Shell-Model Theory of Pb^{206}. II†

WILLIAM W. TRUE*
Laboratory of Nuclear Science and Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts
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The eigenvalues and eigenfunctions of the low-lying levels of Pb^{206} have been calculated using a singlet-even potential, and a P_2-type force which is known to arise when the particle states are weakly coupled to a phonon. All the neutron orbitals between $N=82$ and $N=126$ have been considered. The ground-state wave function is found to be in much better agreement with experiment than has been the case for previous calculations.

I. INTRODUCTION

IN the years following the introduction of the nuclear shell model,[1,2] models describing the nucleus have become more sophisticated and more detailed in their descriptions. A partial list of references is given in Ref. 3–9. Also, in the last few years there has been a great improvement in the accelerators and detecting equipment used in the study of nuclear structure. These advances in experimental equipment has allowed for more detailed investigations into the spectroscopy and structure of the ground states and excited states of various nuclei.

In the regions around nuclei consisting of doubly closed shells, one expects that the nuclei in these regions should be fairly well described by nucleons moving in a central potential and interacting with a residual force. In particular, the TF calculation[5] did a fairly good job of describing energy-level structure and observed transition rates in the Pb^{206} nucleus. As experiments began to probe the structure of the lower-lying states of the nuclei in the lead region,[10,11] some discrepancies have appeared between the experimental results and the calculations of TF. Some of the recent theoretical calculations[12,13] of the structure of Pb^{206} do not appear to have explained the experimental results of Mukherjee and Cohen.[10] However, Plastino, Arvieu, and Moszkowski,[14] using a "surface delta function" as the residual force, found very good agreement between their ground-state wave function and the results of Mukherjee and Cohen. On the other hand, the fit to the experimental energy levels with the results of the calculation by Plastino *et al.* was not too good.

This paper will describe a calculation of the level structure of Pb^{206} where all the neutron hole orbitals between the magic numbers of 82 and 126 have been considered. It will be seen in Sec. III that the wave function for the ground state of Pb^{206} is now in much closer agreement with the experimental results than the ground-state wave function of the TF calculation.

Calculations of the isotopes in the lead region are important because this region of the periodic table appears to be one of the few where the shell model can be tested in some detail. This is in contrast to the oxygen and calcium regions, where, generally speaking, experiments and calculations done in recent years on nuclei indicate that the spherical shell model is inadequate to describe the structure of nuclei in these regions.

In the Pb region, the dominant feature of the residual force is its pairing aspect. The parts of the force giving rise to collective modes of motion appear to have much less influence on the structure of the low-lying states of the Pb nuclei than in lighter nuclei (except possibly for the case of Pb^{208}). These strong pairing aspects will enable one to estimate the pairing strength parameter G from the matrix elements of the residual force.[4] Such a comparison is made in Sec. III.

II. FORCE PARAMETERS

The calculation of the level structure of Pb^{206} was essentially the same as the TF calculation except that more neutron orbitals were included. The energy-level spacings of the neutron orbitals were assumed to be the same as is observed in Pb^{207} and are given in Table I. The experimental energy levels of Pb^{206} used in this paper are given in Table II.

As in TF, harmonic-oscillator wave functions were assumed for the radial dependence with $\nu=0.1842$ fm^{-2} where $\psi\sim e^{-\nu r^2/2}$. First, a pure singlet-even potential of

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* On leave of absence from Department of Physics, University of California, Davis, Calif.
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[9] V. Gillet, A. M. Green, and E. A. Sanderson, Nucl. Phys. **88**, 321 (1966).
[10] P. Mukherjee and B. L. Cohen, Phys. Rev. **127**, 1284 (1962).
[11] W. R. Hering, M. Dost, and A. D. Achterath (to be published).
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the form

$$V=V_0e^{-\beta r^2}, \tag{1}$$

where $V_0=-32.5$ MeV and $\beta=0.2922$ fm^{-2}, was considered. A comparison of the energy levels between the TF singlet-even calculation, this singlet-even calculation, and the experimental levels are given in Fig. 1. There seems to be some improvement in this calculation over the TF calculation for the lower spins, but the agreement gets increasingly worse for the high-spin states.

