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Muon Capture in Helium-3*

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The partial capture rate $\mu^- + He^3 \rightarrow H^3 + \nu$ is calculated, including terms through $O(1/M^2)$ and a phenomenological treatment of the exchange effects. The wave function of the three-nucleon system is largely determined by recent analyses of the electromagnetic form factors. The exchange effects are found to be very important: If they are neglected, two sets of values of F_P consistent with experiment are 0 ± 3 and 35 ± 3. Including the exchange effects, $6 \leq F_P/F_A \leq 33$.

I. INTRODUCTION

In the past several years a large amount of effort, both experimental and theoretical, has been spent on the investigation of the three-nucleon doublet, H^3 and He^3. There have been recent experiments on the β decay[1] of H^3, muon capture[2–4] in He^3, and on the photodisintegration[5–8] of both nuclei. The electromagnetic form factors[9] have been measured over the range $1 \leq q^2 \leq 8$-F^{-2}, and some data on the inelastic process[10] $e + He^3 \rightarrow d + p + e'$ have been obtained. Thus, a large amount of experimental information is available to determine the nuclear wave functions.

The process of determining the wave functions was begun by an attempt to calculate the binding energy by variational methods.[11] There has been recent work on this problem using both two-body[12] and separable[13–15] potential models for the nucleon-nucleon interaction. There have also been a number of calculations of the photodisintegration[16–20] process using various nuclear wave functions and calculations of the inelastic scattering process.[21–23] The major effort, however, has been directed at the problem of explaining the electromagnetic form factors.[24–26] This problem provides the cleanest test of the nuclear wave function since the interaction is well known and there are no strong final-state effects involved.

There have also been a large number of calculations of the muon capture process in He^3. Several[27–29] of these have been based on the "elementary particle" model, in which the H^3-He^3 system is treated in analogy with the neutron-proton system. This model predicts a reasonable capture rate, but detailed calculations are not possible due to the fact that there is no information available about the q^2 dependence of the axial-vector form factor for He^3 (the value at $q^2=0$ is fixed by the triton β decay rate). Current-commutator techniques[30] have also been applied to the muon capture problem, but these involve rather rough approximations. Other calculations[31–33] have been based on a nuclear-physics approach, using the impulse approximation to define an effective Hamiltonian, which is the method used in the present calculation.

The main theoretical interest in muon capture is centered on the question of the magnitude of the in-

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duced pseudoscalar coupling constant, F_P. The original prediction by Goldberger and Treiman of $F_P=(7\text{–}8)F_A$, which can also be obtained from the PCAC hypothesis, has proved to be very difficult to verify. The value is predicted for an isolated nucleon, corresponding to muon capture in H^1. Unfortunately, however, the capture process in hydrogen is complicated by molecular effects, and a precise determination of F_P is not yet feasible. A more precise value can be derived from the process of radiative muon capture, which is a relatively rare phenomenon (the rates are about 10^4 smaller than those for ordinary capture), but one which is particularly sensitive to the value of F_P. The values derived from experiments in Ca^{40} are $F_P=13.3\pm2.7$,[34] using harmonic-oscillator wave functions, and $F_P=16.5\pm3.1$[35], using the Foldy-Walecka[36] technique. The latter method takes the dominant dipole matrix element from experimental evidence and uses specific models only to calculate corrections. The errors quoted are purely experimental in origin. It should be noted, however, that a direct comparison between these values and the Goldberger-Treiman result is not clearly indicated. It is quite possible that the form of the pion propagator is materially altered in nuclear matter, leading to values of F_P other than $(7\text{–}8)F_A$. The aim of this calculation is to use the available information about the wave function of He^3 and to assume the validity of the universal Fermi interaction (including the conserved-vector-current hypothesis and the absence of class II currents) in order to determine whether the value of F_P in the three-nucleon system can be consistent with the radiative-capture results. The conclusion is that if one adopts a reasonable wave function and includes mesonic exchange contributions, the results are consistent with a wide range of values for F_P, including both the radiative-capture and Goldberger-Treiman values. If the exchange contributions are neglected, however, the calculation is inconsistent with both sets of values.

In the following section, the expression for the capture rate is presented and discussed. In Sec. III, the particular form of the three-body wave function used is presented, as well as a discussion of the experimental and theoretical data that are used to determine it. The next section includes a discussion of exchange effects in the three-nucleon system, and an approximate method for including their contribution. The final section presents the results of the calculation.

II. CAPTURE RATE

The expression used for the capture rate is that of Friar,[37] who used a Foldy-Wouthuysen transformation to reduce the muon-capture Hamiltonian through terms of order (1/nucleon mass)^2. The most important terms that are neglected in his treatment are of order $\frac{1}{2}Z\alpha$, where $\alpha=1/137$. Keeping only those terms which represent at least a 1% effect,

$$\Lambda=\frac{|\phi_\mu|^2_{\text{av}}}{2\pi}\frac{k_\nu^2G^2}{(1+k_\nu/m_{\text{He}})}\int\frac{d\hat{k}_\nu}{4\pi}\frac{1}{2}\sum_{\text{spins}}\{G_V^2|\textstyle\int 1|^2+G_A^2|\int\boldsymbol{\sigma}|^2+(G_P^2-2G_PG_A)|\hat{k}_\nu\cdot\int\boldsymbol{\sigma}|^2$$
$$-(G_VF_V/M)[2\,\text{Re}(\textstyle\int 1)(\hat{k}_\nu\cdot\int\mathbf{p})^*]-[F_A(G_A-G_P)/M][2\,\text{Re}(\hat{k}_\nu\cdot\int\boldsymbol{\sigma})(\int\boldsymbol{\sigma}\cdot\mathbf{p})^*]$$
$$+(G_AF_V/M)\hat{k}_\nu\cdot[2\,\text{Im}(\textstyle\int\boldsymbol{\sigma})\times(\int\mathbf{p})^*]. \quad (1)$$

With a minor addition to G_V, this is the expression derived by Fujii and Primakoff[32]:

$$G_V=F_V(1+k_\nu/2M)-k_\nu m_\mu F_M/2M\,, \quad (2)$$

$$G_A=F_A-k_\nu(F_V+2MF_M)/2M\,, \quad (3)$$

$$G_P=(k_\nu/2M)[F_P-F_A-(F_V+2MF_M)]. \quad (4)$$

G is the weak-interaction coupling constant $G=1.14\times10^{-11}/M^2$,[38] where M is the proton mass. For any nuclear operator $\hat{x}$,

$$\int\hat{x}\equiv\langle H^3|\sum_{i=1}^{3}\exp(-i\mathbf{k}_\nu\cdot\mathbf{x}_i)\tau^-(i)\hat{x}(i)|He^3\rangle.$$

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The momentum of the ejected neutrino k_ν is given by[32]

$$k_\nu=m_\mu\left[1-\frac{E_{\max}}{m_\mu}-\frac{1}{2}\left(\frac{Z}{137}\right)^2\right]\left[1-\frac{m_\mu}{2(m_\mu+m_{\text{He}})}\right],$$

where $Z=2$ for helium, and $E_{\max}=0.529$ MeV. The numerical value is $k_\nu=103.21$ MeV. Using the conserved-vector-current (CVC) hypothesis and the values of the nucleon electromagnetic form factors,[39] $F_V=0.973$ and $F_M=3.59/2M$. We assume that the axial-vector form factor has the same value for muon capture as for β decay, $F_A=-1.18\pm0.02$. F_P is to be determined.

