Stepan and Niske[6] found the plutonium-241 β half-life to be 13.63±0.36 yr by neutron absorption measurements, which were independent on the half-life of amercium-241. However, the value of 14.03 yr was preferred here since it was determined from three investigations[3] by observing the americium growth rate.

The difference in the average β energy as determined from the β spectrum of Shlyagin[2] and the calorimetric measurements of this work is surprisingly large. It is difficult to explain the magnitude of the error in the integrated value for the average β energy. In the absence of an accurate and precise β spectrum for plutonium-241, we may, by using a half-life value of 14.03 yr, take the average β energy of plutonium-241 to be 5.78±0.31 keV.

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Odd-Parity States of Even Tin Isotopes in the Quasiparticle Second Tamm-Dancoff Approximation*

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The negative-parity 3^-, 5^-, 6^-, and 7^- states of the even isotopes of tin are studied with the quasiparticle second Tamm-Dancoff (QSTD) theory which treats the two- and four-quasiparticle excitations on the same footing. The static magnetic moments and the $E\lambda$ transition probabilities are calculated. The QSTD eigenvalues, and particularly the QSTD eigenvectors, obtained with phenomenological residual nuclear forces are found to be more sensitive to the residual nuclear force and to the unperturbed single-particle energy levels assumed than are the corresponding results for the even-parity states of the same isotopes. It is shown, however, that a semiquantitative agreement of the QSTD predictions with experiment can be achieved. In particular, rather good agreement is obtained in one of our cases with the observed values of the static magnetic moments of the 5_1^- states of Sn^{116} and Sn^{120}. Corresponding calculations with a realistic nucleon-nucleon potential are most desirable.

1. INTRODUCTION

THE quasiparticle second Tamm-Dancoff (QSTD) theory gives a microscopic description of the low-lying states of even-even nuclei in terms of zero-, two-, and four-quasiparticle (qp) excitations.[1–4] This theory has enjoyed a considerable amount of success in describing the even-parity states (0^+, 2^+, and 4^+) of the even tin isotopes.[1,2] The Sn nuclei are believed to belong to the so-called vibrational region, and are considered to be essentially spherical. In fact, even the large observed values[5] of the quadrupole moments of the first excited 2_1^+ states seem to be understood in terms of collective contributions of four-qp excitation alone, without being forced to assume a stable deformation for the average nuclear (Hartree-Fock) field.[4] Unfortunately, all the results of Refs. 1–4 and the ones reported below are not based on Hartree-Fock-Bogolubov (HFB) self-consistency. In fact, the HFB is a formidable task in itself, almost prohibitive in our case, and we are forced to choose a more or less "reasonable" phenomenological single-particle (s.p.) basis (the unperturbed s.p. energies E_n^0 and the corresponding s.p. wave functions). In fact, we have based our calculations on sets of $\{E_n^0\}$ taken from the literature and limited

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ourselves to the use of harmonic-oscillator radial wave functions. Unfortunately, our results can, in many cases, be rather sensitive to the choice of $\{E_n^0\}$. Nevertheless, many important qualitative and semiquantitative conclusions can be drawn from our results. Actually, we are mainly interested in investigating various different aspects and characteristics of the theory itself rather than in simply attempting to fit particular experimental data. In particular, we are studying the implications of our more complete approximation to the exact solution of our many-body problem for the usefulness of the much simpler but widely used microscopic or semimicroscopic models.

We should point out that in the pure two-qp QTD theory the odd-parity states of the tin isotopes involve only two, three, or at most four configurations. This is in contrast to the even-parity states, where there are many more pure two-qp excitation modes involved. In this situation the QTD theory takes into account only a few of the possible excited configurations, i.e., of the effects of the residual interactions. In particular, it cannot be expected to properly account for the collective character of some of the odd-parity states. For this reason we think the application of the QTD method to these states rather uninteresting.

In the present paper, we report on some of our results for the odd-parity low-lying states 3^-, 5^-, 6^-, and 7^- (and some 1^-) for the tin isotopes, with $A=116$ and 120. Our method, techniques, and notations are all those of Refs. 1 and 2. Most of the FORTRAN computer codes used in the work of Refs. 1 and 2 and due to P. L. Ottaviani have been employed for our present computations with only slight modifications in some of them.

