Electromagnetic Fields of Accelerated Nonradiating Charge Distributions*

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Exact expressions for the retarded electromagnetic potentials are derived for certain extended, rigid, accelerated, but nonradiating charge-current distributions. An unexpected result is that the oscillating parts of these potentials vanish identically for all points of space-time which are outside the distributions. Since each of these nonradiating distributions undergoes time-periodic orbital motion, with the "radius" of the orbit less than the radius of the distribution, this result means that the oscillating part of the electromagnetic field energy is localized within a volume of the same order as the volume of the distribution.

INTRODUCTION

In an article in 1964, one of us (GHG)[1] explicitly demonstrated several charge-current distributions which would not radiate while in rigid periodic motion, orbiting about some fixed point in space and/or spinning about some fixed axis through the center of the distribution. A general criterion was derived in I by considering the radiation to infinity from a localized source. The rate of radiation of energy to infinity is defined by

$$R\equiv\lim_{x\to\infty}\int d\Omega\, x^2\hat{x}\cdot\mathbf{S},$$

where $d\Omega$ is the solid-angle element, $d\Omega=\sin\theta\, d\theta\, d\phi$, $\hat{x}$ is the outward pointing unit normal to the spherical surface of radius x, and $\mathbf{S}$ is the Poynting vector. Using this expression, a sufficient condition for nonradiation is easily determined to be

$$\mathbf{J}(\omega_n\hat{x},n)=0,\quad (n>0)$$

where $\mathbf{J}(\mathbf{k},n)$ is the Fourier transform of the current given by

$$\mathbf{J}(\mathbf{k},n)=\frac{1}{T}\int_0^T dt\int d^3x\, \mathbf{j}(\mathbf{x},t)\, \exp i(\mathbf{k}\cdot\mathbf{x}-\omega_n t),$$

with $\mathbf{j}(\mathbf{x},t)$ the source current distribution and $\omega_n\equiv 2\pi n/T$, n integer>0. Using this criterion, several nonradiating distributions were derived in I; one of the conditions imposed by $\mathbf{J}(\omega_n\hat{x},n)=0$ is that the assumed spherical extent of the distribution must be an integer multiple of cT (c=speed of light).

Here, we wish to show that all those nonradiating distributions previously found in I generate no oscillating fields outside themselves. To be exact, we shall show that, at any point which is *always* outside one of these distributions, there exist no oscillating fields. This result implies that a nonradiating distribution acts somewhat as a resonant cavity, trapping all its oscillating electromagnetic field energy entirely within the total spatial volume which it sweeps out during the course of its motion. Since the radius of an orbit is less than the extent of a distribution (1), this total volume is of the same order as the volume of the distribution itself. This attribute of our distributions is to be contrasted with the field patterns of typical radiating arrays, which possess near fields markedly different in character from their radiating fields.

GENERAL EXPRESSIONS FOR VECTOR AND SCALAR POTENTIALS

In the expression for R defined above, only the leading $1/x$ terms of the retarded vector and scalar potentials, $\mathbf{A}(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$, appear. In order to investigate the condition $\mathbf{J}(\omega_n\hat{x},n)=0$ further, the general expressions for $\mathbf{A}(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$ are considered. $\mathbf{A}(\mathbf{x},t)$ is given by

$$\mathbf{A}(\mathbf{x},t)=\int d^3x'\, |\mathbf{x}-\mathbf{x}'|^{-1}\mathbf{j}(\mathbf{x}',\, t-|\mathbf{x}-\mathbf{x}'|).$$

Here and in what follows, we set $c=1$; we use Gaussian units throughout. Using the Fourier transform of $\mathbf{j}(\mathbf{x},t)$ and making a change of variables,

$$\mathbf{A}(\mathbf{x},t)=(2\pi)^{-3}\sum_{n=-\infty}^{\infty}\int d^3k\, \mathbf{J}(\mathbf{k},n)\, \exp[-i(\mathbf{k}\cdot\mathbf{x}-\omega_n t)]\times\int d^3u\, u^{-1}\exp[-i(\mathbf{k}\cdot\mathbf{u}+\omega_n u)].$$