As pointed out in TF, the observed $E2$ transitions in Pb^{206} and Pb^{207} are not explained by these neutron configurations, but that one must include admixtures of higher-lying proton orbitals in these lower-lying levels in the lead region. TF showed that a P_2 type residual force arose when the neutron orbitals were weakly coupled to $L=2$ phonon excitations of the core. This residual P_2 force,

$$V_{\mathrm{WC}}=(-1)^{\Sigma n_i+1}\left(\frac{k^2}{C}\right)\sum_{\mu} Y_{2\mu}(\theta_1,\phi_1)Y_{2\mu}{}^*(\theta_2,\phi_2)$$

$$=(-1)^{\Sigma n_i+1}\left(\frac{k^2}{C}\right)P_2(\cos\theta_{12}), \tag{2}$$

will be called the weak-coupling force, WC. The sum over the radial quantum numbers n_i takes into account the fact that the radial wave functions may have different signs at the nuclear surface. It is interesting to note that if $(-1)^{\Sigma n_i+1}(k^2/C)$ was replaced by a constant, one would have the surface-delta-function force used so successful by Moszkowski and collaborators[14–16] as the residual force between nucleons.

Recent calculations by Brown and Kuo,[8] using free nucleon-nucleon potentials, determined from scattering experiments, for the forces between nucleons, indicate that a correction should be made to the two-body matrix elements to include effects of exciting particles from the core into higher-lying orbitals than are normally considered in shell-model calculations. In the nickel isotopes, these corrections can be simulated by adding a P_2 and a P_4 force of the form

$$V_2=G_2r_1{}^2r_2{}^2P_2(\cos\theta_{12}) \tag{3}$$

and

$$V_4=G_4r_1{}^4r_2{}^4P_4(\cos\theta_{12}), \tag{4}$$

respectively. The value which Brown and Kuo found for G_2 compares favorably with the G_2 used by Kisslinger and Sorensen.[4] One would expect the P_4 force to be less important in the lead region where the nucleons are further apart on the average.

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TABLE I. Neutron hole energies (Ref. 9) in Pb^{207}.

Orbital	Energy (MeV)
$3p_{1/2}$	0.00
$2f_{5/2}$	0.57
$3p_{3/2}$	0.90
$1i_{13/2}$	1.63
$2f_{7/2}$	2.35
$1h_{9/2}$	3.47

Table XIV of TF shows that $|\langle r^2\rangle|$ for the p, f, and i neutron orbitals in the lead region are approximately constant around 35 fm^2. If one assumes that $|\langle r^2\rangle|$ is a constant, then the P_2 force in Eq. (3) reduces to the surface-delta-function force of Moszkowski and to the WC force of Eq. (2). So there seems to be some justification for including a WC force in the singlet-even residual force in Pb^{206}.

The value of $k^2/C=1.4$ was adjusted in TF to give the correct order of magnitude for the observed $E2$ transitions between the first excited states and the ground states of Pb^{207} and Pb^{206}. It would appear from the above discussion that this value of k^2/C would describe a P_2 force which was due to excitations to higher proton orbitals only. Calculations indicate that 0+ to 2+ spacing in Pb^{206}, the spacing expected to be

TABLE II. Experimental[a–c] energy levels of Pb^{206}.

Energy	$J\pi$	Reference
0	0+	a, b
0.803	2+	a, b
1.165	0+	a
1.338	3+	a, b
1.465	2+	a, b
1.682	4+	a, b
1.73	1+	b
1.785	(2+)	d
1.997	4+	a, b
2.149	(1+,2+)	a, b
2.197	7−	a, b
2.314		a, e
2.385	6−	b
2.421		a, e
2.526	3−	b
2.650	(9−)	a
2.776	5−	a, b
2.924	4+	a, f
3.010	5−	a, b
3.116	3+	a, f
3.191	(5−)	a
3.253	(6+)	a
3.404	5−	b

