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Thomas-Fermi Theory of Nuclei*

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A Thomas-Fermi theory of large, finite nuclei is developed. Realistic nuclear forces with repulsive core are assumed, and maximum use is made of the theory of nuclear matter. Simplifications are introduced wherever permissible. The local-density approximation with a certain correction is found to be valid. Tensor forces are replaced by a density-dependent, effective central force, the repulsive core by a density-dependent δ-function interaction. The Thomas-Fermi expression for kinetic energy is shown to be good whenever the density is at least 17% of nuclear-matter density; under the same conditions, the Slater approximation to the mixed density $\rho(r_1,r_2)$ is valid. From the total energy of the nucleus, an integral equation is derived for the density $\rho(r)$. This is approximated by a differential equation which is solved analytically. The resulting density distribution has both similarities with and differences from the conventional, Fermi-type distribution. Our density agrees as well with electron-scattering experiments as the Fermi type does. The thickness of the nuclear surface comes out about 10% too large from our theory; the surface energy is in good agreement with the semiempirical value. So far, the number of neutrons and protons has been assumed equal, and the Coulomb force has been neglected.

1. INTRODUCTION

THE use of the Thomas-Fermi (TF) method for nuclei is an old idea.[1] All the attempts before 1955, however, used pure exchange forces between nucleons and thus achieved saturation automatically. Actually, as we know since 1949, nuclear forces are of Serber type rather than pure exchange, and saturation is achieved by a strong, short-range repulsive core of some sort. In nuclear matter, such forces are best treated by the Brueckner-Goldstone method,[2] which describes the strong correlations between two nucleons at short distances. The theory of finite nuclei, and the TF theory in particular, should use the nuclear-matter results as a starting point.

A Hartree-Fock (HF) theory using nuclear-matter theory was developed by Brueckner and collaborators[3,4]; it will be discussed at the end of Sec. 2 and in Sec. 3. On the basis of this HF theory, statistical (TF) theories were developed by Hara[5] and by Kumar, LeCouteur, and Roy.[6] These will be discussed below.

This work was preceded by several more phenomenological theories which, however, used some fundamental ideas of nuclear-matter theory. These include the work of Skyrme[7] and of Berg and Wilets.[8] The latter authors define an "energy density of nuclear matter" $\epsilon(\rho)$, which is a function of the density ρ. This function contains an attractive and a repulsive part, plus kinetic energy, each represented as a suitable power of the density. Coefficients are chosen to give the correct binding energy and equilibrium density. In many respects, our theory in this paper is similar, except that we try to fix constants as far as possible from the theory of nuclear matter, and rely less on empirical data.

In addition to $\epsilon(\rho)$, Berg and Wilets introduce a term which takes into account the effect on the energy of the variation of nuclear density with position. They

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[1] E. Majorana, Z. Physik **82**, 137, (1963); C. F. v. Weizsäcker, *ibid.* **96**, 431 (1935); H. Euler, *ibid.* **105**, 553 (1937). For other references, see Ref. 5.

[2] K. A. Brueckner and C. A. Levinson, Phys. Rev. **97**, 1344 (1955); K. A. Brueckner, *ibid.* **97**, 1353 (1955); J. Goldstone. Proc. Roy. Soc. (London) **A293**, 265 (1957); summary in G. Brown, *Unified Theory of Nuclear Models* (North-Holland Publishing Co., Amsterdam, 1967), 2nd ed.

[3] K. A. Brueckner, J. L. Gammel, and H. Weitzner, Phys. Rev. **110**, 431 (1958).

[4] K. A. Brueckner, A. M. Lockett, and M. Rotenberg, Phys. Rev. **121**, 255 (1961).

[5] Y. Hara, Progr. Theoret. Phys. (Kyoto) **24**, 1179 (1960). This paper contains extensive references to the work preceding nuclear-matter theory.

[6] K. Kumar, K. J. LeCouteur, and M. K. Roy, Nucl. Phys. **42**, 529 (1963). This paper contains references to earlier work.