Integrating over the u variables, $\mathbf{A}(\mathbf{x},t)$ becomes

$$\mathbf{A}(\mathbf{x},t)=\frac{1}{(2\pi)^2}\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{k}\mathbf{J}(\mathbf{k},n)e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_n t)}\times\left[\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right].$$

The corresponding expression for $\phi(\mathbf{x},t)$ is

$$\phi(\mathbf{x},t)=\frac{1}{(2\pi)^2}\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{k}\frac{\mathbf{k}\cdot\mathbf{J}(\mathbf{k},n)}{\omega_n}e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_n t)}\times\left[\lim_{L\to\infty}\left(\frac{e^{ik(-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-ik(+\omega_n)L}-1}{k+\omega_n}\right)\right],$$

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[1] G. H. Goedecke, Phys. Rev. 135, B281 (1964); hereafter referred to as I.

where the expression for the Fourier transform of the charge density is required by continuity, except for the $n=0$ term, as discussed in I. The expressions may now be explicitly evaluated using the Fourier transforms of the nonradiating charge-current distributions.

NONRADIATING CHARGE-CURRENT DISTRIBUTIONS

In I, the nonradiating charge-current distributions included the spherical shell,

$$\rho(z)=e(4\pi r^2)^{-1}\delta(z-r)\,,$$

where $\mathbf{z}=\mathbf{x}-\mathbf{a}(t)$, $z=|\mathbf{z}|$, $\mathbf{a}(t)$ is the position vector of the "center" of the distribution, assumed periodic with period T, and r is the radius of the shell;

$$\begin{aligned}\rho(z)&=Az^{-1}\cos\omega_q z \quad && z\leq b\\ &=0, && z>b\end{aligned}$$

where $\omega_q=2\pi q/T$, $q=$ integer, and A is a constant;

$$\begin{aligned}\mathbf{j}(\mathbf{z})&=(\mathbf{\Omega}\times\mathbf{z})g(z) \quad && z\leq b\\ &=0, && z>b\end{aligned}$$

where $\mathbf{\Omega}$ is a constant angular velocity and $g(z)$ is a spherically symmetric distribution; and

$$\begin{aligned}\mathbf{j}(\mathbf{z},t)&=(\mathbf{\Omega}\times\mathbf{z})g(z)(z_1\cos\Omega t+z_2\sin\Omega t) \quad && z\leq b\\ &=0, && z>b\end{aligned}$$

which includes asymmetry as well as spin.

Using the Fourier transforms of the nonradiating distributions, the general expressions for $\mathbf{A}(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$ may be evaluated. We do this in detail in the Appendix. For example, for the spherical shell, we get

$$\mathbf{A}(\mathbf{x},t)=-er^{-1}T^{-1}\sum_{n=-\infty}^{\infty}\int_0^T dt' \times[\exp i\omega_n(t-t'-z)]\dot{\mathbf{a}}(t')\sin\omega_n r\,,$$

valid for all x such that x is always outside the distribution. For the condition for nonradiation, $r=lT$, $l=$ integer>0, this expression vanishes. Similarly, $\mathbf{A}(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$ vanish, for the conditions given in I, for all the other distributions given in I.

DISCUSSION

The above results demonstrate that the nonradiating distributions found in I have only static external fields; that is, for all points x which are *always* outside the distribution, only static electric and magnetic fields are present. Thus, that part of the electromagnetic field energy which is locked in the oscillating fields is entirely localized in the near neighborhood of the distribution. If we were to attempt, as in I, an interpretation of these theoretical particles in terms of observed particles, we would surely want the electromagnetic field energy to be so localized, in order that it could be counted as part of the rest mass of the particle. Furthermore, if the oscillating fields of a real particle were indeed as localized as these, we could never observe them; indeed, we do not observe oscillating electromagnetic fields associated with a free massive particle.

We should remark that we have been unable to show the general result which we might expect to be valid: namely, that $\mathbf{J}(\omega_n\hat{x},n)=0$ implies localized oscillating fields. All we have done above is to show that all the nonradiating distributions which were found in I possess localized oscillating fields.