The square of the average value of the muon wave function over the nuclear volume is given by

$$|\phi_\mu|^2_{\text{av}}=(1/\pi)[2m_\mu'/137]^3R\,,$$

$$m_\mu'=\frac{m_\mu}{(1+m_\mu/m_{\text{He}})}, \quad (5)$$

[39] L. N. Hand, D. G. Miller, and R. Wilson, Rev. Mod. Phys. **35**, 335 (1963).

where R is a reduction factor which accounts for the averaging process and for the finite size of the nucleus. Assuming a hollow-exponential-well charge distribution for He^3, this quantity was calculated and found to be $R=0.9704$. The calculation is outlined in Appendix A.

By making use of the fact that, except for a very small component of isotopic spin $\frac{3}{2}$ in the He^3 wave function, the initial- and final-state nuclear wave functions are identical, one can greatly simplify the terms containing $\int \mathbf{p}$ by an integration by parts. The result is

$$\int \mathbf{p}=\tfrac{1}{3}\mathbf{k}_\nu\int 1\,,$$

so that

$$[2\,\mathrm{Re}(\int 1)(\hat{k}_\nu\cdot\int\mathbf{p})^*]=\tfrac{2}{3}k_\nu|\int 1|^2,$$
$$[2\,\mathrm{Re}(\hat{k}_\nu\cdot\int\boldsymbol{\sigma})(\int\boldsymbol{\sigma}\cdot\mathbf{p})]=\tfrac{2}{3}k_\nu|\hat{k}_\nu\cdot\int\boldsymbol{\sigma}|^2,$$
$$[\hat{k}_\nu\cdot(\int\boldsymbol{\sigma})\times(\int\mathbf{p})^*]=0\,.$$

When the integration over neutrino angles is performed,

$$\int\frac{d\hat{k}_\nu}{4\pi}|\hat{k}_\nu\cdot\int\boldsymbol{\sigma}|^2=\tfrac{1}{3}|\int\boldsymbol{\sigma}|^2,$$

so that the final expression for the capture rate is

$$\Lambda=\frac{R}{2\pi^2}\left[\frac{2m_\mu'}{137}\right]^3\frac{k_\nu^2G^2}{(1+k_\nu/m_{\mathrm{He}})}\times\tfrac{1}{2}\sum_{\mathrm{spins}}\{A|\int 1|^2+B|\int\boldsymbol{\sigma}|^2\},\quad(6)$$

where

$$A=G_V^2-2G_VF_Vk_\nu/3M\,,\quad(7)$$

$$B=G_A^2+\tfrac{1}{3}G_P(G_P-G_A)+2F_Ak_\nu(G_P-G_A)/9M\,.\quad(8)$$

There is one additional important term of order $(1/M)^2$ in the Friar Hamiltonian which involves the quantity $\mathbf{k}_\nu\cdot\int\boldsymbol{\sigma}\times\mathbf{p}$. This vanishes in the present case, but could be important in other processes.

III. WAVE FUNCTIONS

If the ground state of He^3 is assumed to have $J^P=\frac{1}{2}^+$ and isospin $T=\frac{1}{2}$, it has been shown that there are 10 possible components in the wave function. We will include only five of these in this calculation: the fully symmetric ^2S state (No. 1), the ^2S state of mixed symmetry (No. 2), also denoted by S', and the three mixed symmetry ^4D states (Nos. 6, 7, and 8). The state numbering follows Sachs,[40] and Gibson and Schiff.[25] The completely antisymmetric states and the remaining P states are not thought to be present to any appreciable extent. If we allow the He^3 wave function to contain terms with $T=\frac{3}{2}$ as well, there are a number of additional possible states. The state considered by Gibson[26] is a mixed symmetry S state (No. 9) and is the most plausible choice.[41]

The form of the states is based on that of Sachs, and Gibson and Schiff. They are constructed from space, spin, and isotopic-spin functions, all of which have definite and similar transformation properties under particle exchange. All functions subscripted s are completely symmetric, and if ϕ_1 and ϕ_2 represent the two components of a mixed symmetry state,

$$P_{12}\phi_1=\tfrac{1}{2}(\sqrt{3}\phi_2-\phi_1),\quad P_{13}\phi_1=-\tfrac{1}{2}(\sqrt{3}\phi_2+\phi_1),\quad P_{23}\phi_1=\phi_1\,,$$
$$P_{12}\phi_2=\tfrac{1}{2}(\phi_2+\sqrt{3}\phi_1),\quad P_{13}\phi_2=\tfrac{1}{2}(\phi_2-\sqrt{3}\phi_1),\quad P_{23}\phi_2=-\phi_2\,.$$

From two sets of functions satisfying the above equations, for example, $\lambda_1,\lambda_2;\mu_1,\mu_2$, we can construct completely symmetric and antisymmetric functions and two additional mixed symmetry states:

$$\phi_s=(\lambda_1\mu_1+\lambda_2\mu_2)/\sqrt{2}\,,$$
$$\phi_a=(\lambda_2\mu_1-\lambda_1\mu_2)/\sqrt{2}\,,$$
$$\phi_1=(\lambda_2\mu_2-\lambda_1\mu_1)/\sqrt{2}\,,$$
$$\phi_2=(\lambda_2\mu_1+\lambda_1\mu_2)/\sqrt{2}\,.$$

The specific sets of these functions used are

$$\chi_1=[(++-)+(+-+)-2(-++)]/\sqrt{6},$$
$$\chi_2=[(++-)-(+-+)]/\sqrt{2}\,,$$

the two doublet spin functions for three nucleons. A $+$ (or $-$) in, say, the second position of a parenthesis, means that nucleon 2 has spin up (or down). The isospin functions η_1, η_2 for He^3, $T=\frac{1}{2}$, are identical except that $+$ (or $-$) means proton (or neutron). We also need, for the $T=\frac{3}{2}$ state, the completely symmetric isospin function

$$\eta_s=[(++-)+(+-+)+(-++)]/\sqrt{3}\,.$$

The space wave functions are constructed from $\mathbf{R}_1=(\frac{4}{3})^{1/2}\boldsymbol{\rho}$ and $\mathbf{R}_2=-\mathbf{r}$, where $\mathbf{r}$ is the vector from nucleon 2 to nucleon 3, and $\boldsymbol{\rho}$ is the vector from the midpoint of $\mathbf{r}$ to nucleon 1:

$$\mathbf{r}=\mathbf{r}_3-\mathbf{r}_2\,,$$
$$\boldsymbol{\rho}=\mathbf{r}_1-(\mathbf{r}_2+\mathbf{r}_3)/2\,.$$

Three scalar functions are based on these:

$$S_1=\mathbf{R}_2^2-\mathbf{R}_1^2,$$
$$S_2=2\mathbf{R}_1\cdot\mathbf{R}_2,$$
$$S_s=\mathbf{R}_1^2+\mathbf{R}_2^2=\tfrac{2}{3}(\mathbf{r}_{12}^2+\mathbf{r}_{13}^2+\mathbf{r}_{23}^2)\,.$$

$\mathbf{R}_1$, $\mathbf{R}_2$, S_1, S_2, and S_s transform under particle interchange as indicated by their subscripts. There are also three space-spin functions, constructed in Ref. 25, that

[40] R. G. Sachs, *Nuclear Theory* (Addison-Wesley Publishing Co., Inc., Cambridge, Mass., 1953), pp. 180–187.