[7] T. H. R. Skyrme, Phil. Mag. **1**, 1043 (1956).

[8] R. A. Berg and L. Wilets, Phys. Rev. **101**, 201 (1956); L. Wilets, *ibid.* **101**, 1805 (1956); Rev. Mod. Phys. **30**, 542 (1958).

choose for this the form

$$\alpha(\nabla\rho)^2, \tag{1.1}$$

which had also been chosen by Skyrme.[7] We have a similar term, in the (approximate) differential form of our theory (Sec. 8), but we find that

$$-\alpha\rho\nabla^2\rho \tag{1.2}$$

follows more naturally from the structure of the theory. Integration by parts converts (1.2) into (1.1).

With these largely phenomenological assumptions, and fixing the constants in agreement with observations, Berg and Wilets obtain the density distribution near the surface, and several interesting constants. They find that the potential extends about 0.7 F farther, and falls off more slowly, than the density.

Hara makes much more use of nuclear-matter theory. He takes from that theory the idea that there is a long-range attractive potential, and a δ-function repulsion at the surface of the core. The latter is assumed to depend on density as $\rho^{1/3}$, the former is independent of ρ. There are also nonlocal terms, similar to but more complicated than (1.1). He chooses specific analytic forms for the attractive potentials, either square wells or Yukawas, and fixes parameters to give the correct density and binding energy of nuclear matter. The density is assumed to fall off linearly at the surface, and the surface thickness is determined by a variational calculation. It comes out 10–20% too big, the surface energy is found to be 23–28 MeV (observed value about 18 MeV, see Sec. 10).

Kumar *et al.* have even closer contact with the work of Brueckner *et al.*[3,4] on finite nuclei. Their expression for the energy density has somewhat similar form as that of Berg and Wilets, but they attempt to connect the parameters to those found by the Brueckner group. They find that the attractive force used in Brueckner's work does not have enough density dependence (see our Sec. 3) and accordingly determine their own parameters for this. Otherwise, they make a minimum of assumptions on the nuclear interaction, and find good agreement with experimental data.

Seyler and Blanchard[9] use a TF theory with a velocity-dependent interaction, fitting parameters in this interaction to observed data on specific finite nuclei. They find the radial distribution of neutrons and protons separately from two coupled integral equations, similar to our Sec. 7. The result is also similar to our (7.15). The velocity dependence gives the saturation; the potential is only used in first order which, from other experience, may not be justified. The potential is not connected with two-nucleon forces.

In recent years, the theory of nuclear matter has made much progress. Binding energy and density have been calculated in fair agreement with experiment by Sprung and Bhargava[10] and by Dahlblom.[11] The detailed study of the contributions of various two-nucleon states LSJ to the nuclear-matter energy has given us much insight into the mechanism of saturation of nuclear forces. The general theory of finite nuclei has been advanced especially by Brandow.[12,13] Many authors have done specific calculations of finite nuclei which will be discussed in Sec. 2 of this paper, insofar as they influence our considerations.

In this paper, we outline a theory of finite nuclei which makes maximum use of nuclear-matter theory. We attempt to make this theory as simple as is consistent with realistic assumptions about nuclear forces. Our chief aim is to develop a statistical (TF) theory. However, as a first step, we have to discuss some general simplifications which can be applied to any theory of finite nuclei, and could be used as a basis for a simplified Hartree-Fock treatment (Sec. 2). We find that the local-density approximation (LDA) should be good, even in the nuclear surface, provided a certain, easily applied correction is made.

In Sec. 3, we consider the various complications in the nuclear force and devise simplified treatments: the tensor force is replaced by a central, density-dependent force, the forces in the various D states are replaced by suitable approximations, and consideration is given to the fact that the attraction in two-particle S states is stronger than in D states.

In Sec. 4 we show that the momentum dependence of the nuclear force arises mainly from its exchange character, rather than from the repulsive core. It is shown that the simplest way to ensure Hermiticity is to take exchange forces explicitly.