APPENDIX

Here, the nonradiating charge-current distributions are used to evaluate the expressions for the vector and scalar potentials given above. All calculations are for $z>b$ or $z>r$. For the spherical shell,

$$\mathbf{A}(\mathbf{x},t)=-\frac{e}{(2\pi)^2Tr}\sum_{n=-\infty}^{\infty}\int_0^T dt'\,e^{i\omega_n(t-t')}\dot{\mathbf{a}}(t')\left[\int\frac{d^3k}{k^2}\sin kr\,e^{-i(\mathbf{k}\cdot\mathbf{z})}\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}\right].$$

For further evaluation, only the portion of the expression in the square brackets is used.

$$\begin{aligned}\Big[\quad\Big]&=4\pi\int_0^\infty dk\,\sin kr\frac{\sin kz}{kz}\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}\\
&=-\pi\int_{-\infty}^\infty dk\,\frac{[\cos k(z+r)-\cos k(z-r)]}{kz}\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}\\
&=-\pi\int_{-\infty}^\infty dk\left\{\lim_{L\to\infty}\left[\left(\frac{e^{ik(z+r)}-e^{ik(z-r)}}{2kz}\right)\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}\right)+\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}\right)\right.\right.\\
&\quad\left.\left.\times\left(\frac{e^{-ik(z+r)}-e^{-ik(z-r)}}{2kz}\right)+\left(\frac{e^{ik(z+r)}+e^{ik(z-r)}}{2kz}\right)\left(\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)+\left(\frac{e^{-ikz(+r)}-e^{-ik(z-r)}}{2kz}\right)\left(\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right]\right\}.\end{aligned}$$

This may be treated as an integration in the complex k plane with simple poles on the real axis at $k=\pm\omega_n$. The first and fourth terms give no contribution. The second and third terms give

$$=-4\pi^2\omega_n^{-1}z^{-1}\exp(-i\omega_n z)\sin\omega_n r.$$

This is zero for the condition given in I.

The scalar potential $\phi(\mathbf{x},t)$ for the spherical shell is given by

$$\phi(\mathbf{x},t)=-\frac{e}{(2\pi)^2Tr}\sum_{n=-\infty}^{\infty}\int_0^T dt'\frac{e^{i\omega_n(t-t')}}{\omega_n}\dot{\mathbf{a}}(t')\cdot\left[\int\frac{d^3k}{k^2}e^{-i\mathbf{k}\cdot\mathbf{z}}\mathbf{k}\sin kr\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}\right].$$

The part of the expression in the square brackets is

$$4\pi i\hat{z}\int_0^\infty\frac{dk}{k}\left(\frac{k\sin kr\cos kz}{z}-\frac{\sin kr\sin kz}{z^2}\right)\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\},$$

where $\hat{z}\equiv\mathbf{z}/|\mathbf{z}|$. This can be written

$$\pi i\hat{z}\int_{-\infty}^\infty\frac{dk}{k}\left[\frac{k}{z}[\sin k(z+r)-\sin k(z-r)]+\frac{1}{z^2}[\cos k(z+r)-\cos k(z-r)]\right]\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}$$

$$=\pi i\hat{z}\int_{-\infty}^\infty dk\left\{\lim_{L\to\infty}\left[\frac{e^{ik(z+r)}-e^{ik(z-r)}}{2iz}+\frac{e^{ik(z+r)}-e^{ik(z-r)}}{2kz^2}-\frac{e^{-ik(z+r)}-e^{-ik(z-r)}}{2iz}+\frac{e^{-ik(z+r)}-e^{-ik(z-r)}}{2kz^2}\right]\right.$$

$$\left.\times\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}=-4\pi^2 i\hat{z}\exp(-i\omega_n z)(iz^{-1}+\omega_n^{-1}z^{-2})\sin\omega_n r.$$

Again, this is zero for the conditions in I.

