Nuclear Symmetry Energy*

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The energy of nuclear matter is determined for $\alpha=(N-Z)/(N+Z)$ in the range of 0 to 0.60. The results are applicable to superheavy nuclei and to problems of astrophysical interest.

I. INTRODUCTION

THE properties of nuclear matter with equal neutron and proton densities have been determined in a variety of ways by several research groups.[1] The symmetry energy for small neutron excess was estimated first by Brueckner and Gammel[2] and later was more accurately determined by Brueckner and Dabrowski.[3] The properties of the neutron gas have been determined by Brueckner, Gammel, and Kubis.[4] The symmetry energy for large neutron excess is of interest in heavy nuclei which may exist well beyond the region of normal nuclei presently known and also in astrophysical problems. In this paper we determine the energy as a function of density for a range of neutron-proton densities. The result gives the equilibrium energy and density and nuclear compressibility. The interaction used is the Gammel-Thaler phenomenological potential[5] which gives excellent values for equilibrium density, energy, and symmetry energy of nuclear matter. The more recently studied potentials due to Hamada and Johnson[6] and the Yale group[7] are not used, since they fail to bind nuclear matter at the observed density and energy. The use of the Gammel-Thaler potential therefore provides a semiphenomenological procedure for extending the prediction of nuclear properties beyond those presently determinable from known nuclei.

No explicit correction is made in the following calculation for the three-body effects analyzed in detail by Bethe[8] since his recent results show that a correct treatment of all effects gives a correction of approximately 1 MeV per particle, which is probably less important than other uncertainties in the method.

II. EQUATIONS OF REACTION MATRIX THEORY FOR NUCLEAR MATTER WITH A NEUTRON EXCESS

The equations to be solved are very similar to those in Brueckner and Gammel[2] (referred to hereafter as BG) with the added complication of two different Fermi momenta: k_n for neutrons and k_p for protons. These equations are obtained by summation of all particle-particle ladder diagrams and by an approximation to the insertion of self-energy diagrams (of first order in the reaction matrix) into particle and hole lines. For the single-particle energies for states below the Fermi level, the on-the-energy-shell reaction matrix has been used. For the excited states we have made the assumption of an average excitation Δ, which is set equal to the difference in energies of particles at the top and the bottom of the Fermi sea. The equations are familiar and will only be sketched here (compare with BG and Ref. 3).

We assume that there is no spin excess, i.e., every momentum state is occupied by two neutrons with spin up and down, and/or by two protons with spin up and down, or otherwise the momentum state is empty. In this section we neglect spins to simplify the presentation.

For the total energy E of the system we have

$$E=E_{\rm kin}+E_{\rm pot}, \tag{1}$$

where

$$E_{\rm pot}=\sum_{m_1<k_n} V_n(m_1)+\sum_{q_1<k_p} V_p(q_1) \tag{2}$$

and

$$E_{\rm kin}=\frac{3}{5}\frac{k_F^2}{2M}\left\{\frac{(1+\alpha)^{5/3}+(1-\alpha)^{5/3}}{2}\right\}. \tag{3}$$

The Fermi momentum of Eq. (3) is defined by the relation

$$\tfrac{2}{3}\pi^{-2}k_F^3=A/\Omega=(\tfrac{4}{3}\pi r_0^3)^{-1}. \tag{4}$$

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[1] See, for example, K. A. Brueckner and K. S. Masterson, Phys. Rev. **128**, 2267 (1962); S. A. Moszkowski and B. L. Scott, Ann. Phys. (N. Y.) **11**, 65 (1960); P. N. Bhargava and D. W. L. Sprung, *ibid.* **42**, 222 (1967).

[2] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).

[3] K. A. Brueckner and Janusz Dabrowski, Phys. Rev. **134**, B722 (1964).

[4] K. A. Brueckner, John L. Gammel, and Joseph T. Kubis, Phys. Rev. **118**, 1095 (1960).

[5] See Ref. 2 for the details of the potential and references to earlier work.

[6] T. Hamada and I. D. Johnson, Nucl. Phys. **34**, 382 (1962).

[7] K. E. Lassila, M. H. Hull, Jr., H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. **126**, 881 (1962).

[8] H. A. Bethe, Phys. Rev. **138**, 805 (1965).

