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New Formulation of the Axially Symmetric Gravitational Field Problem. II

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The coupled Einstein-Maxwell field equations are reformulated in terms of a pair of complex functions which have especially simple forms in the case of known axially symmetric stationary solutions. The formalism affords, in particular, a simple derivation of a solution previously guessed by Newman *et al.*

I. INTRODUCTION

IN an earlier paper,[1] it was shown that any axially symmetric stationary solution of Einstein's vacuum field equations may be described in terms of a single complex function $\mathcal{E}$ which is independent of azimuth and which satisfies the simple equation

$$(\mathrm{Re}\mathcal{E})\nabla^2\mathcal{E}=\nabla\mathcal{E}\cdot\nabla\mathcal{E}, \qquad (1)$$

where the three-dimensional Laplacian operator is understood. All known solutions, including the Kerr metric,[2] assume extremely simple forms when expressed in terms of the $\mathcal{E}$ field.

In the present paper, we demonstrate that any axially symmetric stationary solution of the coupled Einstein-Maxwell equations may be described in terms of a pair of complex functions $\mathcal{E}$ and Φ, both of which are independent of azimuth. These functions are found to satisfy a simple generalization of Eq. (1), namely,

$$(\mathrm{Re}\mathcal{E}+|\Phi|^2)\nabla^2\mathcal{E}=(\nabla\mathcal{E}+2\Phi^*\nabla\Phi)\cdot\nabla\mathcal{E}, \qquad (2)$$

$$(\mathrm{Re}\mathcal{E}+|\Phi|^2)\nabla^2\Phi=(\nabla\mathcal{E}+2\Phi^*\nabla\Phi)\cdot\nabla\Phi. \qquad (3)$$

If one makes the additional assumption that $\mathcal{E}$ is an analytic function of Φ, then Eqs. (2) and (3) reduce to a single complex equation, which may be transformed into a form completely equivalent to Eq. (1). Thus any solution of Eq. (1) generates a corresponding solution of Eqs. (2) and (3). In particular, when this transformation is applied to the Kerr metric, one obtains a solution of the coupled Einstein-Maxwell equations which was obtained previously by Newman *et al.* by methods which transcend logic.[3] Consequently the present theory may be regarded as a rationalization of their method. Furthermore, it is a generalization, since we may apply the same technique with respect to the perturbation solutions of Eq. (1) found in I, for example.

[1] F. J. Ernst, Phys. Rev. **167**, 1175 (1968); hereafter referred to as I.
[2] R. P. Kerr, Phys. Rev. Letters **11**, 237 (1963).

II. DERIVATION OF FIELD EQUATIONS

If the metric is written in the form

$$ds^2=f^{-1}[e^{2\gamma}(dz^2+d\rho^2)+\rho^2d\phi^2]-f(dt-\omega d\phi)^2,$$

the relevant field equations may be derived from the Lagrangian density

$$\begin{aligned}\mathfrak{L}=&-\tfrac{1}{2}\rho f^{-2}\nabla f\cdot\nabla f\\&+\tfrac{1}{2}\rho^{-1}f^2\nabla\omega\cdot\nabla\omega+2\rho f^{-1}A_4\cdot\nabla A_4\cdot\nabla A_4\\&-2\rho^{-1}f(\nabla A_3-\omega\nabla A_4)\cdot(\nabla A_3-\omega\nabla A_4),\end{aligned}$$

where A_3 and A_4 are the ϕ and t components of the electromagnetic 4-potential, respectively. The electromagnetic potentials satisfy the pair of equations

$$\nabla\cdot[\rho^{-2}f(\nabla A_3-\omega\nabla A_4)]=0, \qquad (4)$$

$$\nabla\cdot[f^{-1}\nabla A_4+\rho^{-2}f\omega(\nabla A_3-\omega\nabla A_4)]=0, \qquad (5)$$

while the gravitational potentials satisfy the pair of equations

$$\nabla\cdot[\rho^{-2}f^2\nabla\omega-4\rho^{-2}fA_4(\nabla A_3-\omega\nabla A_4)]=0, \qquad (6)$$

$$\begin{aligned}f\nabla^2f=&\nabla f\cdot\nabla f-\rho^{-2}f^4\nabla\omega\cdot\nabla\omega+2f\nabla A_4\cdot\nabla A_4+2\rho^{-2}f^3\\&\times(\nabla A_3-\omega\nabla A_4)\cdot(\nabla A_3-\omega\nabla A_4). \qquad (7)\end{aligned}$$

[3] E. T. Newman, E. Couch, K. Chinapared, A. Exton, A. Prakash, and R. Torrence, J. Math. Phys. **6**, 918 (1965).