[a] E. R. Flynn, P. D. Barnes, G. J. Igo, and R. Woods, Bull. Am. Phys. Soc. **12**, 539 (1967); (private communication).
[b] W. W. True and K. W. Ford, Phys. Rev. **109**, 1675 (1958), Table III.
[c] See also: G. Vallois, J. Saudinos, and O. Beer, Phys. Letters **24B**, 512 (1967); A. M. Hellwege and K. H. Hellwege, *Energy Levels of Nuclei A =5 to A =257 in Landolt Bornstein* (Springer-Verlag, Berlin, 1961); J. H. Bjerregaard, O. Hansen, O. Nathan, and S. Hinds (to be published); R. B. Day, A. E. Johnsrud, and D. A. Lind, Bull. Am. Phys. Soc. **1**, 56 (1956); J. A. Harvey, Can. J. Phys. **31**, 278 (1953); D. E. Alburger and M. H. L. Pryce, Phys. Rev. **95**, 1482 (1954); A. M. Bernstein and S. M. Smith (private communication).
[d] Reference a gives this as (0,2) and Ref. b gives a (2+) level at 1.83. It is assumed that these levels are the same and have a (2+) assignment.
[e] Not plotted in Figs. 1–3.
[f] These levels are believed to be 3+ and 4+ doublet as discussed in Ref. b.

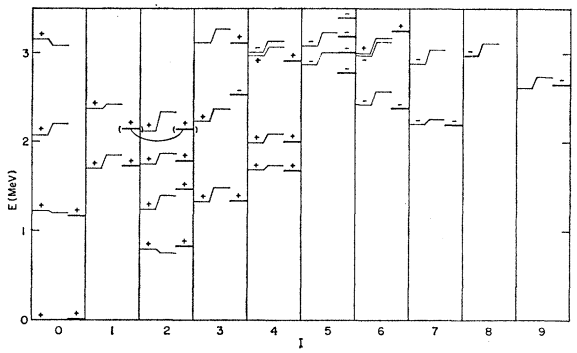

FIG. 1. Comparison of the energies from the True and Ford (Ref. 5) singlet-even calculation (first column for each spin), the singlet-even calculation of this paper (second column), and the empirical energies from Table II (third column).

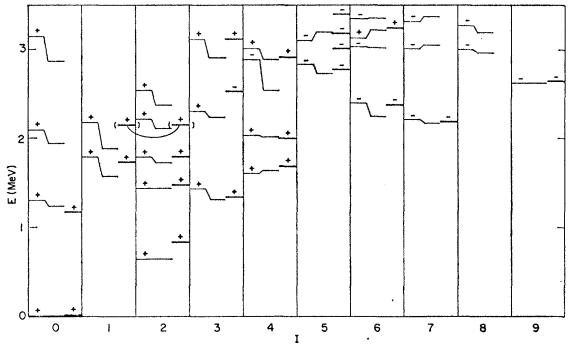

FIG. 3. Comparison of the energies from the singlet-even plus weak-coupling calculation of this paper (first column for each spin), the singlet-even plus triplet-odd plus weak-coupling calculation of this paper (second column), and the empirical energies from Table II (third column).

most sensitive to this type of a force, remains essentially constant when k^2/C is varied from 1.3 to 1.6.

As in the TF calculation, a residual singlet-even force plus the weak-coupling force in Eq. (2) with $k^2/C=1.4$ was used as an "effective" residual force in the Pb^{206} calculation. With the addition of the weak-coupling force, the strength of the singlet-even force should be decreased as indicated in TF. It was found that when the weak-coupling force was included in the residual force, the singlet-even strength of Eq. (1) should be reduced by 30% (versus 25% in the TF calculation).

Kisslinger[17] has pointed out that this is just about the same order of magnitude that one would expect the pairing force strength to decrease if one changed the configuration space in the lead region from including one major shell to including two major shells.

There is also further evidence that this reduction in the singlet-even strength in the lead region is consistent. Gillet, Green, and Sanderson[9] have described the energy levels in Pb^{208} by doing a random-phase calculation. We have used the force of Gillet *et al.* as a residual force in Pb^{206} and find that the agreement with the observed energy levels is very bad. However, when we add the weak-coupling force of Eq. (2) to the force of Gillet *et al.* the eigenvalue results are essentially the same as those discussed below. It is also interesting to note at this point that if the triplet-even force plus singlet-even force of Carter *et al.*[18] is reduced by 45%,[19] the lowest-lying 3-state in Pb^{208} has an eigenvalue and eigenfunction almost identical to that obtained by Gillet *et al.*

A calculation of the Pb^{206} energy levels has been done with a singlet-even force 70% as strong as that given by Eq. (1) and the weak-coupling force of Eq. (2) with $k^2/C=1.4$ (hereafter called the 0.7 SE+WC force). A comparison between these results, the 0.75 SE+WC calculation of TF, and the experimental levels of Pb^{206} is given in Fig. 2.