The Fermi momentum of protons and of neutrons is related to k_F and the neutron excess parameter $\alpha=(N-Z)/A$ as follows:

$$k_n=k_F(1+\alpha)^{1/3},\quad k_p=k_F(1-\alpha)^{1/3}. \tag{5}$$

The single-particle potentials for neutrons V_n and protons V_p for states below the respective Fermi levels are

$$V_n(m_1)=\sum_{m_2<k_n}[(\mathbf{m}_1\mathbf{m}_2|K_{nn}|\mathbf{m}_1\mathbf{m}_2)-\text{exchange}]$$
$$=\sum_{q_2<k_p}(\mathbf{m}_1\mathbf{q}_2|K_{np}|\mathbf{m}_1\mathbf{q}_2), \tag{6}$$

$$V_p(q_1)=\sum_{q_2<k_p}[(\mathbf{q}_1\mathbf{q}_2|K_{pp}|\mathbf{q}_1\mathbf{q}_2)-\text{exchange}]$$
$$+\sum_{m_2<k_n}(\mathbf{m}_2\mathbf{q}_1|K_{np}|\mathbf{m}_2\mathbf{q}_1). \tag{7}$$

We have temporarily used the notation of **m** for the unexcited states of the neutrons, **n** for neutron excited states, **q** for proton unexcited states, and **p** for proton excited states.

The single-particle potentials are derived from the diagonal elements of the K matrices for neutron-neutron scattering, proton-proton scattering, and neutron-proton scattering. These satisfy the equations

$$(\mathbf{m}_1\mathbf{m}_2|K_{nn}|\mathbf{m}_1\mathbf{m}_2)=(\mathbf{m}_1\mathbf{m}_2|v|\mathbf{m}_1\mathbf{m}_2)+\sum_{n_1',n_2'}\frac{(\mathbf{m}_1\mathbf{m}_2|v|\mathbf{n}_1'\mathbf{n}_2')Q_{nn}(\mathbf{n}_1'\mathbf{n}_2')(\mathbf{n}_1'\mathbf{n}_2'|K_{nn}|\mathbf{m}_1\mathbf{m}_2)}{e_n(m_1)+e_n(m_2)-e_n(n_1')-e_n(n_2')}, \tag{8}$$

$$(\mathbf{q}_1\mathbf{q}_2|K_{pp}|\mathbf{q}_1\mathbf{q}_2)=(\mathbf{q}_1\mathbf{q}_2|v|\mathbf{q}_1\mathbf{q}_2)+\sum_{p_1'p_2'}\frac{(\mathbf{q}_1\mathbf{q}_2|v|\mathbf{p}_1'\mathbf{p}_2')Q_{pp}(\mathbf{p}_1',\mathbf{p}_2')(\mathbf{p}_1'\mathbf{p}_2'|K_{pp}|\mathbf{q}_1\mathbf{q}_2)}{e_p(q_1)+e_p(q_2)-e_p(p_1')-e_p(p_2')}, \tag{9}$$

$$(\mathbf{m}_1\mathbf{q}_2|K_{np}|\mathbf{m}_1\mathbf{q}_2)=(\mathbf{m}_1\mathbf{q}_2|K_{np}|\mathbf{m}_1\mathbf{q}_2)+\sum_{n_1'p_2'}\frac{(\mathbf{m}_1\mathbf{q}_2|v|\mathbf{n}_1'\mathbf{p}_2')Q_{np}(\mathbf{n}_1'\mathbf{p}_2')(\mathbf{n}_1'\mathbf{p}_2'|K_{np}|\mathbf{m}_1\mathbf{q}_2)}{e_n(m_1)+e_p(q_2)-e_n(n_1')-e_p(p_2')}, \tag{10}$$

on the energy shell.