[41] The other possibilities are either antisymmetric states, which are expected to be mixed in more weakly as their associated energy denominator must be larger, or P states.

are used to build up the ^4D states. These are

$$D_1=[(\boldsymbol{\sigma}_1\cdot\mathbf{R}_2)(\boldsymbol{\sigma}_{23}\cdot\mathbf{R}_2)-(\boldsymbol{\sigma}_1\cdot\mathbf{R}_1)(\boldsymbol{\sigma}_{23}\cdot\mathbf{R}_1)-\tfrac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_{23})S_1]\chi_2\,,$$

$$D_2=[(\boldsymbol{\sigma}_1\cdot\mathbf{R}_2)(\boldsymbol{\sigma}_{23}\cdot\mathbf{R}_1)+(\boldsymbol{\sigma}_1\cdot\mathbf{R}_1)(\boldsymbol{\sigma}_{23}\cdot\mathbf{R}_2)-\tfrac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_{23})S_2]\chi_2\,,$$

$$D_s=[(\boldsymbol{\sigma}_1\cdot\mathbf{R}_2)(\boldsymbol{\sigma}_{23}\cdot\mathbf{R}_2)+(\boldsymbol{\sigma}_1\cdot\mathbf{R}_1)(\boldsymbol{\sigma}_{23}\cdot\mathbf{R}_1)-\tfrac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_{23})S_s]\chi_2\,,$$

where $\boldsymbol{\sigma}_1$ is the Pauli matrix (with unit elements) for nucleon 1, and $\boldsymbol{\sigma}_{23}=\boldsymbol{\sigma}_2-\boldsymbol{\sigma}_3$. Using these functions, the component wave functions for the three-body nucleus are as follows:

$$\psi_1=(\chi_2\eta_1-\chi_1\eta_2)f_1(S_s)\,,$$
$$\psi_2=[(\chi_2S_2-\chi_1S_1)\eta_2-(\chi_1S_2+\chi_2S_1)\eta_1]f_2(S_s)\,,$$
$$\psi_6=[(5D_sS_2-2D_2S_s)\eta_1-(5D_sS_1-2D_1S_s)\eta_2]f_6(S_s)\,,$$
$$\psi_7=S_s(D_2\eta_1-D_1\eta_2)f_7(S_s)\,,$$
$$\psi_8=[(D_2S_1+D_1S_2)\eta_1-(D_2S_2-D_1S_1)\eta_2]f_8(S_s)\,,$$
$$\psi_9=(\chi_1S_2-\chi_2S_1)\eta_sf_9(S_s)\,.$$

Here the f_i represent, in principle, any function of $S_s=\mathbf{R}_1^2+\mathbf{R}_2^2$. In practice, their forms were chosen to be the same, and restricted to one of the following:

Gaussian:

$$f_i(S_s)=N_i\exp[-\tfrac{1}{2}\alpha_i^2(2\boldsymbol{\varrho}^2+\tfrac{3}{2}\mathbf{r}^2)];$$

Irving:

$$f_i(S_s)=N_i\exp[-\tfrac{1}{2}\alpha_i(2\boldsymbol{\varrho}^2+\tfrac{3}{2}\mathbf{r}^2)^{1/2}];$$

Irving-Gunn:

$$f_i(S_s)=N_i(2\boldsymbol{\varrho}^2+\tfrac{3}{2}\mathbf{r}^2)^{-1/2}\exp[-\tfrac{1}{2}\alpha_i(2\boldsymbol{\varrho}^2+\tfrac{3}{2}\mathbf{r}^2)^{1/2}].$$

The complete wave functions are

$$|\mathrm{H}^3\rangle=\textstyle\sum_i A_i(\mathrm{H}^3)\psi_i(\mathrm{H}^3)\,,$$
$$|\mathrm{He}^3\rangle=\textstyle\sum_i A_i(\mathrm{He}^3)\psi_i(\mathrm{He}^3)\,.$$

The sum runs over $i=1$, 2, 6, 7, 8 for H^3 and $i=1$, 2, 6, 7, 8, 9 for He^3. The amplitudes A_i, which are real if the forces are time-reversal invariant,[40] are taken to be the same for H^3 and He^3 with the exception of the small $T=\frac{3}{2}$ admixture in He^3.

Information concerning the amounts of S' and D states included has been taken from the results of variational calculations. The two-body potential calculations of Blatt[11,12] predict large admixtures of the D states. Calculations with the best available potentials and a huge amount of parameter searching result in predictions of 7–9% D state and 0.5–1.5% S' state. The binding energies from these calculations are, however, appreciably lower than the experimental value, and the authors of Ref. 12 point out that their ^4D and S' state probabilities should not be taken literally. Somewhat lower values of both percentages are predicted by Bhakar and Mitra,[13] who use a separable-potential approach. The best calculation of this type predicts a 5% admixture of ^4D states, and about 1% S'. This calculation overbinds the triton by 0.5 MeV, but it is hoped that a combination of hard-core and relativistic effects can correct this.[15] Both calculalations predict negligible amounts of P state (less than 0.1%). It is clear, however, that the strong nuclear tensor force mixes in appreciable amounts of D state, probably on the order of 5 to 6%.

There is also information available which limits the amount of S' and $T=\frac{3}{2}$ admixtures present. The work of Meister and Rhada[42] on slow neutron capture in deuterium limits the S' state probability to 2%, and is most consistent with values of 1% or less. Werntz and Valk[43] give an upper limit of a few tenths of a percent for the $T=\frac{3}{2}$ state probability.

A great deal of information on the photodisintegration of the three-nucleon system is now available. There have been three recent experiments that cover a wide energy range,[5,6,8] and supporting data[7] at low energies are also available. The analysis of these data, however, is complicated by the presence of final-state effects, and by the fact that the tail of the nuclear wave function largely determines the magnitude of the cross section. Many of the early calculations[16] use Gaussian wave functions which give a very bad fit to the data due to their unreasonable asymptotic forms. In general, the most acceptable fits to the two-body disintegration have used the Irving-Gunn[16] form of the wave function, but these neglect the final-state interactions. The final-state interactions have been examined[17] and amount to about 25% at the peak cross section for the two-body disintegration, and to about 200–300% for the three-body case.[18] The inclusion of these effects would probably destroy the Irving-Gunn fit. The only application of the Irving function to the two-body data used an unreasonable value for the radius parameter.[19]

The analysis of the experiment on inelastic electron scattering is also clouded. Two models, a nuclear-physics model[21] and a pole model,[22] have been applied by Griffy and Oakes—both fit the peak data but disagree by 70% on the integrated cross section. In the nuclear-physics calculation, the Gaussian wave function gives too small a peak, the Irving function is better, and the Irving-Gunn function fits the data. Again, however, final-state effects are neglected, and it appears that two-particle correlations in He^3 are important as well. There is the additional factor that the experiments are difficult to perform (they involve counting the scattered electron and the ejected proton in coincidence) and difficult to analyze.[23]

The most reliable data concerning the three-body wave function are obtained from the analysis of the elastic electron scattering form factors. The initial

[42] T. K. Radha and N. T. Meister, Phys. Rev. **136**, B388 (1964).

