A2a), (A12) to see that it possesses the properties remarked upon in Sec. VII.

A direct calculation of $x^2(s)$ shows that *any* trajectory is spacelike, and that d^2 is the minimum value of x^2: This is, of course, due to the infinite potential barrier, which does not let particles penetrate into the region of timelike separation. A direct calculation of $h\equiv p^2/2\mu$, $\mathbf{n}\equiv\mathbf{x}p^0-\mathbf{p}t$, and $\mathbf{j}\equiv\mathbf{x}\times\mathbf{p}$ shows that they are conserved quantities:

$$h=p_a^2/2\mu\,,\quad \mathbf{n}=\mathbf{d}p_a^0-\mathbf{p}_a d^0\,,\quad \mathbf{j}=\mathbf{d}\times\mathbf{p}_a\,. \tag{A13}$$

The final momenta of the two particles are, from Eqs. (6) and (A11):

$$\begin{aligned} p_{1b}^\nu&=p_{1a}^\nu+d^\nu 2\mu(p_{1a}/m_1-p_{2a}/m_2)\cdot d/d^2\,,\\ p_{2b}^\nu&=p_{2a}^\nu-d^\nu 2\mu(p_{1a}/m_1-p_{2a}/m_2)\cdot d/d^2\,, \end{aligned} \tag{A14}$$

from which one can directly compute the final masses $m_{1b}^2=-p_{1b}^2$, $m_{2b}^2=-p_{2b}^2$ of the two particles [or one may use Eqs. (25)]:

$$\begin{aligned} m_{1b}^2&=m_1^2-(2m_1/m)(p_a\cdot d)(P\cdot d)\,,\\ m_{2b}^2&=m_2^2+(2m_2/m)(p_a\cdot d)(P\cdot d)\,. \end{aligned} \tag{A15}$$

The initial conditions that particles 1 and 2 pass through the space-time points x_{1a}^ν, x_{2a}^ν with momenta p_{1a}^ν, p_{2a}^ν, respectively, are not a complete set of initial conditions for this problem, because there is still freedom to choose the values of s—call them s_{1a} and s_{2a}—at which each trajectory passes through the specified space-time point. Of these two free parameters, one is inessential because the theory is translationally invariant in s, but it is determined by the added requirement that the collision occur at $s=0$. The remaining degree of freedom appears in the solution for $x^\nu(s)$, in that d^ν can be chosen so that d^0 is a free parameter, while $\mathbf{d}$ is a function of the initial conditions and d^0. Depending upon our choice of d^0, which is the relative time at which the particles collide, the scattering can be made elastic, inelastic, or unphysical.

If we choose d^0 so that $P\cdot d=\mathbf{P}\cdot\mathbf{d}-P^0d^0$ vanishes, then according to Eq. (A15), the scattering will be elastic. In the center-of-mass reference frame where $\mathbf{P}=0$, this becomes the condition:

$$d^0=0\,. \tag{A16}$$

From Eqs. (A2a), (A12), and (A16) we find that for elastic scattering in the center-of-mass frame

$$\begin{aligned} \mathbf{x}&=(\mathbf{p}_a s/\mu+\mathbf{d})\Theta(-s)+(\mathbf{p}_b s/\mu+\mathbf{d})\Theta(s)\,,\\ t&=p_a^0 s/\mu\,. \end{aligned} \tag{A17}$$

The final momentum,

$$\mathbf{p}_b=\mathbf{p}_a-\mathbf{d}(2\mathbf{p}_a\cdot\mathbf{d})/\mathbf{d}^2\,, \tag{A18}$$

has the same magnitude as the initial momentum. The "angle of incidence" that $-\mathbf{p}_a$ makes with $\mathbf{d}$ [or with $\mathbf{n}$, since by Eq. (A13) $\mathbf{n}=\mathbf{d}p_a^0$] is the same as the "angle of reflection" that $\mathbf{p}_a$ makes with $\mathbf{d}$ (i.e., $-\mathbf{p}_a\cdot\mathbf{d}=\mathbf{p}_b\cdot\mathbf{d}$). The collision occurs at $t=0$ where x^2 has its minimum value $\mathbf{d}^2$. All the other special properties of elastic scattering in the center-of-mass frame which were pointed out in Sec. VII may be verified for the solution (A17).

If $d^0\neq 0$, there is inelastic or unphysical scattering. For example, in the center-of-mass frame, one can choose the initial conditions so that $\mathbf{p}_a\cdot\mathbf{d}=0$. From Eq. (A15), the final masses of the particles are then

$$\begin{aligned} m_{1b}^2&=m_1^2-(2m_1/M)p_a^0P^0(d^0)^2\,,\\ m_{2b}^2&=m_2^2+(2m_2/M)p_a^0P^0(d^0)^2\,. \end{aligned} \tag{A19}$$

In this case, one sees that inelastic scattering trajectories occur for $0<(d^0)^2\leq Mm_1/2p_a^0P^0$, that particle 1 goes off with zero mass when $(d^0)^2=Mm_1/2p_a^0P^0$, and that nonphysical scattering occurs if $d^{0^2}>Mm_1/2p_a^0P^0$.