The exclusion-principle operators Q_{nn}, Q_{pp}, and Q_{np} are defined by

$$Q_{nn}(\mathbf{n}_1\mathbf{n}_2)=1,\quad \text{for}\quad n_1>k_n \text{ and } n_2>k_n$$
$$=0,\quad \text{otherwise}; \tag{11}$$

$$Q_{pp}(\mathbf{p}_1\mathbf{p}_2)=1,\quad \text{for}\quad p_1>k_p \text{ and } p_2>k_p$$
$$=0,\quad \text{otherwise}; \tag{12}$$

$$Q_{np}(\mathbf{n}_1\mathbf{p}_2)=1,\quad \text{for}\quad n_1>k_n \text{ and } p_2>k_p$$
$$=0,\quad \text{otherwise}. \tag{13}$$

The single-particle energies appearing in the denominators are

$$e_n(m_1)=m_1^2/2M+V_n(m_1),$$
$$e_p(q_1)=q_1^2/2M+V_p(q_1). \tag{14}$$

The set of equations connecting the single-particle energies and K-matrix elements off the energy shell has exactly the same structure. The only difference is the replacement of $e_n(m_1)+e_n(m_2)$ by $e_n(m_1)+e_n(m_2)-\Delta_{nn}$ in the denominator of Eq. (8) and similar substitutions in Eqs. (9) and (10).

III. SOLVING THE EQUATIONS

To solve the K-matrix equations (8)–(10) we use the procedure of BG with the additional approximation of Brueckner and Masterson[1] (in the following referred to as BM). From now on we use k to denote all momenta. We introduce the total and relative momenta

$$\mathbf{K}=\mathbf{k}_1+\mathbf{k}_2 \quad\text{and}\quad \mathbf{k}=\tfrac{1}{2}(\mathbf{k}_1-\mathbf{k}_2) \tag{15}$$

and get

$$(\mathbf{k}_1\mathbf{k}_2|K_{np}|\mathbf{k}_1'\mathbf{k}_2')=\delta_{\mathbf{K}\mathbf{K}'}(k|K_{np}|k'), \tag{16}$$

for example.

The approximation of BM is the assumption that the difference of energies in the denominators of Eqs. (8)–(10) is independent of the total momentum. This approximation is good if $e_n(k_1)$ has a quadratic dependence on k_1 or if the relative momentum is large compared with the total momentum. We should examine this approximation for the more complex case of neutron-proton scattering. Assume that $e_n(k_1)=a_nk_1^2$ and $e_p(k_1)=a_pk_1^2$. Then the denominator

$$e_n(k_1)+e_p(k_2)-e_n(k_1')-e_p(k_2')$$

becomes

$$a_n(k^2+\mathbf{k}\cdot\mathbf{K}+\tfrac{1}{4}K^2)+a_p(k^2-\mathbf{k}\cdot\mathbf{K}+\tfrac{1}{4}K^2)$$
$$-a_n(k'^2+\mathbf{k}'\cdot\mathbf{K}+\tfrac{1}{4}K^2)-a_p(k'^2-\mathbf{k}'\cdot\mathbf{K}+\tfrac{1}{4}K^2)$$
$$=a_n(k^2-k'^2)+a_p(k^2-k'^2)$$
$$+\mathbf{k}\cdot\mathbf{K}(a_n-a_p)-\mathbf{k}'\cdot\mathbf{K}(a_n-a_p). \tag{17}$$

For nn scattering and pp scattering, $a_n=a_p$, so that the denominator is independent of K and the approximation is exact. If $N=Z$, $a_n=a_p$ for np scattering also and the approximation is exact (always assuming a single-particle energy quadratic in the momentum). For a neutron excess, the approximation is equivalent to dropping the last two terms in Eq. (17).

The total momentum appears in the exclusion-principle operators. We replace the total momentum by its average value compatible with a given value of relative momentum k for the three types of K matrix. For the cases of nn scattering and pp scattering, the

formula is the same as that used in BM and will be quoted only for *nn* scattering. To use these formulas for *pp* scattering, change only the Fermi momentum. By definition

$$(K^2(k)_{nn})_{\rm av} = \frac{\int_{k_1<k_n} d\mathbf{k}_1 \int_{k_2<k_n} d\mathbf{k}_2\, K^2\delta(k-\tfrac{1}{2}|\mathbf{k}_1-\mathbf{k}_2|)}{\int_{k_1<k_n} d\mathbf{k}_1 \int_{k_2<k_n} d\mathbf{k}_2\, \delta(k-\tfrac{1}{2}|\mathbf{k}_1-\mathbf{k}_2|)}. \quad (18)$$