[43] Carl Werntz and H. S. Valk, Phys. Rev. Letters **14**, 910 (1965).

analysis by Schiff[24] required only an admixture of the S' state (4%) to fit the form factors. This, however, was found to be inconsistent with slow neutron capture in deuterium,[42] and an analysis motivated by the variational calculations was undertaken.[25,26] The conclusions were that the form factors can be calculated assuming a mixture of 2% S', 6% ^4D, and 0.25% $T=\frac{3}{2}$ states, provided that one includes both an isoscalar and isovector exchange contribution to the magnetic moment form factors. The fit was made using the Irving form for the f_i's, and the value of α_s (the range parameter for the dominant S state) was found to be some 5% larger than the value determined from the Coulomb energy of He^3. This reflects the effect of the finite size of the proton on the Coulomb energy.[26] The analysis also showed that a reasonable value for α_D is $\alpha_D=\sqrt{2}\alpha_s$. The state described above, called the Gibson state in what follows, seems to be the most reasonable form available for the wave function.

The analogy between the electron scattering analysis and the muon capture calculation is actually quite close—in both calculations the matrix elements $\int 1$ and $\int \boldsymbol{\sigma}$ appear, although with different isospin directions. If the isovector charge and magnetic form factors of the three-nucleon system were known (at $q^2=m_\mu^2$) to sufficient accuracy, the matrix elements for muon capture could be determined directly. By choosing a wave function which fits the electron scattering data, we have essentially chosen an extrapolation procedure which makes full use of the available experimental data. This procedure is only weakly dependent on the specific form chosen for the wave function, as long as the form is used for both analyses.

IV. EXCHANGE EFFECTS

The most striking example of nuclear exchange effects[44] occurs in the analysis of the β decay of H^3. If one assumes that H^3 and He^3 are identical except for the $T=\frac{3}{2}$ state in He^3, the axial-vector matrix element can be calculated neglecting exchange effects:

$$|M_A|^2=3[p_1-\tfrac{1}{3}p_2+\tfrac{1}{3}p_D+\tfrac{2}{3}p_9+\tfrac{2}{3}A_2A_9]^2. \quad (9)$$

Here the $p_i(A_i)$ represent the probabilities (amplitudes) of the states considered in this paper. The quantity p_D is the total mixed symmetry ^4D probability, $p_D=p_6+p_7+p_8$. For any reasonable mixture of component states, $|M_A|^2<3$; for the Gibson state, $|M_A|^2=2.59$. This is in serious disagreement with the triton decay rate. This is most easily seen from the ratio of the expressions for the neutron and triton ft values[45]:

$$\frac{(ft)_n}{(ft)_{H^3}}=\frac{1+F_A^2|M_A|^2}{1+3F_A^2}=\frac{1228\pm35}{1137\pm20},$$

which leads to

$$|M_A|_{\text{expt}}^2=3.3\pm0.2.$$

The possible relativistic corrections have been estimated[45] and lead to a 2% reduction of the calculated matrix element. There have been attempts to calculate the magnitude of the exchange contribution for β decay.[46,47] These are based on the Chew-Low static model and use a series of canonical transformations to eliminate pion-nucleon coupling terms through fourth order. The result depends strongly on the choice of the nucleon-nucleon hard-core radius and appears to be a little too small even for an extreme limit on this parameter. The calculation does confirm, however, that exchange effects have the correct sign and general order of magnitude to explain the discrepancy.

The isovector magnetic moment of the H^3-He^3 system also exhibits an exchange effect. This can be seen by comparing the experimental moments with values computed from the expectation of the simple moment operator

$$\hat{\mu}=\sum_{i=1}^{3}\{\tfrac{1}{2}[1+\tau_z(i)][\mu_p\boldsymbol{\sigma}(i)+\mathbf{l}(i)]+\tfrac{1}{2}[1-\tau_z(i)]\mu_n\boldsymbol{\sigma}(i)\}.$$

The expressions for the isoscalar and isovector moments calculated from $\hat{\mu}$ are

$$\mu_s=(\mu_p+\mu_n)(p_1+p_2-p_D+\tfrac{1}{2}p_9)+p_D=0.835\text{ nm}, \quad (10)$$

$$\mu_v=(\mu_p-\mu_n)(p_1-\tfrac{1}{3}p_2+\tfrac{1}{3}p_D+\tfrac{1}{6}p_9)-\tfrac{1}{3}p_D=4.38\text{ nm}, \quad (11)$$

where the numerical values (in nuclear magnetons) are for the Gibson state. The last term in both expressions is the small contribution of the orbital motion of the protons, estimated by assuming that the ^4D state is symmetric in $\mathbf{r}$ and $\boldsymbol{\rho}$.[48] The isoscalar moment is in reasonable agreement with the experimental value of 0.851 nm, but the isovector moment is about 17% smaller than its experimental value of 5.11 nm. This has been largely explained by Villars,[49] who found an exchange contribution of about $\frac{2}{3}$ nm based on pseudoscalar-meson theory.[50] It is probably also worth noting that no acceptable fit to the H^3 and He^3 magnetic moment form factors could be found without including an isovector and (small) isoscalar exchange moment form factor.

One fact that can be inferred from the preceding dis-

[44] A more descriptive title is "mesonic" effects, that is, everything that can be attributed to the failure of the impulse approximation.

[45] R. J. Blin-Stoyle and S. Papageorgiou, Nucl. Phys. **64**, 1 (1965).

[46] J. S. Bell and R. J. Blin-Stoyle, Nucl. Phys. **6**, 87 (1958).

[47] R. J. Blin-Stoyle, V. Gupta, and H. Primakoff, Nucl. Phys. **11**, 444 (1959).

[48] R. G. Sachs, Phys. Rev. **72**, 312 (1947).

[49] Felix Villars, Helv. Phys. Acta **20**, 476 (1947). A full discussion of exchange magnetic moments is included in Ref. 40, pp. 241–258. There is also evidence for exchange contributions to forbidden magnetic dipole transitions in heavy nuclei; Marc Ross, Phys. Rev. **88**, 935 (1952).

[50] Exchange effects in other nuclei are often not this large (or have this sign). See, for example, S. D. Drell and J. D. Walecka, Phys. Rev. **120**, 1069 (1960); R. J. Adler and S. D. Drell, Phys. Rev. Letters **13**, 349 (1964); E. M. Nyman, Nucl. Phys. **B1**, 535 (1967).

cussion is that exchange effects are important in the H^3-He^3 system, and that a calculation that ignores them is very likely to be incorrect. The question that remains is how to include them in muon capture. The most plausible way seems to be phenomenologically, using the reasonably clear-cut situation in β decay as a model. The nuclear matrix elements for β decay and for muon capture are almost identical; only the factor $\exp(-i\mathbf{k}_\nu\cdot\mathbf{x}_i)$ distinguishes them. Roughly speaking, the $\int\boldsymbol{\sigma}$ and $\int 1$ "form factors" are evaluated at $q^2=0$ for β decay and at $q^2=k_\nu^2$ for muon capture. An obvious choice, then, is to include the exchange effect by assuming that its relative magnitude is not too different at $q^2=0.25$ F^{-2} than it was found to be at $q^2=0$ in β decay. Using the Gibson state, this relative magnitude is $(27\pm8)\%$, and this rather large uncertainty should cover the uncertainty in the "scaling" assumption. To be specific, we have assumed that the exchange effect is present in muon capture, and that it represents (to within the 30% experimental uncertainty in its magnitude) the same relative correction to the muon capture $|\int\boldsymbol{\sigma}|^2$ as it does to the β decay $|M_A|^2$.[51] The results of the calculation are presented, however, with and without this correction.