Considering now the volume distribution, the vector potential is given by

$$\mathbf{A}(\mathbf{x},t)=\frac{A}{\pi T}\sum_{n=-\infty}^{\infty}\int_0^T dt'\,e^{i\omega_n(t-t')}\dot{\mathbf{a}}(t')\left[\int\frac{d^3k}{k}\left[\frac{\cos kb-1}{k^2-\omega_q^2}\right]e^{-i\mathbf{k}\cdot\mathbf{z}}\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}\right].$$

The portion of the above expression in the square brackets is

$$4\pi\int_0^\infty dk\frac{\sin kz}{z}\left[\frac{\cos kb-1}{k^2-\omega_q^2}\right]\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}$$

$$=\pi\int_{-\infty}^\infty dk\left[\frac{\sin k(z+b)+\sin k(z-b)-2\sin kz}{z(k^2-\omega_q^2)}\right]\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}$$

$$=\pi\int_{-\infty}^\infty dk\left[\frac{e^{ik(z+b)}+e^{ik(z-b)}-2e^{ikz}}{2iz(k^2-\omega_q^2)}-\frac{e^{-ik(z+b)}+e^{-ik(z-b)}-2e^{-ikz}}{2iz(k^2-\omega_q^2)}\right]\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}.$$

Here, there are simple poles at $\pm\omega_q$ as well as $\pm\omega_n$, giving

$$\pi^2\left(\frac{e^{-i\omega_n z}4(\cos\omega_n b-1)}{z(\omega_n^2-\omega_q^2)}+\left(\frac{\cos\omega_q b-1}{\omega_q z}\right)\left\{-\frac{4\omega_q\cos\omega_q z}{\omega_n^2-\omega_q^2}-2i\sin\omega_q z\left[\lim_{L\to\infty}\left(\frac{e^{i(\omega_q-\omega_n)L}}{\omega_q-\omega_n}-\frac{e^{-i(\omega_q+\omega_n)L}}{\omega_q+\omega_n}\right)\right]\right\}\right),$$

which vanishes since $\cos\omega_n b=\cos\omega_q b=1$.

For the corresponding scalar potential,

$$\phi(\mathbf{x},t)=\frac{A}{\pi T}\sum_{n=-\infty}^{\infty}\int_0^T dt'\frac{e^{i\omega_n(t-t')}}{\omega_n}\dot{\mathbf{a}}(t')\cdot\left[\int\frac{d^3k}{k}\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{z}}\left[\frac{\cos kb-1}{k^2-\omega_q^2}\right]\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}\right].$$

Again the terms in the square brackets amount to

$$2\pi i\hat{z}\int_{-\infty}^{\infty} dk\left[\frac{k\cos kb\cos kz-k\cos kz}{z(k^2-\omega_q{}^2)}-\frac{\sin kz\cos kb-\sin kz}{z^2(k^2-\omega_q{}^2)}\right]\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}$$

$$=2\pi i\hat{z}\int_{-\infty}^{\infty} dk\left[\frac{k(\cos k(z+b)+\cos k(z-b)-2\cos kz)}{2(k^2-\omega_q{}^2)z}-\frac{\sin k(z+b)}{2z^2(k^2-\omega_q{}^2)}\right.$$

$$\left.+\frac{\sin k(z+b)-2\sin kz}{2z^2(k^2-\omega_q{}^2)}\right]\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}$$

$$=2\pi i\hat{z}\int_{-\infty}^{\infty} dk\left[\frac{k}{z}\,\frac{e^{ik(z+b)}+e^{ik(z-b)}-2e^{ikz}}{4(k^2-\omega_q{}^2)}-\frac{e^{ik(z+b)}+e^{ik(z-b)}-2e^{ikz}}{4iz^2(k^2-\omega_q{}^2)}\right.$$

$$\left.+\frac{k}{z}\,\frac{e^{-ik(z+b)}+e^{-ik(z-b)}-2e^{-ikz}}{4(k^2-\omega_q{}^2)}+\frac{e^{-ik(z+b)}+e^{-ik(z-b)}-2e^{-ikz}}{4iz^2(k^2-\omega_q{}^2)}\right]\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}$$

$$=4\pi^2 i\hat{z}\left(e^{-i\omega_n z}\frac{(\cos\omega_n b-1)}{\omega_n{}^2-\omega_q{}^2}(-i\omega_n-z^{-1})+\frac{(\cos\omega_q b-1)}{\omega_n{}^2-\omega_q{}^2}\left(\frac{\omega_q}{z}\sin\omega_q z+\frac{\cos\omega_q z}{z^2}\right)\right.$$

$$\left.-(\cos\omega_q b-1)i\left(\frac{\cos\omega_q z}{2z}+\frac{\sin\omega_q z}{\omega_q z^2}\right)\left\{\lim_{L\to\infty}\left[e^{-i\omega_n L}\left(\frac{e^{i\omega_q L}}{\omega_q-\omega_n}-\frac{e^{-i\omega_q L}}{\omega_q+\omega_n}\right)\right]\right\}\right),$$

which is zero for the conditions used above.