The total energy of the nucleus is expressed in terms of attractive ordinary and exchange forces, repulsive forces of δ-function character, and kinetic energy (Sec. 5). The Thomas-Fermi-Slater approximation is used for the kinetic energy and the exchange term (Sec. 6); this is justified by numerical calculations of a model in Appendices B and C, respectively. By variation, an integral equation for the density as a function of position is derived (Sec. 7) and it is shown that far inside the nucleus, the density may be represented by

$$\rho/\rho_0=1-e^{x/a}, \tag{1.3}$$

where x is the distance from the surface. The relation of (1.3) to the conventional Fermi-Woods-Saxon distribution is discussed.

The integral equation is then approximated by a differential equation (Sec. 8) which can easily be solved, giving an analytical expression for the density, (8.23). At large r for a spherical nucleus, this is replaced by an expression resulting from the exponential decay of the

[9] R. G. Seyler and C. H. Blanchard, Phys. Rev. **131**, 355 (1963).

[10] P. N. Bhargava and D. W. L. Sprung, Ann. Phys. (N. Y.) 42, 222 (1967).

[11] T. Dahlblom (private communication).

[12] B. H. Brandow, Phys. Rev. **152**, 863 (1966).

[13] B. H. Brandow, Rev. Mod. Phys. (to be published).

wave functions, (8.29). The combined distribution is compared with electron-scattering results (Sec. 9). The agreement seems to be as good as for the conventional "Fermi" shape. The surface thickness derived from theory tends to be larger than that from electron-scattering experiments, and is quite sensitive to the details of the nuclear force. Luckily, our best assumption on nuclear forces agrees with experiment within about 10%, considerably better than the earlier attempt of Hara.[5]

The surface energy is calculated from our incomplete integral and the more complete differential theory; both calculations give excellent agreement with the experimental value of 18 MeV. Previous calculations[5,6] tended to give too high surface energy.

The density distribution obtained from our theory differs significantly from the conventional "Fermi" distribution. Our density approaches full nuclear-matter density only very slowly as we go toward the center of the nucleus (Fig. 5). This tendency is opposed by the well-known tendency of the Coulomb force to concentrate protons near the surface; if both are taken into account, the result may be a rather flat distribution. This was actually found in some preliminary calculations but a reliable result has not yet been obtained.

We intend to pursue this theory in three directions: (1) to include the "symmetry energy" which tends to make neutron and proton density to go in parallel, and the Coulomb energy, (2) to use a better quantitative representation of nuclear forces, and (3) to apply the same principles to a Hartree-Fock calculation.

2. LOCAL-DENSITY APPROXIMATION

The energy[14] of a finite nucleus, in the Brueckner-Goldstone theory is given by

$$W=\sum_{m<n}\langle mn|G|mn\rangle+\cdots, \tag{2.1}$$

where the sum goes over all occupied states m and n and represents two-body interactions, while the $\cdots$ denotes all interactions involving more than two bodies. The G matrix satisfies the equation

$$\langle m'n'|G|mn\rangle=\langle m'n'|v|mn\rangle-\sum\langle m'n'|v|ab\rangle \times\frac{Q(ab)}{E(a)+E(b)-E(m)-E(n)}\langle ab|G|mn\rangle, \tag{2.2}$$

where the Pauli operator $Q(ab)=1$ if a and b are both unoccupied, $Q=0$ otherwise. In principle, (2.2) can be solved; in the process, the energies of all occupied and all empty nucleon states can (and must) be determined, as well as the one-particle wave functions ϕ_m. The Pauli operator is probably best considered as one minus the projection on occupied states.

[14] The reader who is not interested in theoretical detail needs only to read Sec. 2 D and may otherwise omit Secs. 2 and 3.

While the solution of (2.2) is possible in principle, it is very difficult in practice, except perhaps for light nuclei for which oscillator wave functions are a good approximation for the $\phi(m)$. In general, it is desirable to replace (2.2) by a more manageable approximation.