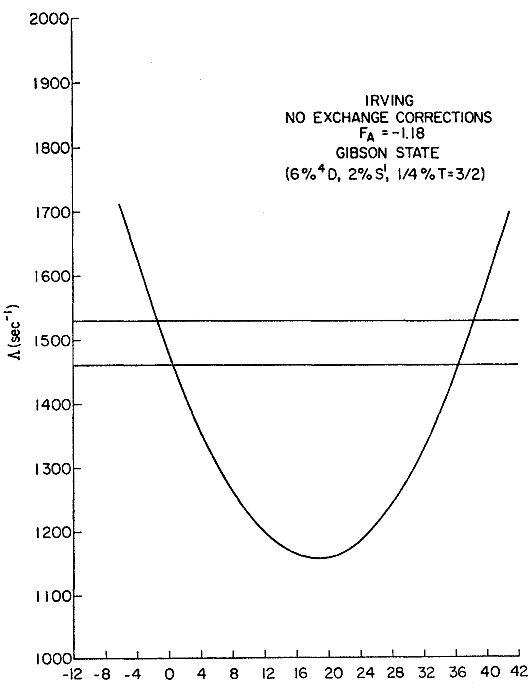

FIG. 1. Calculated He^3 muon capture rate for various values of F_P/F_A assuming UFI, CVC, and no exchange correction. Irving wave functions with $\alpha_s=1.34$ F^{-1}, $\alpha_D=\sqrt{2}\alpha_s$. Experimental results are indicated by the horizontal band.

[51] This involves the assumption that the q^2 dependence of the exchange term is the same as that of the direct term. This is consistent (for small q^2) with the empirical exchange-moment form factors found by Gibson, within their large associated uncertainties.

The existence and magnitude of this correction is supported by the magnetic moment data. If the dominant exchange process is to renormalize the $(\boldsymbol{\tau\sigma})$ nuclear operator, the axial-vector matrix element in β decay should be proportional to the square of the isovector magnetic moment. This predicts an exchange effect of about 36% in the squared matrix element, somewhat larger than that derived from the β decay alone. While the renormalization assumption is approximate, it does support the β decay result and indicate that the exchange effects are probably quite large.

V. RESULTS

The evaluation of the various matrix elements of $\int 1$ and $\int\boldsymbol{\sigma}$ involves a considerable amount of computation, none of which will be discussed here. The analytic expressions are quoted in Appendix B, and the integrated matrix elements are tabulated for the Irving, Irving-Gunn, and Gaussian functions used in this calculation.

The results for the Irving function assuming the Gibson parameters with no exchange correction are shown in Fig. 1. The upper and lower bounds for the experimental capture rate are $1460\ \text{sec}^{-1}\leq\Lambda_{\text{expt}}\leq1530\ \text{sec}^{-1}$, derived from Refs. 3 and 4. The weighted average of the three experiments is $1470\ \text{sec}^{-1}$. The uncorrected (for exchange effects) theoretical predictions are moderately sensitive to the amounts of ^4D, S', and $T=\frac{3}{2}$ states included. Roughly speaking, the 6% ^4D mixture lowers the rates by $120\ \text{sec}^{-1}$ and the 2% S' mixture by another $50\ \text{sec}^{-1}$, so that the predictions for the Gibson state lie about $170\ \text{sec}^{-1}$ below those of a pure S-state wave function.[52] Even allowing a reasonable margin for the uncertainty from the wave function, the values predicted for F_P/F_A are unreasonable: $F_P=0\pm3$ or $F_P=35\pm3$.

The uncorrected results are comparable to those of Pascual and Pascual,[33] who use a different set of basic wave functions and slightly different coupling constants. The effect of the S' state is typical: A 4% mixture of this state lowers the Pascuals' predictions by about $130\ \text{sec}^{-1}$ compared to $100\ \text{sec}^{-1}$ for this calculation. A detailed comparison of their results with those shown in Fig. 1 is not possible since the range parameters and exact state mixtures do not match.

The situation changes markedly when the exchange contribution is included. Figure 2 presents the results for the Gibson state assuming the Irving form of the wave functions. The theoretical error indicated (approximately $\pm5\%$) is solely from the uncertainty in $|\int\boldsymbol{\sigma}|^2$ at $q^2=0$. The uncertainties due to the nuclear wave function are small. Gibson's analysis of the electron scattering determines the range parameter α to within $2\frac{1}{2}\%$, which limits the error in this calculation to about 1% from this source. The calculation is also

[52] State 7 is an exception. A given percentage of this state, which has a particularly simple form, lowers the rate 70% more than states 6 or 8. The Gibson state is composed of 2% each of states 6, 7, and 8; the $120\ \text{sec}^{-1}$ quoted above applies to it.

relatively insensitive to the admixtures of the various states involved, in all cases explored to within the exchange effect error. In particular, the rates predicted for the alternate state suggested by Gibson in the case of a nonzero neutron charge from factor (0.6% S', 6% ^4D, and no $T=\frac{3}{2}$) lie about 30 sec^{-1} below those in Fig. 2. The rates are more sensitive to the value of F_A: They vary by about $\pm 1\frac{1}{2}\%$ for $|F_A|=1.18\pm 0.02$, still within the indicated margin.

The fact that the corrected calculation is not very sensitive to admixtures is easily seen from the form of the corrected $\int\boldsymbol{\sigma}$:

$$|\textstyle\int\boldsymbol{\sigma}|_{\text{corr}}{}^2=(3.3\pm0.2)\dfrac{|\int\boldsymbol{\sigma}|_{\text{uncorr}}{}^2}{|\int\boldsymbol{\sigma}(q^2=0)_{\text{uncorr}}{}^2|},$$

where the uncorrected matrix element at $q^2=0$ is the β-decay expression, Eq. (9). Except for pathological cases, the recipe was found to be not very different at $q^2=0.25\ F^{-2}$ than at $q^2=0$.

The close relationship between this calculation and the form factor calculation of Gibson is indicated in Figs. 3 and 4, where the results for the Gaussian and Irving-Gunn forms are presented. At small momentum transfer these functions overestimate and underestimate the form factors, respectively, and this is mirrored in the muon-capture predictions. For this reason, no conclusions concerning the magnitude of F_P should be taken from these figures.

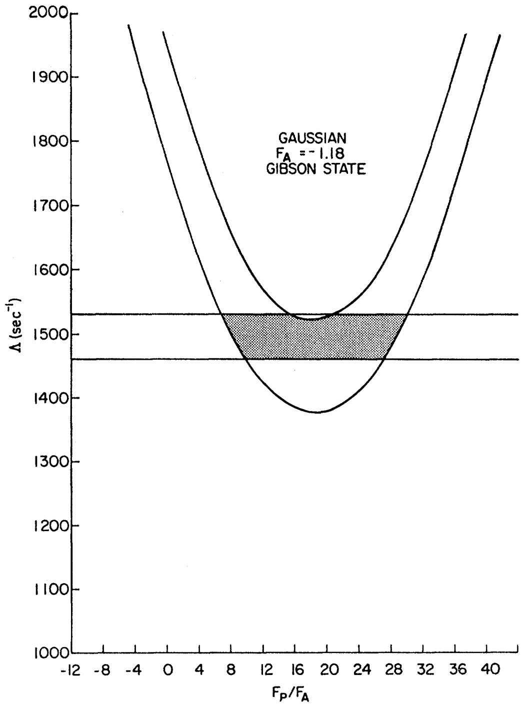

FIG. 3. Illustrative results for Gaussian wave functions, all $\alpha_i=0.384\ F^{-1}$.