As in Refs. 1 and 2, we consider the 50–50 "magic" core to be inert, and only the valence neutrons belonging to the five subshells $2d_{5/2}$, $1g_{7/2}$, $3s_{1/2}$, $2d_{3/2}$, and $1h_{11/2}$ to be active in the pairing and the configuration-mixing interactions. This implies that we are working with phenomenological "effective" residual interaction potentials involving the core-polarization effects rather than with any realistic nucleon-nucleon potentials. In the same sense, we must introduce the neutron effective charge for our electromagnetic transition probabilities which takes into account implicitly the effects of the excited core-proton configurations.

Finally, we compare critically our numerical results with those of simple pure two-qp theories of Arvieu.[6]

The importance of the octupole collective character of the first excited 3^- (3_1^-) states of even tin isotopes was first revealed in the experiments by Cohen *et al.*[7] on the so-called anomalous inelastic scattering [cf., however, the critical remarks by Schneid, Hamburger, and Cohen (Ref. 7, p. 1210)]. The states 3_1^-, 5_1^-, and 7_1^- for $A=116$, 118, 120, 122, and 124 have been reported by several groups.[7–13] The 6_1^- state of $A=116$ has been reported by Bodenstaedt *et al.*[17] Several experimental groups[9–13] have reported values of half-lives [or of reduced transition probabilities $B(E\lambda, I_i \to I_f)$ for the following transitions: $E3$: $5_1^- \to 2_1^+$ and $7_1^- \to 4_1^+$ in $A=116$, 118, 120; $E2$: $5_1^+ \to 3_1^-$ and $7_1^- \to 5_1^-$ in $A=116$, 120; and $E1(?)$: $5_1^- \to 4_1^+$]. We shall give below some of our predictions for some of the above $B(E\lambda)$'s. Unfortunately, the operator $E1$ vanishes identically in our Hilbert space, and we cannot calculate the $B(E1, 5_1^- \to 4_1^+)$ in our model. On the other hand, the observed value of this quantity appears to be very small indeed, as expected. Similarly, we have no direct simple $1_n^- \to 0_1^+$ transitions in our model. On the other hand, our low-lying 1_n^- states are of little interest for two reasons: (a) They lie rather far below the giant dipole resonance; (b) because their eigenvalues lie above 4 MeV, the six-qp excited configuration may be important for them. Finally, no such 1^- levels have as yet been identified experimentallv.

2. CALCULATIONS AND NUMERICAL RESULTS

In our numerical work we have kept all the parameter values of Ref. 2. The harmonic-oscillator parameter is $\sqrt{\nu}=(M\omega_0/\hbar)^{1/2}=0.454$ F^{-1}. The s.p. unperturbed energies $\{E_n^0\}$ of Ref. 2 are those of Kuo and Baranger [Ref. 14, Table 5(a)]. The set $\{E_n^0\}$ seems to give a rather good fit of the spectra of the odd isotopes of tin.

The residual interaction potential $V(1,2)$ is of the Gaussian form[2,14]

$$V(1,2)=-V_0\exp(-r_{12}^2/r_0^2)(P^s+tP^t), \qquad (1)$$

where V_0 is fixed at 31.0 MeV, $r_0=2.0$ F, $r_{12}=|\mathbf{r}_1-\mathbf{r}_2|$, and P^s and P^t are the singlet-even and triplet-odd projection operators, respectively. We present below results for the two extreme values of the parameter t: (a) $t=1$ (a Wigner force) and (b) $t=-0.555$ (a Rosenfeld mixture). The independent qp energies

$$E_n=[(E_n^0-\lambda)^2+\Delta_n^2]^{1/2}$$

are found first for each one of the cases (a) and (b) by solving the appropriate usual BCS equations.[15]

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The construction of our complete orthonormal basis involving all the possible two- and four-qp excitations is performed as in Ref. 1. (The fractional parentage coefficients orthonormalizing the QSTD basis are found directly by the computer.) The dimensions of the complete QSTD secular matrices are, respectively: for $J^\pi=1^-$, 57×57; for 3^-, 132×132; for 5^-, 179×179; for 6^-, 179×179; for 7^-, 165×165. As we see, the dimensions here are comparable to those for the QSTD even-parity states of Ref. 2. Because of very long matrix-computation times, we have had to apply important truncations of our original Hilbert spaces. The criterion for each such truncation was a very large separation in energy (diagonal elements) of the four-qp configurations to be left out (usually diagonal elements $\gtrsim 7$ MeV) and rather weak couplings of such four-qp modes to the two-qp modes. (A special computer program has been applied for this purpose.) In addition, we have verified *a posteriori* the numerical stability of a truncation against the inclusion of a number of additional basis vectors associated with larger diagonal elements. For example, for the Gaussian Wigner ($t=1$) force, we have finally obtained the following dimensions of our secular problems:

1^-: 57×57 (no truncation necessary here),

3^-: 61×61, 5^-: 33×33,

6^-: 50×50, 7^-: 39×39.