A. Short-Range Forces

Such an approximation is suggested by the Moszkowski-Scott (MS) theory,[15] which has been thoroughly investigated in the case of nuclear matter, i.e., constant density. In this theory, the interaction v is separated into a short- and a long-range part by a separation distance d which is about 1 F. It can then be shown[15] that the long-range potential v_l can be treated in the first Born approximation; the second Born approximation for central forces contributes less than 1 MeV per particle (tensor forces will be discussed in Sec. 3). The short-range part, $r<d$, gives a repulsive contribution G_s (called the dispersion term by MS) which increases with density ρ. There is also an interference term between short and long range which is small, again about 1 MeV at normal nuclear density for central forces.

Whether we use Moszkowski-Scott or not, the interaction matrix G is related to the two-body wave function Ψ by

$$G\Phi=v\Psi=v(\Phi-\zeta), \tag{2.3}$$

where Φ is the unperturbed wave function of the two nucleons (product of two shell-model orbitals), and ζ is the "wave-function defect." This defect is most important[16–18] near the surface of the repulsive core, $r=0.4$ or 0.5 F, then rapidly falls to zero close to the separation distance d, and then is negative and small for larger r. The important point is that ζ is large only for small distances r; this behavior, known as "rapid healing" of the wave function, is the main reason for the success of the MS method. The largest contributions to G_s come from the inside and the surface of the repulsive core.[16] These contributions obviously depend very little on the long-distance behavior of the two-nucleon wave function Φ but mainly on its local magnitude when the two nucleons are close together. Therefore, to a very good approximation, G_s should be determined by local conditions alone, in particular by the local density. For a given pair m, n of interacting nucleons, G_s depends of course also on their energies E_m+E_n.

The importance of the density is shown by the theory of nuclear matter. The easiest way to see this is the formulation of MS themselves, who show that G_s

[15] S. A. Moszkowski and B. L. Scott, Ann. Phys. (N. Y.) **11**, 65 (1960).

[16] H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963).

[17] M. W. Kirson, Ann. Phys. (N. Y.) (to be published).

[18] G. Dahll, E. Østgaard, and B. Brandow, Nucl. Phys. (to be published).

is nearly proportional to the difference

$$e-e_0=U(a)+U(b)-U(m)-U(n), \tag{2.4}$$

where e is the denominator in (2.2), e_0 the same denominator for free particles of the same momenta, and $U(a)$ the potential energy of a particle in state a. Recent work on the three-body problem[19–21] has shown that the "particle" potential energies $U(a)$, $U(b)$ are quite small; the "hole" potential energies $U(m)$, $U(n)$ are of the order of the average potential energy of a nucleon and therefore nearly proportional to the density, with some saturation setting in near normal nuclear density. Thus we conclude that approximately

$$G_s\sim\rho^\lambda, \tag{2.5}$$

with λ somewhat less than 1; $\frac{2}{3}$ is probably a reasonable choice.[22,23] Greater accuracy can be achieved by using numerical results from nuclear matter,[11] and this procedure will be followed in most of this paper.

B. Unperturbed Wave Functions

We have been led to the conclusion that the main part of the interaction in nuclei can be treated by the Born approximation while most of the remainder depends only on the local density. We shall now consider this problem in more detail. The interaction depends (1) on the unperturbed wave function on which it acts, (2) on the energy denominators in Eq. (2.2), and (3) on the Pauli operators. We shall consider these points separately. We may consider G as an operator, operating on the two-particle wave function

$$|mn\rangle\equiv\Phi_{mn}=\phi_m(\mathbf{r}_1)\phi_n(\mathbf{r}_2), \tag{2.6}$$

where ϕ_m and ϕ_n are one-particle waves in the Hartree-Fock (HF) potential[24] chosen for the nucleus. (Since the HF theory includes exchange, the HF potential will in general be nonlocal, cf. Sec. 4). Then we may write

$$G\Phi_{mn}(r_1,r_2)=v_l(|\mathbf{r}_1-\mathbf{r}_2|)\phi_m(\mathbf{r}_1)\phi_n(\mathbf{r}_2)+\int K(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_1',\mathbf{r}_2')\phi_m(\mathbf{r}_1')\phi_n(\mathbf{r}_2')d\tau_1'd\tau_2', \tag{2.7}$$

where the kernel K is given by (2.2), i.e.,

$$K=v_s(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_1',\mathbf{r}_2-\mathbf{r}_2')-v(Q/e)G. \tag{2.8a}$$