We get Eq. (2.2) of BM:

$$(\tfrac{1}{4}K^2(k)_{nn})_{\rm av} = \tfrac{3}{5}k_n^2\left(1-\frac{k}{k_n}\right)\frac{(1+\frac{1}{2}k/k_n+\frac{1}{6}k^2/p_n^2)}{(1+\frac{1}{2}k/k_n)}, \quad k<k_n. \quad (19)$$

For $k\leq k_n$, we have set $K_{\rm av}=0$. For *np* scattering, the definition is

$$(K_{np}^2(k))_{\rm av} = \frac{\int_{k_1<k_n} d\mathbf{k}_1 \int_{k_2<k_p} d\mathbf{k}_2\, K^2\delta(k-\tfrac{1}{2}|\mathbf{k}_1-\mathbf{k}_2|)}{\int_{k_1<k_n} d\mathbf{k}_1 \int_{k_2<k_p} d\mathbf{k}_2\, \delta(k-\tfrac{1}{2}|\mathbf{k}_1-\mathbf{k}_2|)}. \quad (20)$$

The value of this integral is a little long and requires some new notation. Let $s=(k_n-k)$, $t=(k_p+k)$, $x=(k_n+k)$, and $y=(k_p-k)$. If $k_n<3k_p$ (this includes the range of neutron excess discussed in this paper), we get

$$(K_{np}^2(k))_{\rm av}=(12/5)k_p^2+4k^2, \qquad 0<k<\tfrac{1}{2}(k_n-k_p)$$

$$=\left[\frac{32}{5}(s^5+y^5)+\frac{(ty+sx)^3}{3k}-\frac{4(ty^5+xs^5)}{k}+\frac{8}{3k}(y^6+s^6)\right]\Big/$$

$$\left[\frac{8}{3}(s^3+y^3)+\frac{(ty+sx)^2}{2k}-\frac{2(ty^3+xs^3)}{k}+\frac{1}{k}(y^4+s^4)\right], \quad \tfrac{1}{2}(k_n-k_p)<k<\tfrac{1}{2}(k_n+k_p)$$

$$=0. \qquad \tfrac{1}{2}(k_n+k_p)<k \quad (21)$$

If $\alpha>0.8$, then $k_n>3k_p$ and we get

$$(K_{np}^2(k))_{\rm av}=(12/5)k_p^2+4k^2, \qquad 0<k<k_p$$

$$=\left(\frac{32s^5}{5}+\frac{4yt^5}{k}+\frac{4sxt^4}{k}-\frac{4s^5x}{k}+\frac{8}{3}\frac{s^6}{k}-\frac{16}{3}\frac{t^6}{k}\right)\Big/$$

$$\left(\frac{8s^3}{3}+\frac{2yt^3}{k}+\frac{2sxt^2}{k}-\frac{2s^3x}{k}+\frac{s^4}{k}-\frac{2t^4}{k}\right), \qquad k_p<k<\tfrac{1}{2}(k_n-k_p)$$

$$=\left[\frac{32}{5}(s^5+y^5)+\frac{(ty+sx)^3}{3k}-\frac{4(ty^5+xs^5)}{k}+\frac{8}{3k}(y^6+s^6)\right]\Big/$$

$$\left[\frac{8}{3}(s^3+y^3)+\frac{(ty+sx)^2}{2k}-\frac{2}{k}(ty^3+xs^3)+\frac{1}{k}(y^4+s^4)\right], \quad \tfrac{1}{2}(k_n-k_p)<k<\tfrac{1}{2}(k_n+k_p). \quad (22)$$

We insert the average total momentum in the angle-average $\bar{Q}(k,k)$ of the exclusion principle operator $Q(k,k)$. For *nn* scattering the result is

$$\bar{Q}_{nn}(K,k)=0, \qquad \text{for} \quad k<(k_n^2-\tfrac{1}{4}K^2)^{1/2}$$

$$=\frac{\frac{1}{4}(K^2+k^2-k_n^2)}{kK}, \qquad \text{for} \quad (k_n^2-\tfrac{1}{4}K^2)^{1/2}<k<\tfrac{1}{2}K+k_n$$