Figure 2 indicates that except for the lowest 2+ level there is a small over-all improvement with the observed energy levels. The results for the 0.7 SE+WC calculation will be discussed further in Sec. III.

For identical nucleons, only the singlet-even and triplet-odd force contribute to the matrix elements for central force. Usually the odd-state forces are considered small when compared to the even-state forces and often neglected as have been done above. Recent calculations[20] of the level structure of several odd-odd nuclei in the mass-120 region indicate, however, that the residual neutron-proton central force should contain a triplet-odd force about equal to the singlet-even force. Figure 3 shows the comparison between the results of the 0.7 SE+WC calculation above, a 0.7(SE+TO)+WC force, and the experimentally observed levels of Pb^{206}. Some levels are moved closer to the observed levels and about an equal number are

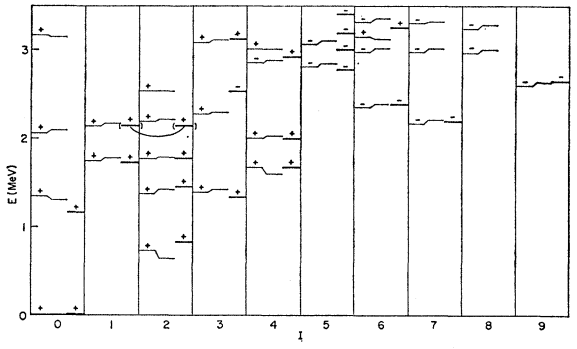

FIG. 2. Comparison of the energies from the True and Ford (Ref. 5) singlet-even plus weak-coupling calculation (first column for each spin), the singlet-even plus weak-coupling calculation of this paper (second column), and the empirical energies from Table II (third column).

[17] L. S. Kisslinger (private communication).

[18] J. C. Carter, W. T. Pinkston, and W. W. True, Phys. Rev. **120**, 504 (1960). [The singlet-even potential in this reference is the same as given by Eq. (1) above, and the triplet-even potential was 1.5 times stronger.]

[19] These calculations on Pb^{208} have been done in collaboration with W. T. Pinkston.

[20] W. W. True, L. S. Kisslinger, and V. K. Thankappan (to be published).

moved farther away. Consequently, no definite conclusion can be drawn as to whether to include a triplet-odd force or not.

Since the ground-state wave function of Pb^{206} for the 0.7 SE+WC calculation agrees most closely with the experimental results of Mukherjee and Cohen,[10] only these results will be discussed in detail in Sec. III.

III. DISCUSSION OF RESULTS

The results for the energy levels of the 0.7 SE+WC calculation are compared with experiment in Fig. 2. The eigenvalues and eigenfunctions for the lower-lying levels for each spin up to $I=9$ are given in Table III.

In Table IV, the square of the amplitudes for each of the configuration components of the ground-state wave function of the TF calculation and the 0.7 SE+WC calculation are compared with the experimental results of Mukherjee and Cohen.[10] It is seen from Table IV that the agreement between the theory and experiment is much better.

One can see from Fig. 2 that this present calculation increases the discrepancy between calculated and observed position of the first 2+ level, as compared to

TABLE III. Eigenvalues and eigenfunctions for Pb^{206} below 3.4 MeV.