K contains the short-range reaction matrix G_s, the MS interference term which is of medium range, the second-order Born term from the long-range forces, the corresponding terms from the tensor force, etc. The most important central, long-range force has been removed as the first term in (2.7), so most of K is short or medium range. This effectively minimizes the part of K which acts on states of relative angular momentum $L\neq0$: this is true even for the tensor force, which is really important only in its action on the ^3S state. (The tensor force on the ^3P state is large and may be of some importance for spin-orbit splitting, and hence for HF energies. But its effect on the binding energy of nuclear matter is negligible.[10,11])

Two situations may now be considered, the interior and the surface of the nucleus. In the interior, the density changes slowly. Then it is a good approximation to replace ϕ_m and ϕ_n in the integral in (2.7) by plane waves, just as in the Thomas-Fermi theory or the Slater approximation[25] to the HF exchange term. Doing this reduces the wave functions $\phi_m(r_1')\phi_n(r_2')$ to the nuclear-matter case. Of course, in many cases, an individual wave function ϕ_m may have a node in the spatial region $\mathbf{r}_1'$ in which the kernel K is large: then ϕ_m should be considered as the sum of two or more plane waves.[26,27]

In the surface, the density ρ changes rapidly, and the wave functions are not close to plane waves. We may in particular look at the region near the classical turning point of states m and n. Here we use the facts (a) that the S-state part of $\phi_m\phi_n$ is by far the most important, and (b) that the wave functions $\phi_m\phi_n$ do not have large curvature. Neglecting terms of order r^2 where

$$\mathbf{r}=\mathbf{r}_1'-\mathbf{r}_2' \tag{2.8b}$$

is the relative coordinate, we may write[27a]

$$\phi_m(\mathbf{r}_1')\phi_n(\mathbf{r}_2')=\phi_m(\mathbf{R})\phi_n(\mathbf{R}), \tag{2.9a}$$

where

$$\mathbf{R}=\tfrac{1}{2}(\mathbf{r}_1'+\mathbf{r}_2') \tag{2.9b}$$

[19] H. A. Bethe, Phys. Rev. **138**, 805 (1965); **158**, 941 (1967).
[20] M. W. Kirson, Ann. Phys. (N. Y.) (to be published).
[21] R. Rajaraman and H. A. Bethe, Rev. Mod. Phys. **39**, 745 (1967).
[22] A similar conclusion is reached on the basis of the BBP reference spectrum method. According to their Eq. (5.28), the contribution of the repulsive core volume is proportional to (a) $(\gamma^2+k_0^2)c^3$, where c is the core radius, k_0 the relative momentum of the two interacting particles, and γ a measure of the separation in energy between occupied and unoccupied states, BBP (3.8). Using BBP (2.5), (8.22)–(8.24) one easily shows that (b) $\gamma^2+k^2\approx2k_F^2\Delta$, where Δ has been shown, by several numerical calculations (Ref. 11) to be nearly independent of density and about 0.8. Thus (a) is proportional to $k_F^2\sim\rho^{2/3}$. The core surface term in BBP (5.28), which is usually several times the volume term, depends less on k_F, a linear dependence on k_F being reasonable. However, the repulsive contribution of the core is partly compensated in G_s by the attractive force acting between core radius and separation distance, and this does not depend much on density. The net repulsion G_s will therefore have a stronger relative density dependence than the core (volume+surface) contribution alone, and (2.5) with $\lambda=\frac{2}{3}$ is reasonable also from this point of view.
[23] J. Nemeth has drawn the same conclusion from a study of the numerical results in Ref. 10.
[24] For a discussion of the construction of the HF potential, see Ref. 12.
[25] J. C. Slater, Phys. Rev. **81**, 385 (1951).
[26] Similar arguments have been used by B. D. Day, Phys. Rev. **136**, B1594 (1964).
[27] B. Brandow, thesis, Cornell University, 1964, University Microfilm 64-8090 (unpublished).
[27a] Strictly speaking, the right-hand side is equal to $\frac{1}{2}\phi_m(\mathbf{r}_1')\phi_n(\mathbf{r}_2')+\frac{1}{2}\phi_n(\mathbf{r}_1')\phi_m(\mathbf{r}_2')$.