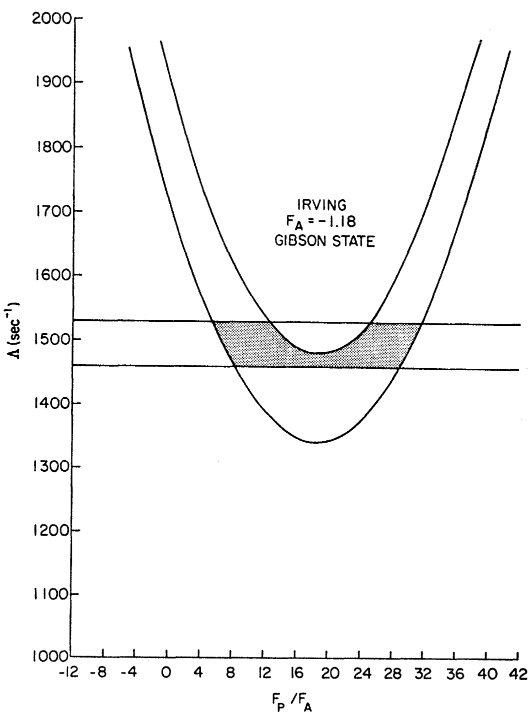

FIG. 2. Calculated He^3 muon capture rate for various values of F_P/F_A assuming UFI, CVC, and the phenomenological exchange contribution of Sec. IV. Irving wave functions with $\alpha_s=1.34\ F^{-1}$, $\alpha_D=\sqrt{2}\alpha_s$. The shaded area is the intersection of the experimental uncertainty and the theoretical uncertainty from the exchange contribution.

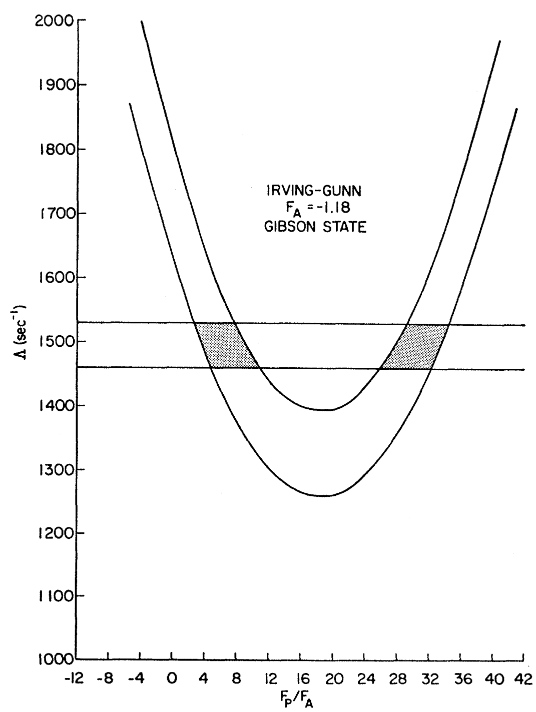

FIG. 4. Illustrative results for Irving-Gunn wave functions, $\alpha_s=0.813\ F^{-1}$, $\alpha_D=\sqrt{2}\alpha_s$.

The most probable value of F_P/F_A from Fig. 2 is about $F_P=(11\text{–}12)F_A$, which is in good agreement with the results from radiative muon capture. If the isovector magnetic moment is used to estimate the exchange contribution, the minimum acceptable value is $12F_A$ and the most probable $15F_A$, the range being $12\leq F_P/F_A\leq 25$. Strictly on the basis of β decay, however, the allowable range of values is very broad,

$$6\leq F_P/F_A\leq 32\,,$$

and the calculation is consistent with the Goldberger-Treiman value. As noted by Oakes,[31] for a pure S state the class-II axial-vector current (also called the tensor current) enters in the same way as the pseudoscalar term. This is also true for the complete state including the $1/M^2$ corrections. Thus, only the combination of the two effects can be determined. The comparison between Figs. 1 and 2 does indicate, however, that exchange effects are important in muon capture.

VI. CONCLUSIONS

This calculation leans heavily on the analysis of the 3-body electromagnetic form factors to demonstrate the necessity of an accurate value of the axial-vector matrix element including the mesonic exchange effect. With this correction, the muon capture rate in He^3 is consistent with values of the pseudoscalar coupling in the range $6\leq F_P/F_A\leq 32$, with a most probable value of 11.

ACKNOWLEDGMENTS

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APPENDIX A

Previous calculations of muon capture in He^3 have neglected the effects of the spatial extension of the nuclear charge distribution on the atomic wave function of the muon. This Appendix outlines a calculation of this effect.

The capture process takes place almost entirely from the ground state of the muonic atom, so that the lowest S-state wave function is required. As the muon is no more relativistic than a corresponding electron ($v/c\approx 0.01$), the Schrödinger equation should be sufficiently accurate.

The potential which the muon moves in is that from the charge distribution of He^3, which has been measured by Collard.[53] The most tractible of three "best fits" is the hollow-exponential-well II distribution (H.E. II),

$$\rho(r)=(Ze)\frac{75b}{8\pi a^2}\left(\frac{r}{a}\right)^2 e^{-br/a}\,,$$

[53] H. Collard, thesis, Stanford University, 1966 (unpublished).

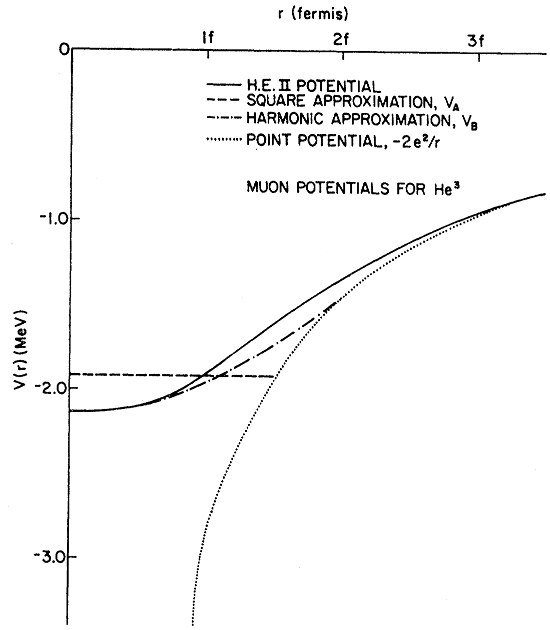

FIG. 5. A plot of the H.E. II potential of Collard (Ref. 53) for He^3 together with the two approximate potentials considered in Appendix A. At $r=1.5$ F(2F), $V_A(V_B)$ joins the point Coulomb potential, which is also indicated.