The procedure developed in I is now applied to both pairs of equations. Equation (4) may be regarded as the integrability condition for the existence of a magnetic scalar potential A_3' such that

$$\rho^{-1}f(\nabla A_3-\omega\nabla A_4)=\hat{n}\times\nabla A_3'. \tag{8}$$

Here $\hat{n}$ is a unit vector in the azimuthal direction. It follows that

$$\rho^{-1}\hat{n}\times\nabla A_3=-[f^{-1}\nabla A_3'-\rho^{-1}\omega\hat{n}A_4],$$

and hence that

$$\nabla\cdot[f^{-1}\nabla A_3'-\rho^{-1}\omega\hat{n}\times\nabla A_4]=0.$$

Comparing this equation with Eq. (5), which may be written in the form

$$\nabla\cdot[f^{-1}\nabla A_4+\rho^{-1}\omega\hat{n}\times\nabla A_3']=0,$$

one sees that it is advantageous to introduce a complex potential

$$\Phi=A_4+iA_3', \tag{9}$$

which satisfies the equation

$$\nabla\cdot[f^{-1}\nabla\Phi-i\rho^{-1}\omega\hat{n}\times\nabla\Phi]=0. \tag{10}$$

A "duality rotation" simply corresponds to the replacement

$$\Phi\rightarrow\Phi e^{i\alpha}.$$

Equation (6) may now be written in the form

$$\nabla\cdot[\rho^{-2}f^2\nabla\omega-2\rho^{-1}\hat{n}\times\operatorname{Im}(\Phi^*\nabla\Phi)]=0,$$

an equation which may be regarded as the integrability condition for the existence of a new potential φ, such that

$$\rho^{-1}f^2\nabla\omega-2\hat{n}\times\operatorname{Im}(\Phi^*\nabla\Phi)=\hat{n}\times\nabla\varphi. \tag{11}$$

It follows that

$$\rho^{-1}\hat{n}\times\nabla\omega=-f^{-2}[\nabla\varphi+2\operatorname{Im}(\Phi^*\nabla\Phi)],$$

and hence that

$$\nabla\cdot\{f^{-2}[\nabla\varphi+2\operatorname{Im}(\Phi^*\nabla\Phi)]\}=0. \tag{12}$$

On the other hand, Eq. (7) assumes the form

$$f\nabla^2f=\nabla f\cdot\nabla f-[\nabla\varphi+2\operatorname{Im}(\Phi^*\nabla\Phi)]\cdot[\nabla\varphi+2\operatorname{Im}(\Phi^*\nabla\Phi)]+2f\nabla\Phi\cdot\nabla\Phi^*. \tag{13}$$

If one introduces the complex function

$$\mathcal{E}=(f-|\Phi|^2)+i\varphi, \tag{14}$$

Eqs. (12) and (13) combine to yield Eq. (2), while Eq. (10) yields Eq. (3).

III. SYSTEMATIC SOLUTION OF FIELD EQUATIONS

We shall at this point make the additional assumption that $\mathcal{E}$ is an analytic function of Φ. From Eqs. (2) and (3) it follows that

$$(\operatorname{Re}\mathcal{E}+|\Phi|^2)\frac{d^2\mathcal{E}}{d\Phi^2}\nabla\Phi\cdot\nabla\Phi=0,$$

and hence that $\mathcal{E}$ is a linear function of Φ. Using the boundary conditions $\mathcal{E}\rightarrow 1$ and $\Phi\rightarrow 0$ at infinity, we are led to write

$$\mathcal{E}=1-2q^{-1}\Phi, \tag{15}$$

where q is a complex constant. If, following I, we write

$$\mathcal{E}=(\xi-1)/(\xi+1), \tag{16}$$

then Eq. (15) implies

$$\Phi=q/(\xi+1). \tag{17}$$

These expressions may now be substituted into Eq. (2) or Eq. (3) in order to obtain the single complex equation