is the center of mass. The corrections of order r^2 are small, due to the small curvature of the ϕ. But also the density is small, so that the plane waves which would be used in nuclear-matter theory at the local density, could also be approximated by (2.9a). So for any position $\mathbf{R}$, we may replace $\phi_m(\mathbf{r}_1)\phi_n(\mathbf{r}_2)$ by a product of plane waves which have the same value of $\phi_m(\mathbf{R})\phi_n(\mathbf{R})$.

In (2.2), there also occur the intermediate states a, b. Wong[28] has pointed out that there is great freedom in their choice; they only need to be a complete set. Therefore it is certainly permissible to choose ϕ_a, ϕ_b to be locally plane waves—this is in fact a much better approximation than the same assumption for ϕ_m, ϕ_n which we have justified. Then the matrix elements $\langle a,b|v|m,n\rangle$ will be the same as in nuclear matter.

C. Energy Denominators

We turn now to the operator e in (2.8a), viz.,

$$e=E(a)+E(b)-E(m)-E(n). \tag{2.10}$$

The hole energies $E(m)$, $E(n)$ must be obtained by solving the HF equation in the potential of the nucleus as a whole.[29] The intermediate-state energy is, in accord with Brandow's proposal,[30]

$$E(a,\mathbf{R})=(\hbar^2/2m)k_a^2+U(a,\mathbf{R}). \tag{2.11}$$

The potential energy $U(a,\mathbf{R})$ is calculated in the Brueckner-Goldstone theory by calculating the interaction of the particle a with all the nucleons present in the neighborhood of $\mathbf{R}$, and depends on the local density at $\mathbf{R}$. Of course, since a is a "particle state," its interaction with the Fermi-sea nucleons must be calculated off the energy shell, as described in BBP. It should be noted that we have made full use of Wong's freedom[28] to choose the intermediate states: Instead of taking constant $E(a)$ and following the change of wave number with $\mathbf{R}$ due to change of $U(a,\mathbf{R})$, we consider a state of definite $\mathbf{k}_a$ at R and determine its energy by (2.11).

According to the latest calculations[11,21] of the three-body correlations, the potential $U(a)$ of intermediate states seems to be very small so that $U(a,\mathbf{R})$ in (2.11) could well be neglected. This procedure of setting

$$U(a,\mathbf{R})=0 \tag{2.12}$$

has been recommended by Kuo and Brown.[31] But even if we do not assume (2.12), the method of constructing U shows that

$$U(a,\mathbf{R})\approx U(k_a,\rho(\mathbf{R})) \tag{2.13}$$

depends essentially only on the local density at $\mathbf{R}$. Further refinement is unjustified because (2.12) is already such a good approximation.