for $a=1.85$ F and $b=\sqrt{30}$. The potential from this distribution is

$$V(r)=-\frac{e^2}{12}\left\{\frac{24}{r}(1-e^{-br/a})-\frac{b}{a}\left[\left(\frac{br}{a}\right)^2+\frac{6br}{a}+18\right]e^{-br/a}\right\}.$$

To simplify the analytical work, two approximations to this potential were considered: V_a (square well) and V_b (harmonic oscillator):

$$\begin{aligned}V_a(r)&=-1.92\text{ MeV} & r&\leq 1.5\text{ F}\\ &=-2e^2/r\,, & r&>1.5\text{ F}\\ V_b(r)&=[-2.132+\tfrac{1}{2}\kappa r^2]\text{ MeV} & r&\leq 2\text{ F}\\ &=-2e^2/r\,, & r&>2\text{ F}\\ \kappa&=0.346\text{ MeV F}^{-2}.\end{aligned}$$

The H.E. II potential and the two approximations are shown in Fig. 5. The Schrödinger equation was solved analytically for both approximations and normalized numerically. The resulting values of the wave function at the origin agree to better than 2 parts in 10 000 and the average values over the range $0\leq r\leq 1.8$ F to 3 parts in 10 000. The result can be expressed as

$$|\phi_\mu|^2{}_{\text{av}}=(1/\pi)[2m_\mu'/137]^3R\,,\; R=0.9704$$

where the first factors represent the value of the "point nucleus" wave function at the origin.

APPENDIX B

The normalization constants for the various states, C_i, are defined by

$$C_i^{-2}=\int d^3\mathbf{r}d^3\boldsymbol{\varrho}\,|\psi_i|^2.$$

For the dominant S state the integrals are easily evaluated and lead to the following:

$$\begin{aligned}&\text{Irving:} && C_1=[\sqrt{3}\alpha_1^6/80\pi^3]^{1/2};\\ &\text{Irving-Gunn:} && \quad=[\sqrt{3}\alpha_1^4/4\pi^3]^{1/2};\\ &\text{Gaussian:} && \quad=[3\sqrt{3}\alpha_1^6/2\pi^3]^{1/2}.\end{aligned}$$

For the remaining states, the normalization integrals were done numerically; the integrands are

$$C_i^{-2}\equiv\int d^3\mathbf{r}d^3\boldsymbol{\varrho}\,|f_i|^2N_i,$$

where

$$\begin{aligned}N_2&=2(S_1^2+S_2^2),\\ N_6&=2[25(S_1^2+S_2^2)^2-21(S_1^2+S_2^2)S_s^2+8S_s^4],\\ N_7&=4S_s^2[S_s^2-\tfrac{1}{3}(S_1^2+S_2^2)],\\ N_8&=4[S_s^2(S_1^2+S_2^2)-\tfrac{1}{3}(S_1^2+S_2^2)^2],\\ N_9&=S_1^2+S_2^2.\end{aligned}$$

The S_i^2 and f_i are defined in Sec. III of the text.

After summing over the nuclear spin states, the expression for $|\int 1|^2$ can be written

$$\tfrac{1}{2}\sum_{\text{spins}}|\textstyle\int 1|^2=\left|\displaystyle\int d^3\mathbf{r}d^3\boldsymbol{\varrho}\,\exp(-i\tfrac{2}{3}\mathbf{k}_\nu\cdot\boldsymbol{\varrho})\sum_{i,j}f_if_jI_{ij}\right|^2,$$

where i and j run over the possible H^3 and He^3 states, respectively. The f's are supposed to carry the associated (real) phases. Since the operator 1 is a scalar in ordinary spin space, there are no cross terms between the doublet states (1,2,9) and the quartet states (6,7,8). Thus the only nonzero integrands are

$$\begin{aligned}I_{11}&=2,\\ I_{22}&=2(S_1^2+S_2^2),\\ I_{66}&=2(75S_1^4+50S_1^2S_2^2-25S_2^4\\ &\qquad-47S_1^2S_s^2+5S_2^2S_s^2+8S_s^4),\\ I_{77}&=4(S_1^2S_s^2-1\tfrac{2}{3}S_2^2S_s^2+S_s^4),\\ I_{88}&=4(S_1^4-8\tfrac{2}{3}S_1^2S_2^2+S_2^4+S_1^2S_s^2+S_2^2S_s^2),\\ I_{19}&=\sqrt{2}S_1,\\ I_{12}+I_{21}&=8S_1,\\ I_{67}+I_{76}&=16(4S_1^2S_s^2-S_s^4),\\ I_{68}+I_{86}&=10\tfrac{2}{3}S_1S_2^2S_s,\\ I_{78}+I_{87}&=16(\tfrac{4}{3}S_1S_2^2S_s-S_1S_s^3),\\ I_{29}&=\sqrt{2}(S_2^2-S_1^2).\end{aligned}$$

TABLE I. Irving wave functions, $\alpha_1=\alpha_2=\alpha_9=1.34$ F^{-1}, $\alpha_6=\alpha_7=\alpha_8=\sqrt{2}\alpha_1$.

Matrix elements for $\int\sigma$

	He³ states 1	2	6	7	8	9
H³ 1	0.8898	0	0.0056	−0.1152	0.0051	−0.0457
states 2	a	−0.2389	0.1463	0.1531	0.0661	0.4937
6	a	a	0.2854	0.0224	0.0247	−0.0379
7	a	a	a	0.2906	−0.0270	−0.0414
8	a	a	a	a	0.2975	0.0328

Matrix elements for $\int 1$

	He³ states 1	2	6	7	8	9
H³ 1	0.8898	0.1827	0	0	0	0.0457
states 2	a	0.7380	0	0	0	−0.0053
6	a	a	0.7478	0.0178	0.0246	0
7	a	a	a	0.7458	−0.3220	0
8	a	a	a	a	0.7444	0

[a] The total cross-term contribution is given only once, above the main diagonal.

TABLE II. Irving-Gunn wave functions, all $\alpha_i=0.771$ F^{-1}.

Matrix elements for $\int\sigma$

	He³ states 1	2	6	7	8	9
H³ 1	0.8498	0	0.0157	−0.1047	0.0143	−0.0616
states 2	a	−0.1654	0.2157	0.2013	0.0981	0.3811
6	a	a	0.1772	0.0397	−0.0063	−0.0593
7	a	a	a	0.1672	−0.0980	−0.0680
8	a	a	a	a	0.1937	0.0447

Matrix elements for $\int 1$

	He³ states 1	2	6	7	8	9
H³ 1	0.8498	0.2462	0	0	0	0.0616
states 2	a	0.5632	0	0	0	−0.0168
6	a	a	0.3172	0.1779	0.0482	0
7	a	a	a	0.2952	−0.6521	0
8	a	a	a	a	0.2836	0

[a] The total cross-term contribution is given only once, above the main diagonal.

TABLE III. Gaussian wave functions, all $\alpha_i=0.384$ F^{-1}.

Matrix elements for $\int\sigma$

	He³ states 1	2	6	7	8	9
H³ 1	0.9021	0	0.0029	−0.0859	0.0026	−0.0380
states 2	a	−0.2784	0.1017	0.1075	0.0461	0.5616
6	a	a	0.2934	0.0196	0.0231	−0.0260
7	a	a	a	0.2984	−0.0220	−0.0281
8	a	a	a	a	0.3040	0.0230

Matrix elements for $\int 1$

	He³ states 1	2	6	7	8	9
H³ 1	0.9021	0.1518	0	0	0	0.0380
states 2	a	0.8417	0	0	0	−0.0016
6	a	a	0.7861	0.0114	0.0212	0
7	a	a	a	0.7848	−0.2785	0
8	a	a	a	a	0.7840	0

[a] The total cross-term contribution is given only once, above the main diagonal.