$$[\xi\xi^*-(1-qq^*)]\nabla^2\xi=2\xi^*\nabla\xi\cdot\nabla\xi.$$

Thus, introducing $\xi=(1-qq^*)^{1/2}\xi_0$, one arrives at the same equation

$$(\xi_0\xi_0^*-1)\nabla^2\xi_0=2\xi_0^*\nabla\xi_0\cdot\nabla\xi_0 \tag{18}$$

that one can obtain directly from Eq. (1) by the substitution

$$\mathcal{E}=(\xi_0-1)/(\xi_0+1). \tag{19}$$

The most interesting solutions of Eq. (18) discussed in I were those which arise naturally when one separates the equation in terms of prolate spheriodal coordinates. If the unit of length is chosen appropriately, ρ and z may be expressed in terms of new coordinates x and y such that

$$\rho=(x^2-1)^{1/2}(1-y^2)^{1/2},$$
$$z=xy,$$

where $0\leqslant y\leqslant 1$. Among the simpler solutions was one corresponding to

$$\xi_0=x\cos\lambda+iy\sin\lambda. \tag{20}$$

This turned out to be equivalent to the Kerr solution of Einstein's vacuum field equations.

If the same solution [Eq. (20)] is employed in connection with Eqs. (16) and (17) rather than Eq. (19), one obtains a solution of the coupled Einstein-Maxwell equations. This solution, of course, can be written out directly either in terms of the x-y coordinate system or in terms of the Weyl ρ-z coordinate system. However, to facilitate comparison with the conventional forms of the Schwarzschild, Kerr, and Newman solutions, we replace the parameters q and λ by new parameters m and e defined by

$$\sec\lambda=m(1-qq^*)^{1/2},$$
$$|q|=e/m,$$

and we introduce new coordinates r and θ, such that

$$r=x+m,$$
$$\cos\theta=y.$$

When the entire metric is constructed, one obtains a simple generalization of the Kerr metric, viz.,

$$ds^2=(r^2+a^2\cos^2\theta)\left[d\theta^2+\frac{dr^2}{r^2+a^2-(2mr-e^2)}\right]+\left[(r^2+a^2)\sin^2\theta d\phi^2-dt^2+\frac{2mr-e^2}{r^2+a^2\cos^2\theta}(dt+a\sin^2\theta d\phi)^2\right]. \quad (21)$$

The corresponding electromagnetic potentials are given by

$$A_3=-\frac{ear\sin^2\theta}{r^2+a^2\cos^2\theta},\quad A_4=\frac{er}{r^2+a^2\cos^2\theta}. \quad (22)$$

The constant a is related to m and e by

$$a=\tan\lambda=(1-m^2+e^2)^{1/2}$$

because of our particular choice of the distance scale. In effect, we have chosen our unit of length for ρ, z, r, m, e, and a in such a way that $(m^2-a^2-e^2)^{1/2}\rightarrow 1$. This is of considerable convenience in the intermediate steps of the derivation, but Eqs. (21) and (22) are valid for an arbitrary length scale.

The transformation of the solution [Eq. (21)] to the Weyl-Papapetrou canonical form is rendered by

$$\rho=[r^2+a^2-(2mr-e^2)]^{1/2}\sin\theta,$$
$$z=(r-m)\cos\theta. \quad (23)$$

That this result is completely equivalent to the metric discovered by Newman *et al.*[3] can be seen by performing the transformation

$$u=t+\int\left(1-\frac{2mr-e^2}{r^2+a^2}\right)^{-1}dr,$$
$$\phi'=\phi-\int\frac{adr}{r^2+a^2-(2mr-e^2)}. \quad (24)$$

More complicated solutions may be found by applying perturbation theory to Eq. (18) as in I.

IV. CONCLUSIONS

The consideration of electrostatic and magnetostatic fields within the framework of the $\mathcal{E}$ formulation presents no difficulties. In particular, whenever, $\mathcal{E}$ is an analytic function of the complex scalar potential Φ, the problem reduces to one already treated in I. The singular advantage of the $\mathcal{E}$ formulation over related techniques[4] is particularly well illustrated by the ease with which one extracts the solution corresponding to the Newman *et al.* metric. It is hoped that eventually the formalism can be extended to handle nonstationary axially symmetric problems as well.

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[4] After completing the present research, the author received a report from B. K. Harrison [J. Math. Phys. (to be published)] in which an elaborate and rather involved discussion of such techniques is given. Although in certain respects the work seems to be closely related to the present paper, the utility of the complex potentials $\mathcal{E}$ and Φ with respect to problems involving axial symmetry is not mentioned, and no attempt is made to derive the Newman *et al.* metric. Nevertheless, the question naturally arises whether or not an amalgamation of the two approaches would be useful for time-dependent problems involving axial symmetry.