The spinning distribution leads to the following form for the vector potential:

$$\mathbf{A}(\mathbf{x},t)=\frac{Bi}{\pi T}\sum_{n=-\infty}^{\infty}\int_0^T dt'\,e^{i\omega_n(t-t')}\boldsymbol{\Omega}\times\left[\int\frac{d^3k}{k^5}\mathbf{k}[2(\cos kb-1)+kb\sin kb]e^{-i\mathbf{k}\cdot\mathbf{z}}\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}\right].$$

The terms in the square bracket become

$$4\pi i\hat{z}\int_0^{\infty}\frac{dk}{k^4}\left\{\frac{2k\cos kz(\cos kb-1)}{z}+\frac{bk^2\cos kz\sin kb}{z}-\frac{2\sin kz}{z^2}\right.$$

$$\left.\times(\cos kb-1)-\frac{kb\sin kz\sin kb}{z^2}\right\}\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}$$

$$=2\pi i\hat{z}\int_{-\infty}^{\infty}\frac{dk}{k^4}\left[\frac{k[\cos k(z+b)+\cos k(z-b)-2\cos kz]}{z}+\frac{bk^2}{2}[\sin k(z+b)-\sin k(z-b)]\right.$$

$$\left.-\frac{2}{z^2}[\sin k(z+b)+\sin k(z-b)-2\sin kz]+\frac{bk}{z^2}[\cos k(z+b)-\cos k(z-b)]\right]\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}$$

$$=2\pi i\hat{z}\int_{-\infty}^{\infty}dk\left\{\frac{e^{ik(z+b)}+e^{ik(z-b)}-2e^{ikz}}{2k^3z}+\frac{b}{4i}\,\frac{e^{ik(z+b)}-e^{ik(z-b)}}{k^2z}-\frac{e^{ik(z+b)}+e^{ik(z-b)}-2e^{ikz}}{2ik^4z^2}\right.$$

$$+\tfrac{1}{4}b\frac{e^{ik(z+b)}-e^{ik(z-b)}}{k^3z^2}+\frac{e^{-ik(z+b)}+e^{-ik(z-b)}-2e^{-ikz}}{2k^3z}-\tfrac{1}{4}b\frac{e^{-ik(z+b)}-e^{-ik(z-b)}}{ik^2z}+\frac{e^{-ik(z+b)}}{2ik^4z^2}$$

$$\left.+\frac{e^{-ik(z-b)}-2e^{-ikz}}{2ik^4z^2}+\tfrac{1}{4}b\frac{e^{-ik(z+b)}-e^{-ik(z+b)}}{2k^3z^2}\right\}\left\{\lim_{L\to\infty}\left(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\right)\right\}$$

$$=-4\pi^2 i\hat{z}e^{-i\omega_n z}\{(\cos\omega_n b-1)(i\omega_n{}^{-3}z^{-1}+\omega_n{}^{-4}z^{-2})+(\tfrac{1}{2}b)\sin\omega_n b(i\omega_n{}^{-2}z^{-1}+\omega_n{}^{-3}z^{-2})\}.$$

Referring again to I, we find that this expression is zero.