$$=1, \qquad \text{for} \quad k>\tfrac{1}{2}K+k_n \quad (23)$$

and for np scattering,

$$\begin{aligned}\bar{Q}_{np}(K,k)&=0, && \text{for } k<(\tfrac{1}{2}(k_n^2+k_p^2)-\tfrac{1}{4}K^2)^{1/2}\\ &=\frac{\tfrac{1}{4}K^2+k^2-k_n^2-k_p^2}{kK}, && \text{for } (\tfrac{1}{2}(k_n^2+k_p^2)-\tfrac{1}{4}K^2)^{1/2}<k<\tfrac{1}{2}K+k_p\\ &=\frac{1}{2}\left(1+\frac{\tfrac{1}{4}K^2+k^2-k_n^2}{kK}\right), && \text{for } \tfrac{1}{2}K+k_p<k<\tfrac{1}{2}K+k_n\\ &=1, && \text{for } \tfrac{1}{2}K+k_n<k,\end{aligned} \tag{24}$$

for $k_n<3k_p$.

Having made the approximations of an average total momentum and the angle-average of the exclusion principle operators, we may now transform the K-matrix equations into coordinate space and make a partial-wave expansion. The resulting equations are obvious generalizations of those given in BG and BM and will not be given here.

IV. COMPUTATIONAL PROCEDURE

In our calculation the nucleon-nucleon potential v of Gammel and Thaler[5] has been used. This potential has a hard core of radius $r=0.4$ F with central, tensor, and spin-orbit parts of Yukawa shape for $r>r_c$. The strength and range parameters of this potential are listed in Table III of BG.

To compute the Green's functions, we used the method of Coon and Dabrowski[9] (in the following referred to as CD):

$$\begin{aligned}G_{l,np}{}^k(r,r')=&\frac{1}{2\pi^2}\int_0^{k_{\rm int}}dk''\,k''^2 j_l(k''r)j_l(k''r')\\ &\times\{\bar{Q}_{np}(K_{np}(k),k'')/\\ &\quad[E_n(k)+E_p(k)-E_n(k'')-E_p(k'')]\\ &-1/[E_n(k)+E_p(k)-k''^2/M]\}\\ &+G_{l,np}{}^k(r,r')_{\rm nxd},\end{aligned} \tag{25}$$

where nxd stands for "no exclusion principle and no dispersion effects." For the last part of G_l, we use the analytical expression

$$\begin{aligned}G_{l,np}{}^k(r,r')_{\rm nxd}=&\frac{1}{2\pi^2}\int_0^{\infty}dk''\\ &\times k''^2\frac{j_l(k''r)j_l(k''r')}{E_n(k)+E_p(k)-2k''^2/2M}\\ &=(M/4\pi)ah_l{}^{(1)}(iar_>)j_l(iar_<),\end{aligned} \tag{26}$$

where

$$a=\{-\tfrac{1}{2}M(E_p(k)+E_n(k))\}^{1/2}. \tag{27}$$

[9] Sidney A. Coon and Janusz Dabrowski, Phys. Rev. **140**, B287 (1965).

The first approximation for the energies appearing in the denominators was simply the kinetic energies. As was done in BG and BM but not in CD we used kinetic energies for $E(k'')$ when $k''>3.0$ in units of the appropriate Fermi momentum. On subsequent iterations we used $E(k)$ and $E(k'')$ from previous iterations.

We used the same meshes for the calculation as those given in Eqs. (36)–(40) of CD. Because of our treatment of the last part of the Green's functions, it was only necessary to integrate numerically up to three units of the appropriate Fermi momentum in the calculation of the Green's function. For neutron-proton Green's function we integrated up to $3k_n$.

We calculated single-particle energies for

$$\binom{k_n}{k_p}=0.0(0.25)1. \tag{28}$$

All integrations have been performed by means of Simpson's rule. The K matrix and single-particle energies have been interpolated linearly when necessary.

A major iteration consists of a calculation of Green's functions, plane-wave basis functions, wave functions, and elements of the K matrix for neutron-neutron, proton-proton, and neutron-proton scattering and the determination of single-particle energies for neutrons and for protons from the diagonal elements of the K matrix. These single-particle energies are then used in the next major iteration and the iterations continue until self-consistency of the single-particle energies is achieved. Then the total energy of the system is calculated.