All possible cross terms (excepting the S-S') contribute to $\int\boldsymbol{\sigma}$. The integrands are defined by

$$\tfrac{1}{2}\sum_{\text{spins}}|\textstyle\int\boldsymbol{\sigma}|^2=3\left|\displaystyle\int d^3\mathbf{r}d^3\boldsymbol{\varrho}\,\exp(-i\tfrac{2}{3}\mathbf{k}_\nu\cdot\boldsymbol{\varrho})\sum_{i,j}f_if_jS_{ij}\right|^2,$$

where

$$S_{11}=2\,,$$
$$S_{22}=2S_1^2-3\tfrac{1}{3}S_2^2\,,$$
$$S_{19}=-\sqrt{2}S_1\,,$$
$$S_{29}=\sqrt{2}(S_1^2+\tfrac{1}{3}S_2^2)\,,$$
$$S_{66}=150(S_1^4-\tfrac{1}{3}S_2^4)+16S_s^4+100S_1^2S_2^2-118S_1^2S_s^2+18S_2^2S_s^2$$
$$+R_{1z}^2[300S_1^3+180S_1^2S_s-100S_1S_2^2-80S_1S_s^2-60S_2^2S_s-32S_s^3]$$
$$+R_{2z}^2[-300S_1^3+180S_1^2S_s+100S_1S_2^2+80S_1S_s^2-60S_2^2S_s-32S_s^3]$$
$$+R_{1z}R_{2z}[-600S_1^2S_2+200S_2^3+32S_2S_s^2]\,,$$
$$S_{77}=S_s^2[\tfrac{4}{3}S_1^2+4S_s^2-(52/9)S_2^2-8S_s(R_{1z}^2+R_{2z}^2)+21\tfrac{1}{3}S_2R_{1z}R_{2z}]\,,$$
$$S_{88}=\tfrac{4}{3}(S_1^4+S_2^4)-(232/9)S_1^2S_2^2+4(S_1^2+S_2^2)S_s^2-R_{1z}^2[8(S_1^2+S_2^2)S_s+21\tfrac{1}{3}S_1S_2^2]$$
$$-R_{2z}^2[8(S_1^2+S_2^2)S_s-21\tfrac{1}{3}S_1S_2^2]+42\tfrac{2}{3}R_{1z}R_{2z}S_1^2S_2\,,$$
$$S_{67}+S_{76}=16S_s^2(3S_1^2+\tfrac{1}{3}S_2^2-S_s^2)+40(R_{1z}^2-R_{2z}^2)S_1S_s^2-8(R_{1z}^2+R_{2z}^2)[5(S_1^2-\tfrac{1}{3}S_2^2)S_s-4S_s^3]-58\tfrac{2}{3}R_{1z}R_{2z}S_2S_s^2\,,$$
$$S_{68}+S_{86}=8S_1S_s(S_s^2+\tfrac{1}{3}S_2^2-7\tfrac{2}{3}S_1^2)-21\tfrac{1}{3}R_{1z}R_{2z}S_1S_2S_s$$
$$+40(R_{1z}^2+R_{2z}^2)(S_1^2-\tfrac{1}{3}S_2^2-\tfrac{1}{5}S_s^2)-8(R_{1z}^2-R_{2z}^2)(7S_1^2+3S_2^2)S_s\,,$$
$$S_{78}+S_{87}=16[\tfrac{1}{3}S_1^2+(11/9)S_2^2-S_s^2]S_1S_s+32(R_{1z}^2+R_{2z}^2)S_1S_s^2+16(R_{1z}^2-R_{2z}^2)(S_1^2+\tfrac{1}{3}S_2^2)S_s-64R_{1z}R_{2z}S_1S_2S_s\,,$$
$$S_{16}+S_{61}=8\sqrt{3}[S_1S_s-R_{1z}^2(5S_1+2S_s)+R_{2z}^2(2S_s-5S_1)]\,,$$
$$S_{17}+S_{71}=8\sqrt{3}[\tfrac{1}{3}S_1S_s+(R_{1z}^2-R_{2z}^2)S_s]\,,$$
$$S_{18}+S_{81}=-8\sqrt{3}[\tfrac{1}{3}(S_1^2-S_2^2)+(R_{1z}^2-R_{2z}^2)S_1+2R_{1z}R_{2z}S_2]\,,$$
$$S_{26}+S_{62}=8\sqrt{3}[(S_1^2+\tfrac{1}{3}S_2^2)S_s-2(R_{1z}^2-R_{2z}^2)S_1S_s-5(R_{1z}^2+R_{2z}^2)(S_1^2+\tfrac{1}{3}S_2^2)+\tfrac{4}{3}R_{1z}R_{2z}S_2S_s]\,,$$
$$S_{27}+S_{72}=8\sqrt{3}[\tfrac{1}{3}(S_1^2+\tfrac{1}{3}S_2^2)S_s+(R_{1z}^2-R_{2z}^2)S_1S_s-\tfrac{2}{3}R_{1z}R_{2z}S_2S_s]\,,$$
$$S_{28}+S_{82}=8\sqrt{3}[(5/9)S_1S_2^2-\tfrac{1}{3}S_1^3-(R_{1z}^2-R_{2z}^2)(S_1^2-\tfrac{1}{3}S_2^2)-2\tfrac{2}{3}R_{1z}R_{2z}S_1S_2]\,,$$
$$S_{69}=(4\sqrt{2}/\sqrt{3})[-S_2^2S_s+5(R_{1z}^2+R_{2z}^2)S_2^2-4R_{1z}R_{2z}S_2S_s]\,,$$
$$S_{79}=(4\sqrt{2}/\sqrt{3})[-\tfrac{1}{3}S_2^2S_s+2R_{1z}R_{2z}S_2S_s]\,,$$
$$S_{89}=(4\sqrt{2}/\sqrt{3})[-\tfrac{2}{3}S_1S_2^2-(R_{1z}^2-R_{2z}^2)S_2^2+2R_{1z}R_{2z}S_1S_2]\,.$$

The factors of R_{1z} and R_{2z} are a result of the fact that the expressions quoted above are essentially reduced matrix elements, calculated between states of H^3 and He^3 with $m_s=+\frac{1}{2}$. The detailed spin sum produces rotationally invariant matrix elements but involves four times as much algebra. The angular integrations are only slightly more complicated due to their presence and are easily done analytically. The remaining radial integrals, however, are more complicated and were done (in all cases) numerically. A convenient check on both the algebra and the programming is available: If the neutrino momentum $\mathbf{k}_\nu$ is set to zero, the β-decay expressions for $\int 1$ and $\int \boldsymbol{\sigma}$ result, both of which can be calculated without reference to the detailed form of the wave function. The form for $\int \boldsymbol{\sigma}$ is given by Eq. (9); the $\int 1$ matrix element is unity. Thus the expressions for the matrix elements above and the integrated values in Tables I, II, and III are presented with a good deal of confidence.