The spurious kets due to the nucleon-number nonconservation of the type described in Ref. 1, and which are entirely contained within the respective two- and four-qp Hilbert spaces for each value of J^π, have been constructed explicitly and projected out of our secular matrices before their diagonalizations. In the case $J^\pi=1^-$ there are no two-qp modes, and thus also no spurions of the kind described above in our basis.

In Table I, we give our QTD and QSTD energies of the lowest three levels for $J^\pi=3^-$, 5^-, 6^-, and 7^- for $A=116$ and 120, and for the Wigner ($t=1$) and the Rosenfeld ($t=-0.555$) $V(1,2)$ of Eq. (1). The first excited level energies observed are indicated for each case J^π, A for comparison. The total percentage of all four-qp components (%b of Refs. 1 and 2) is indicated in parentheses for each QSTD level. As for $J^\pi=1^-$, we find, e.g., for $A=120$ and $t=1$, the following three lowest eigenenergies: 4.161, 4.354, and 4.921 MeV (100% four-qp). We realize that some of the odd-parity states are extremely sensitive to the residual nuclear force employed. Particularly in the 3^- case for $A=116$, not only the four-qp percentages of the QSTD eigenvectors, but also the corresponding eigenvalues, change dramatically in passing from our Wigner force ($t=1$) to the Rosenfeld mixture ($t=-0.555$). In this case the differences between the two eigensolutions for the state 3_1^- are due to the most important diagonal matrix elements of the residual nuclear force. In the 3^- case the dramatic difference between the ($t=1$) and ($t=-0.555$) eigenvalues is due mainly to the absence of the two-qp mode $(3s_{1/2}, 1h_{11/2})$, in contrast to all the other cases J^π. For example, in the 6^- case the difference between the four-qp percentage of the ($t=1$) and the ($t=-0.555$) eigenvector $|6_1^-\rangle$ is large, but the corresponding eigenvalues are comparable. In the case ($t=1$) the four-qp percentages are so large as to invalidate completely the pure two-qp QTD theory (cf. Ref. 6). On the other hand, this effect depends very much on the choice of the residual nuclear force (cf., e.g., our corresponding results for $t=-0.555$), and thus no general conclusion can be drawn on this point. Calculations with a realistic nuclear force will be most interesting in this respect.

As for the BCS solutions, the single-qp state which is most important is the state $1h_{11/2}$ because it appears once or thrice in *all* the configurations involved. It is therefore the single-qp energy of $1h_{11/2}$ which is most important in determining the unperturbed excitation

TABLE I. Energy levels (in MeV) calculated for tin isotopes $A=116$, 120, and for two spin-dependent Gaussian potentials of Kuo *et al.* (Ref. 14). The numbers in parentheses in the QSTD columns are the four-qp percentages. The columns labelled 'Expt' give the experimental values.

	116						120			
A / J^π		$t=1$			$t=-0.555$			$t=1$		
	Expt	QTD	QSTD		QTD	QSTD	Expt	QTD	QSDT	
	2.267	4.299	2.348	(~100)	3.568	3.464 (4)	2.391 (?)	4.384	3.077	(99)
3^-	···	4.433	2.996	(~100)	4.044	3.956 (4)	···	4.620	3.644	(95)
	···	···	3.456	(~100)	···	4.308 (98)	···	···	3.975	(69)
	2.364	2.560	2.416	(11)	2.335	2.271 (2)	2.285	2.520	2.432	(4)
5^-	···	2.831	2.682	(46)	2.762	2.671 (3)	···	2.831	2.758	(4)
	···	3.972	2.661	(77)	3.658	3.614 (2)	···	4.224	3.310	(98)
	2.774	2.750	2.344	(98)	2.723	2.585 (4)	···	2.591	2.477	(5)
6^-	···	2.852	2.564	(19)	2.923	2.828 (3)	···	2.929	2.841	(6)
	···	3.990	2.644	(16)	3.603	3.509 (3)	···	4.270	3.063	(97)
	2.909	2.498	2.267	(14)	2.268	2.180 (3)	2.483	2.487	2.371	(5)
7^-	···	3.972	2.745	(97)	3.446	3.390 (2)	···	4.237	3.495	(~100)
	···	4.364	3.029	(99)	4.031	3.635 (~100)	···	4.737	3.729	(89)

energies (in the diagonal elements of our secular matrices).