With the simplest assumption (2.12), the energy denominator now becomes

$$e=(\hbar^2/2m)(P^2+k^2)-E_m-E_n, \tag{2.14}$$

where $\mathbf{P}=\frac{1}{2}(\mathbf{k}_a+\mathbf{k}_b)$ is the average and $\mathbf{k}=\frac{1}{2}(\mathbf{k}_a-\mathbf{k}_b)$ the relative momentum in the intermediate state. In the HF method, the energies E_m and E_n are determined by self-consistency. (As we shall show, no great accuracy is required in this.) The relative momentum k in (2.14) is variable, and in the end is integrated over. The average momentum P, on the other hand, is defined as being identical with that in the initial state, i.e.,

$$\mathbf{P}=\tfrac{1}{2}(\mathbf{k}_m+\mathbf{k}_n), \tag{2.15}$$

where $\mathbf{k}_m$, $\mathbf{k}_n$ are the local wave numbers of ϕ_m, ϕ_n. These depend on the local one-particle potential U, thus:

$$(\hbar^2/2m)k_m^2(R)=E_m-U(E_m,R). \tag{2.16}$$

The construction of the one-particle potential is one of the subjects of this paper; it depends of course on E_m itself, as well as on R. This latter dependence makes the characteristic parameter of the reference spectrum method,

$$\gamma^2=P^2-(2m/\hbar^2)(E_m+E_n), \tag{2.17}$$

dependent on R. The defect wave function ζ will then also depend on R, not just on r, and the theory becomes somewhat unpleasant. To escape this complication, Wong, in his recent paper,[32] has replaced γ^2 by an average, and has given prescriptions to calculate this average. He has shown, in Ref. 32, Table 9, that the value of γ^2 has very little influence (a few tenths of an MeV) on the binding energy of the nucleus, or on the HF energies of the various single-particle states. In the LDA, no such averaging of γ^2 needs to be done at all.

We shall show in Sec. 2 D that LDA, with a certain correction, is probably quite a good approximation. In the LDA we can use the argument of MS,[15] which shows that G_s depends only on the difference in *potential* energy between particle and hole state, viz., on

$$\Delta U=U(k_a,\rho)+U(k_b,\rho)-U(k_m,\rho)-U(k_n,\rho). \tag{2.18}$$

This expression is entirely known from nuclear-matter theory, and does not require the determination of P, which is always tedious.[28]

When calculated from (2.16), the potential $U(k_m)$ depends, of course, only on k_m^2. If k_m is to be deduced from a real wave function ϕ_m (e.g., the radial wave function), this function must be approximated by $\sin(k_m r+\alpha)$, with α a phase.

[28] C. W. Wong, Nucl. Phys. **A104**, 419 (1967).
[29] This has been emphasized by S. Köhler, Nucl. Phys. **32**, 661 (1962) [see especially Eq. (27) of this paper]; also Phys. Rev. **137**, B1145 (1965).
[30] B. H. Brandow, in *International School of Physics "Enrico Fermi," Course 36*, edited by C. Bloch (Academic Press Inc., New York, 1966), p. 528.
[31] T. T. S. Kuo and G. E. Brown, Nucl. Phys. **85**, 40 (1966).
[32] C. W. Wong, Nucl. Phys. **91**, 399 (1967).

In many cases it is sufficient to replace (2.18) by its average over k_m and k_n:

$$(\Delta U)_{\rm av}=2k_F{}^2\Delta(\rho)\,, \tag{2.19}$$

where Δ is the quantity defined in BBP,[16] Eq. (7.3). In the surface region where

$$U(E_m,R)>E_m\,, \tag{2.20}$$

$k_m{}^2$ is negative. But these regions are of little importance, and it should suffice to approximate ΔU by (2.19).

D. Extension of the Local Approximation

So far, we have assumed that the bound-state energies E_m, E_n are the HF eigenvalues. This procedure is satisfactory for a HF calculation. For a Thomas-Fermi (TF) thoery which we want to develop in this paper, a further approximation is useful. We are here interested mainly in the total energy of the nucleus, and in the spatial distribution of the total density. Thus we must sum over all pairs of nucleons m, n. We then make the Thomas-Fermi assumption that at any point R, all states are occupied which have a momentum

$$k_m<k_F=(\tfrac{3}{2}\pi^2\rho)^{1/3}. \tag{2.21}$$

This assumption will be investigated in detail in Sec. 6. Thus we assume in effect that the energy of the highest occupied state is

$$E_F=w_F(\rho(R))\,, \tag{2.22}$$

$$w_F(\rho(R))=(\hbar^2/2m)k_F{}^2(R)+U(w_F,\rho(R))\,, \tag{2.23}$$

where $U(E,\rho)$ is the potential energy of a particle of energy E in nuclear matter of constant density ρ. $w_F(\rho(R))$ is, of course, the total energy[33] of the most loosely bound nucleon in nuclear matter of density $\rho(R)$.