Finally, the asymmetric spinning current distribution gives for the vector potential

$$\mathbf{A}(\mathbf{x},t)=-\frac{G}{\pi T}\sum_{n=-\infty}^{\infty}\int_0^T dt'\, e^{i\omega_n(t-t')}\boldsymbol{\Omega}\times\Bigg[\int\frac{d^3k}{k}\Bigg(\mathbf{kk}\Bigg(\frac{4[2(\cos kb-1)+kb\sin kb]}{k^6}+\frac{kb\sin kb-k^2b^2\cos kb}{k^6}\Bigg)$$
$$-\mathbf{I}\frac{(2(\cos kb-1)+kb\sin kb)}{k^4}\Bigg)\Bigg\{\lim_{L\to\infty}\Bigg(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\Bigg)\Bigg\}\Bigg]\cdot\mathbf{g}(t)\,,$$

where $\mathbf{I}$ is the unit dyadic and $g_1(t)=\cos\Omega t$, $g_2(t)=\sin\Omega t$, and $g_3(t)=0$. The portion of the expression in square brackets is

$$-2\pi\int_{-\infty}^{\infty}dk\,\Bigg\{\Bigg(\frac{8(\cos kb-1)+5kb\sin kb-k^2b^2\cos kb}{k^6}\Bigg)\Bigg[\hat{z}\hat{z}\Bigg(-\frac{k^2\sin kz}{z}-\frac{3k\cos kz}{z^2}+\frac{3\sin kz}{z^3}\Bigg)$$
$$+\mathbf{I}\Bigg(\frac{(\cos kz)k}{z^2}-\frac{\sin kz}{z^3}\Bigg)\Bigg]+\Bigg(\frac{2(\cos kb-1)}{k^4}+\frac{kb\sin kb}{k^4}\Bigg)\mathbf{I}\frac{\sin kz}{z}\Bigg\}\Bigg\{\lim_{L\to\infty}\Bigg(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\Bigg)\Bigg\}\,.$$

Substituting in terms of the expressions $\sin k(z+b)$, $\sin k(z-b)$, $\cos k(z+b)$, and $\cos k(z-b)$, and writing out these functions in terms of exponentials, we get

$$-2\pi\int_{-\infty}^{\infty}dk\,\Bigg\{\Bigg[\hat{z}\hat{z}\Bigg((e^{ik(z+b)}+e^{ik(z-b)}-2e^{ikz})\Bigg(\frac{2}{ik^4z}-\frac{6}{k^5z^2}+\frac{6}{ik^6z^3}\Bigg)+(e^{ik(z+b)}-e^{ik(z-b)})$$
$$\times\Bigg(\frac{5b}{4k^3z}-\frac{15b}{4ik^4z^2}-\frac{15b}{4k^5z^3}\Bigg)+(e^{ik(z+b)}+e^{ik(z-b)})\Bigg(\frac{b^2}{4ik^2z}+\frac{3b^2}{4k^3z^2}-\frac{3b^2}{4ik^4z^3}\Bigg)\Bigg)+\text{c.c.}\Bigg]+\Bigg[\mathbf{I}\Bigg((e^{ik(z+b)}+e^{ik(z-b)}-2e^{ikz})$$
$$\times\Bigg(\frac{2}{k^5z^2}+\frac{2}{ik^6z^3}+\frac{1}{2ik^4z}\Bigg)+(e^{ik(z+b)}-e^{ik(z-b)})\Bigg(\frac{5b}{4ik^4z}+\frac{5b}{4k^5z^3}-\frac{b}{4k^3z}\Bigg)+(e^{ik(z+b)}+e^{ik(z-b)})$$
$$\times\Bigg(\frac{b^2}{4ik^4z^3}-\frac{b^2}{4k^3z^2}\Bigg)\Bigg)+\text{c.c.}\Bigg]\Bigg\}\Bigg\{\lim_{L\to\infty}\Bigg(\frac{e^{i(k-\omega_n)L}-1}{k-\omega_n}+\frac{e^{-i(k+\omega_n)L}-1}{k+\omega_n}\Bigg)\Bigg\}\,,$$

where c.c. denotes complex conjugate. Evaluating the above expression at $k=\pm\omega_n$ and using the values for $\sin\omega_n b$ and $\cos\omega_n b$ as above yields

$$4\pi^2b^2e^{-i\omega_n z}[\hat{z}\hat{z}(\omega_n^{-2}z^{-1}-3i\omega_n^{-3}z^{-2}-3\omega_n^{-4}z^{-3})+\mathbf{I}(i\omega_n^{-3}z^{-2}-\omega_n^{-4}z^{-3})]\,.$$

This is, of course, nonzero by itself, but as was shown in I, it may be made zero by adding another concentric distribution.