Our failure to reproduce the observed level energy of the 7^- state in Sn^{116} (provided that the experimental value of 2.909 MeV refers actually to the *first* 7^- state 7_1^-) is probably due mainly to our inappropriate choice of the original unperturbed single-neutron energies $\{E_n^0\}$, resulting, in particular, in inappropriate values of the single-qp energy of $1h_{11/2}$. In fact, our $E(1h_{11/2})$ are too low to be able to account for the rather high experimental energy of 2.909 MeV.

The impressive difference between our results for $A=116$ and those for $A=120$ is again probably due mainly to the fact that the true Hartree-Fock single-neutron energies $\{E_n^0\}$ should vary from one isotope to another, while we have, instead, kept for $A=120$ the values of $A=116$. Our $\{E_n^0\}$ are actually more appropriate to Sn^{116}. Unfortunately, a search for a better set of $\{E_n^0\}$ for Sn^{120} should require a great amount of expensive extra computer time, which does not seem warranted at the present stage. Rather one should approach the problem either via an HFB self-consistent solution or at least by a search for better $\{E_n^0\}$, but in the case of a realistic nuclear force. One such project is now in preparation. [*Note added in proof.* M. Gmitro and J. Sawicki, Phys. Letters (to be published); M. Gmitro, A. Rimini, J. Sawicki, and T. Weber (to be published).]

We have also calculated the static magnetic moment $\mu(5_1^-)$ of the 5_1^- state of $A=116$ and $A=120$ and several of the characteristic 2^λ-pole electric transition probabilities.

For the magnetic moment $\mu(5_1^-)$, Bodenstaedt *et al.*[12] give the following respective experimental value of the g factors:

$$g_{5_1^-}(A=116) = -0.065\pm0.005,$$
$$g_{5_1^-}(A=120) = -0.068\pm0.007.$$

Lombard[16] has attempted to fit these values with the pure two-qp QTD theory of Arvieu.[6] The eigenvectors $|5_1^-\rangle$ of Ref. 6 have rather strong admixtures of the $(1h_{11/2}, 2d_{3/2})_{5^-}$ configuration. Lombard does not introduce any neutron effective charge and includes only the pure-spin part of the $M1$ operator. His predictions are in a rather sharp disagreement[17] with experiment.[12]

The g factor is defined as

$$g_I=\{I(I+1)(2I+1)\}^{-1/2}\langle I\|\hat{\mu}_{(1)}\|I\rangle, \quad (2)$$

in the usual notation, where the magnetic dipole operator is

$$\hat{\mu}_{(1)}=g_{(s)}s_z+g_{(l)}l_z. \quad (3)$$

When calculated with QSTD eigenvectors $|IM\rangle$, the reduced matrix element $\langle I\|\hat{\mu}\|I\rangle$ is calculated from the formula given in the Appendix, with $\hat{O}_\lambda=\hat{\mu}_{(1)}$.

Assuming a vanishing neutron $M1$ effective charge $e_n^{(M1)}=0$, we find for Sn^{116} with our $|5_1^-\rangle$ eigenvector of the Gaussian $t=1$ force the value

$$g_{5_1^-}(A=116) = -0.0545,$$

which is extremely close to the observed value. ($e_n^{(M1)}\cong-0.04$ would reproduce exactly the experimental value.) The corresponding quantity for $A=120$ is found to be

$$g_{5_1^-}(A=120) = -0.0962,$$

again very close to observation. These results are not very reliable quantitatively, because the element $\langle I\|\hat{\mu}\|I\rangle$ is rather sensitive to the details of our eigenvectors. For example, the above agreement with experiment for $e_n^{(M1)}\cong0$ is lost when we employ the Gaussian force with $t=-0.555$ instead of that with $t=1$.