Energy MeV	Eigenfunctions											
$I=0+$	$p_{1/2}^2$	$f_{5/2}^2$	$p_{3/2}^2$	$i_{13/2}^2$	$f_{7/2}^2$	$h_{9/2}^2$						
0	0.822	0.401	0.363	−0.109	0.130	0.059						
1.314	0.495	−0.834	−0.090	0.152	−0.145	−0.083						
2.085	0.281	0.292	−0.907	0.064	−0.090	−0.007						
3.136	0.001	−0.190	−0.140	−0.955	0.105	0.145						
$I=1+$	$p_{1/2}p_{3/2}$	$f_{5/2}p_{3/2}$	$f_{5/2}f_{7/2}$	$f_{7/2}h_{9/2}$								
1.782	−0.995	0	−0.098	0								
2.170	0	0.999	0	0.045								
$I=2+$	$p_{1/2}f_{5/2}$	$p_{1/2}p_{3/2}$	$f_{5/2}^2$	$f_{5/2}p_{3/2}$	$p_{3/2}^2$	$f_{5/2}f_{7/2}$	$p_{3/2}f_{7/2}$	$i_{13/2}^2$	$f_{5/2}h_{9/2}$	$f_{7/2}^2$	$f_{7/2}h_{9/2}$	$h_{9/2}^2$
0.637	0.724	−0.523	0.278	−0.176	0.200	−0.058	0.185	−0.055	0.092	0.068	0.012	0.029
1.435	0.619	0.766	−0.052	−0.015	−0.149	0.049	−0.027	0.005	0.034	−0.011	0.001	−0.003
1.791	−0.178	0.201	0.945	0.151	−0.070	−0.052	0.004	−0.037	0.023	0.039	−0.012	0.040
2.205	0.196	−0.166	−0.079	0.954	−0.077	0.063	−0.057	0.031	−0.035	−0.044	−0.039	−0.011
2.542	−0.071	0.241	−0.015	0.135	0.949	0.086	0.084	−0.022	0.006	0.058	−0.004	0.004
$I=3+$	$p_{1/2}f_{5/2}$	$f_{5/2}p_{3/2}$	$p_{1/2}f_{7/2}$	$f_{5/2}f_{7/2}$	$p_{3/2}f_{7/2}$	$f_{5/2}h_{9/2}$	$p_{3/2}h_{9/2}$	$f_{7/2}h_{9/2}$				
1.433	0.998	0.035	−0.013	0.018	0.002	0.032	−0.039	0.001				
2.302	−0.036	0.995	−0.025	0.071	−0.007	0.028	0.030	0.029				
3.105	−0.010	−0.018	−0.990	−0.124	−0.055	0.017	0.009	−0.001				
$I=4-$	$f_{5/2}i_{13/2}$	$f_{7/2}i_{13/2}$	$h_{9/2}i_{13/2}$									
2.882	0.999	0.003	0.013									
$I=4+$	$f_{5/2}^2$	$f_{5/2}p_{3/2}$	$p_{1/2}f_{7/2}$	$f_{5/2}f_{7/2}$	$p_{3/2}f_{7/2}$	$i_{13/2}^2$	$p_{1/2}h_{9/2}$	$f_{5/2}h_{9/2}$	$p_{3/2}h_{9/2}$	$f_{7/2}^2$	$f_{7/2}h_{9/2}$	$h_{9/2}^2$
1.597	0.567	−0.727	−0.293	−0.135	0.148	−0.050	0.097	0.077	0.034	0.060	0.019	0.026
2.034	0.813	0.560	0.125	0.037	−0.069	0.007	−0.017	0.023	−0.038	−0.012	−0.021	0.012
3.008	0.090	−0.360	0.900	0.178	−0.133	0.049	0.008	−0.021	0.011	−0.028	−0.005	−0.009
$I=5-$	$f_{5/2}i_{13/2}$	$p_{3/2}i_{13/2}$	$f_{7/2}i_{13/2}$	$h_{9/2}i_{13/2}$								
2.845	0.808	−0.582	−0.084	0.033								
3.096	−0.588	−0.804	−0.089	−0.006								
$I=6-$	$p_{1/2}i_{13/2}$	$f_{5/2}i_{13/2}$	$p_{3/2}i_{13/2}$	$f_{7/2}i_{13/2}$	$h_{9/2}i_{13/2}$							
2.393	0.987	0.144	0.075	−0.000	0.004							
3.019	0.131	−0.979	0.156	0.009	−0.030							
3.352	0.096	−0.144	−0.982	−0.072	−0.005							
$I=6+$	$f_{5/2}f_{7/2}$	$i_{13/2}^2$	$f_{5/2}h_{9/2}$	$p_{3/2}h_{9/2}$	$f_{7/2}^2$	$f_{7/2}h_{9/2}$	$h_{9/2}^2$					
3.131	0.971	0.110	−0.090	−0.153	−0.102	−0.051	−0.025					
$I=7-$	$p_{1/2}i_{13/2}$	$f_{5/2}i_{13/2}$	$p_{3/2}i_{13/2}$	$f_{7/2}i_{13/2}$	$h_{9/2}i_{13/2}$							
2.208	0.952	0.250	−0.169	−0.055	0.023							
3.015	0.268	−0.959	0.067	0.040	−0.050							
3.320	0.149	0.115	0.977	0.105	−0.005							
$I=8-$	$f_{5/2}i_{13/2}$	$p_{3/2}i_{13/2}$	$f_{7/2}i_{13/2}$	$h_{9/2}i_{13/2}$								
2.998	0.987	0.158	−0.010	0.034								
3.275	0.157	−0.986	−0.060	0.006								
$I=9-$	$f_{5/2}i_{13/2}$	$f_{7/2}i_{13/2}$	$h_{9/2}i_{13/2}$									
2.629	0.995	−0.081	0.067									