In the first iteration, single-particle energies consisting of kinetic energies only were used. The calculation was then iterated until self-consistency occurred for the value of $k_F=1.49$ F^{-1} and $\alpha=0$ to compare with the results of BG and BM. A binding energy of -14.97 MeV was obtained, which compares favorably with BG's result of -15.2 MeV and BM's result of -16.9 MeV. At this density, the neutron excess parameter was then varied up to $\alpha=0.8$, in steps of $\alpha=0.2$. We found that if we used the single-particle energies calculated for the previous value of α as an input spectrum we achieved self-consistency of about 1%, with only two major iterations. This procedure was followed for

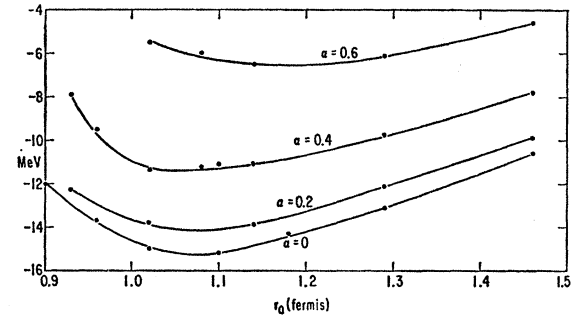

FIG. 1. Energy versus spacing parameter for several values of $\alpha=(N-Z)/(N+Z)$. The solid dots represent the computed points.

all later calculations. Having established the nature of the dependence of binding energy on neutron excess at this fixed density we then varied the density and kept α fixed for each value of α previously calculated at $k_F=1.49$ F^{-1}. Again we used as an input spectrum the single-particle energies obtained for the values of k_F and α nearest the new values of k_F and α for which we desired a binding energy.

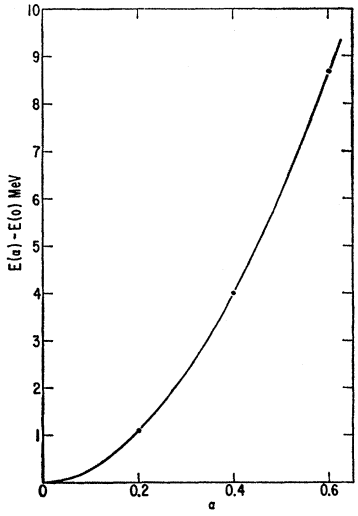

FIG. 2. Energy minimum as a function of α. The solid dots represent the computed points.

All of the numerical calculations were performed on the CDC 3600 computer of the Computer Center of the University of California at San Diego.

V. RESULTS

The computed value of self-consistent energy versus particle spacing parameter is given in Fig. 1 for several values of α. Figure 2 gives the variation of energy with α at the saturation density.

The variation of energy with α can be accurately represented for the range of α from zero to 0.4 by

$$E(\alpha)-E(0)=\tfrac{1}{2}\epsilon_{\rm sym}\alpha^2(1-\lambda\alpha^2)\,, \tag{29}$$

with

$$\epsilon_{\rm sym}=56.0\text{ MeV},\qquad \lambda=0.670. \tag{30}$$

The curve cannot be extended to values of α approaching unity, since the neutron gas is unbound[4] and no energy minimum exists. The variation of equilibrium density with α is only approximately determined by these results, which show that the equilibrium spacing is a slowly increasing function of α, corresponding to a drop in density of roughly 30% for $\alpha=0.6$. A similar result has been found for small values of α by Dworzecka.[10] The above value for the symmetry energy may be compared with the theoretical analysis by Brueckner and Dabrowski, who found $\epsilon_{\rm sym}=64$ MeV. For a pure volume symmetry energy, Green[11] found empirically $\epsilon_{\rm sym}=47$ MeV. By assuming that there is also a surface part of the symmetry energy, Green[11] found $\epsilon_{\rm sym}=61$ MeV and Cameron[12] 63 MeV. The value given by Eq. (30) falls in the range of these results; the difference probably can be resolved only by a more careful *a priori* analysis of the symmetry energy using, for example, the Hartree-Fock method appropriately modified[13] for the strong correlations present in the nucleus.

[10] M. Dworzecka, Acta Phys. Polonica **29**, 783 (1966).
[11] A. E. S. Green, Rev. Mod. Phys. **30**, 569 (1958); Phys. Rev. **95**, 1006 (1954).
[12] A. G. W. Cameron, Can. J. Phys. **35**, 1021 (1957).
[13] K. A. Brueckner, J. L. Gammel, and H. Weitzner, Phys. Rev. **110**, 431 (1958).