As for the reduced transition probabilities, $B(E\lambda)$, we find them extremely sensitive to all the details of the eigenvectors involved. In view of this fact and of the crudeness of our assumptions for the s.p. basis and of our approximations, no quantitatively reliable fit would be warranted at this stage, although it would be possible. Therefore, only some general characteristics could be of some interest.

To give an idea of the numerical situation, we mention a few numbers obtained. For the transition $5_1^-\to3_1^-$ in Sn^{116} we find $B(E2, 5_1^-\to3_1^-, 116)=7.7(e_n^{(E2)})^2\ e^2\ F^4$ for our Gaussian ($t=1$) force and $14.9\ (e_n^{(E2)})^2\ e^2\ F^4$ for the Gaussian ($t=-0.555$) force, where $e_n^{(E2)}$ is the neutron effective charge for the $E2$ operator. The corresponding experimental value appears to be $\cong240\ e^2\ F^4$. For the transition $7_1^-\to5_1^-$ in Sn^{120} we find $B(E2, 7_1^-\to5_1^-, 120)=0.008\ (e_n^{(E2)})^2\ e^2\ F^4$ for the Gaussian ($t=1$) force and $2.94\ (e_n^{(E2)})^2\ e^2\ F^4$ for the Gaussian ($t=-0.555$) force. The corresponding experimental value seems to be $\cong0.17\ e^2\ F^4$.

3. CONCLUSION

We have shown that a semiquantitative description of the general characteristics of the negative-parity states of the even tin isotopes in terms of QSTD eigenstates is possible. On the other hand, we find these states much more sensitive to the numerical details of the assumptions for the s.p. basis and of the residual nuclear force than the corresponding even-parity states of Refs. 2 and 4. For this reason both an HFB self-consistent single-qp basis and a residual force based on a realistic nucleon-nucleon potential would be most

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[17] However, we should point out that in the case of the g factor, in contrast to the case of the quadrupole moment of the first excited 2^+ state 2_1^+, and of typical $B(E2)$'s, the contributions of the two-qp-four-qp mixed terms are rather small here. Essentially, the g factor is determined by the two-qp components alone of the vector $|5_1^-\rangle$. For example, for $g_{5_1^-}(A=116)$ calculated with the Gaussian ($t=1$) force, we find the contribution of the two-qp components alone to be $\Delta_{2qp}g_{5_1^-}(A=116)=-0.046$. When the corresponding calculation is done for the same quantity with the pure two-qp QTD theory, we find $g_{5_1^-}{}^{(QTD)}(A=116)=-0.042$. This shows that, after all, the disagreement with experiment of Ref. 16 is due to the particular structure of the vector $|5_1^-\rangle$ of Arvieu (Ref. 6) rather than to the importance of the specific QSTD effects (four-qp contributions).

desirable. One such series of calculations is now in preparation. One of our most important general findings is the great sensitivity of our QSTD eigenvectors, in particular, of their four-qp percentages to the residual nuclear force assumed. In some cases the pure two-qp QTD theory is clearly invalidated.

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APPENDIX

The reduced matrix element between QSTD states of a tensor operator $\hat{O}_\lambda$, $\lambda>0$, such that the corresponding s.p. reduced matrix elements have the property

$$\langle s\|\hat{O}_\lambda\|r\rangle=(-)^{j_s-j_r}\langle r\|\hat{O}_\lambda\|s\rangle,$$

is given by

$$\langle E'J'\|\hat{O}_\lambda\|EJ\rangle=\sum_{rs}\langle r\|O_\lambda\|s\rangle[U_{rs}(\lambda)(Q_{DA}{}^\lambda(r,s)+Q_{AD}{}^\lambda(r,s)+Q_{BA}{}^\lambda(rs)+Q_{AB}{}^\lambda(r,s))$$
$$+V_{rs}(\lambda)(Q_{AA}{}^\lambda(r,s)+Q_{BB}{}^\lambda(r,s))], \quad \text{(A1)}$$

where

$$U_{rs}(\lambda)=U_rV_s+(-)^\lambda U_sV_r, \quad V_{rs}(\lambda)=U_rU_s-(-)^\lambda V_rV_s. \quad \text{(A2)}$$