TABLE IV. Comparison of the calculated wave functions for the ground-state wave functions of Pb^{206} and experiment.

Configuration	$p_{1/2}^2$	$f_{5/2}^2$	$p_{3/2}^2$	$i_{13/2}^2$	$f_{7/2}^2$	$h_{9/2}^2$
True and Ford calculation[a]	0.731	0.132	0.129	0.009		
This calc. (0.7 SE+WC)	0.677	0.160	0.132	0.012	0.017	0.003
Expt. results[b]	0.54	0.20	0.12	0.12	0.03	

[a] Reference 5. [b] Reference 10.

the TF results. One reason for this is that in this present calculation, the increase in number of 2+ configurations considered is greater than the increase in number of 0+ configurations. As a result of these increases, the first 2+ level is pushed down farther than the ground state is pushed down. This 0+ to 2+ spacing is quite insensitive to variations in any of the force parameters. Consequently, it is concluded that it is unlikely that any additional improvement can be obtained without introducing something new into the picture, like a tensor force and/or explicit consideration of excitation of particles out of the core.

If one assumes that the diagonal matrix elements for $I=0$ of the singlet-even part of the residual force arise from a pure pairing force, then it is possible to determine an effective pairing strength parameter $\bar{G}$ from the expression

$$\langle j^2 I=0 | V | j^2 I=0\rangle = -\bar{G}(2j+1). \quad (5)$$

The $\bar{G}$ for each j^2 configuration in Pb^{206} was determined by using Eq. (5) and the matrix elements from the singlet-even part of the residual force used in the 0.7 SE+WC calculation above. These $\bar{G}$'s and their average are given in Table V. The average $\bar{G}$ of 0.097 MeV is very close to the value of 0.1 MeV used by Kisslinger and Sorensen[4] in the lead region. Note that the definition in Eq. (5) differs by a factor of 2 from the usual definition of the pairing force strength. This is to take into account that the experimental gap parameterizing $\bar{G}$ includes the effects of both long-range and short-range forces, while this is not the case with standard treatments using pairing plus quadrupole interactions.[20]

TABLE V. Comparison of the effective pairing strength of the force used in this paper with the pairing strength used by Kisslinger and Sorensen.[a]

j^2	$\langle j^2 I=0 \| V \| j^2 I=0\rangle$	$\bar{G}$(MeV)
$p_{1/2}^2$	−0.296	0.148
$f_{5/2}^2$	−0.480	0.080
$p_{3/2}^2$	−0.602	0.151
$i_{13/2}^2$	−0.720	0.051
$f_{7/2}^2$	−0.639	0.080
$h_{9/2}^2$	−0.670	0.067
		Average $\bar{G}=0.096$
	Kisslinger and Sorensen $G=0.100$	

[a] Reference 4.

It is interesting to note that the effective $\bar{G}$ for the $p_{1/2}^2$ and $p_{3/2}^2$ configurations is about twice the effective $\bar{G}$ for the other configurations. The use of a constant $\bar{G}$ for all matrix elements would tend to decrease the $p_{1/2}^2$ and $p_{3/2}^2$ amplitudes and increase the $f_{5/2}^2$, $i_{13/2}^2$, $f_{7/2}^2$, and $h_{9/2}^2$ amplitudes in the ground-state wave function of Pb^{206}. These changes would be in a direction to give better agreement between theory and experiment for the ground-state wave function.