The various contributions are given by

$$Q_{DA}{}^\lambda(r,s)=-\tfrac{1}{2}\delta_{\lambda J}\delta_{J'0}(-)^\lambda d^*(E')\sum_{(ab)}\delta_\lambda(rs,ab)\mathbf{a}(EJ;\,ab), \quad \text{(A3)}$$

$$Q_{AD}{}^\lambda(r,s)=-\tfrac{1}{2}\delta_{\lambda J'}\delta_{J0}d(E)\sum_{(a'b')}\delta_\lambda(a'b',rs)\mathbf{a}^*(E'J';\,a'b'), \quad \text{(A4)}$$

$$Q_{BA}{}^\lambda(r,s)=-\tfrac{1}{2}(-)^{\lambda+J+J'}\hat{J}'\hat{\lambda}^{-1}\sum_{\substack{(a'b'c'd')\\J_1'J_2'}}\sum_{(ab)}\mathbf{b}^*(E'J';\,J_1'J_2';\,a'b'c'd')\mathbf{a}(EJ;\,ab)P_{(J_1'J_2')J'(\lambda J)}(a'b'c'd',rsab), \quad \text{(A5)}$$

$$Q_{AB}{}^\lambda(r,s)=-\tfrac{1}{2}(-)^\lambda\hat{J}\hat{\lambda}^{-1}\sum_{(a'b')}\sum_{\substack{(abcd)\\J_1J_2}}\mathbf{a}^*(E'J';\,a'b')\mathbf{b}(EJ;\,J_1J_2;\,abcd)P_{(\lambda J')J(J_1J_2)}(rsa'b',abcd), \quad \text{(A6)}$$

$$Q_{AA}{}^\lambda(r,s)=-(-)^{J+J'+j_r+j_s}\hat{J}\hat{J}'\sum_{(a'b')}\sum_{(ab)}\mathbf{a}^*(E'J';\,a'b')\mathbf{a}(EJ;\,ab)$$
$$\times\{[\delta_{a'r}W(J'j_{b'}\lambda j_s;\,j_rJ)\delta_J(sb',ab)]-(-)^{J'+j_{a'}+j_{b'}}[a'\leftrightarrow b']\}, \quad \text{(A7)}$$

$$Q_{BB}{}^\lambda(r,s)=-(-)^{J+J'+j_r+j_s}\hat{J}\hat{J}'\sum_{\substack{(a'b'c'd')\\J_1'J_2'}}\sum_{\substack{(abcd)\\J_1J_2}}\mathbf{b}^*(E'J';\,J_1'J_2';\,a'b'c'd')\mathbf{b}(EJ;\,J_1J_2;\,abcd)$$
$$\times\{[\hat{J}_1\delta_{sb}\sum_I\hat{I}W(J_2J'J_1\lambda;\,IJ)W(J_1j_a\lambda j_r;\,j_bI)P_{(J_1'J_2')J'(IJ_2)}(a'b'c'd';\,arcd)]-(-)^{J_1+j_a+j_b}[a\leftrightarrow b]$$
$$+(-)^{J_1+J_2+J}[J_1\leftrightarrow J_2,\,a\leftrightarrow c,\,b\leftrightarrow d]+(-)^{J+j_a+j_b+j_c+j_d}[J_1\leftrightarrow J_2,\,a\leftrightarrow d,\,b\leftrightarrow c]\}, \quad \text{(A8)}$$

where

$$\delta_J(abcd)=\delta_{ac}\delta_{bd}-(-)^{J+j_c+j_d}\delta_{ad}\delta_{bc}$$

and

$$P_{(J'J'')J(I'I'')}(abcd,a'b'c'd')=\langle 0|[\mathbf{B}_{(J'J'')JM}(abcd),\mathbf{B}_{(I'I'')JM}{}^+(a'b'c'd')]|0\rangle, \quad \text{(A9)}$$

where the four-qp operators $\mathbf{B}^+$ are defined in Eq. (10) of Ref. 1, $P_{(J'J'')J(I'I'')}(abcd,a'b'c'd')$ can be calculated from Eq. (A5) of Ref. 1, d is the coefficient fo the zero-qp mode (of the qp vacuum itself), and the symbols $\mathbf{a}$ and $\mathbf{b}$ are the coefficients of expansion of the QSTD eigenvectors on the nonorthonormalized two-qp ($\mathbf{A}^+$) and four-qp ($\mathbf{B}^+$) basic operators of Ref. 1, respectively. The summations in $\sum_{(ab)}$ and $\sum_{(abcd)}$ are restricted in order to avoid any repetition of equivalent configurations.