Later on, we shall find a correction to (2.22). For the moment, we assume (2.22) to be correct; then the energy denominators e become exactly the same as for nuclear matter at density $\rho(R)$. Indeed, consider the interaction of two nucleons of local momenta k_m, k_n. Then, in analogy to (2.23),

$$E_m=(\hbar^2/2m)k_m{}^2(R)+U(k_m,\rho(R))\equiv w(k_m,\rho)\,. \tag{2.24}$$

The potential energy U is the interaction of particle m with all nucleons present near point R. In the present approximation this is determined by the local density $\rho(R)$; of course, it also depends on k_m or E_m. For given E_m, $k_m(R)$ can be determined from the implicit Eq. (2.24). The energy denominator (2.10) can now be written

$$\begin{aligned}e_{\rm TF}&=w(k_a,\rho)+w(k_b,\rho)-w(k_m,\rho)-w(k_n,\rho)\\&=(\hbar^2/2m)(k_a{}^2+k_b{}^2-k_m{}^2-k_n{}^2)+U(k_a,\rho(R))\\&\quad+U(k_b,\rho(R))-U(k_m,\rho(R))-U(k_n,\rho(R))\,.\end{aligned} \tag{2.25}$$

This is exactly the value of e for the same momenta in nuclear matter of constant density $\rho(R)$. If we assume that all states of local wave number up to k_F are occupied, (2.25) has reduced the problem to a purely local one from which all reference to the "global" energies E_m, etc., has disappeared.

The question now arises of the accuracy of the TF assumption (2.22). For this purpose, we use the results of Sec. 7. According to (7.2),

$$E_F=w_F(\rho(\mathbf{r}_1))+\frac{3}{4}\int d\tau_2 v_l(\mathbf{r}_1-\mathbf{r}_2)[\rho(\mathbf{r}_2)-\rho(\mathbf{r}_1)]\,. \tag{2.26}$$

Here the second term[34] represents the effect of the long-range, ordinary forces (see Secs. 4–6). Since these do not depend on momentum, we have also for an arbitrary nucleon

$$\begin{aligned}E_m&=w(\rho(\mathbf{r}_1),k_m)+\frac{3}{4}\int d\tau_2 v_l(\mathbf{r}_1-\mathbf{r}_2)[\rho(\mathbf{r}_2)-\rho(\mathbf{r}_1)]\\&\equiv w_m+D_l(\mathbf{r}_1)\,.\end{aligned} \tag{2.27}$$

For constant density, the last term $D_l(\mathbf{r}_1)$ is absent, as is to be expected. If the density curves downward, the last term is positive since the average of the bracket, and v, are both negative. Sections 7 and 8 show that downward curvature is the normal situation (Fig. 5), valid for $\rho>\frac{1}{4}\rho_0$, where ρ_0 is the normal nuclear-matter density; in this case then,

$$E_m>w(\rho,k_m)\,.\quad(\rho>\tfrac{1}{4}\rho_0)\,. \tag{2.28}$$

Only for very low density, the inequality is reversed, and these low densities contribute little to the average potential energy of nucleon m, or the total binding energy of the nucleus.

We may now compare[34a] the correct energy denominator (2.10) with that of the LDA, (2.25):

$$e=e_{\rm TF}-2D_l(R)\,. \tag{2.29}$$

Now consider a G matrix element which involves e,

[33] It is the self-consistent energy in the sense of the Brueckner formalism. It differs from the physical removal energy by rearrangement terms. Some of the terms denoted by this name by Brueckner and Goldman [Phys. Rev. **116**, 424 (1959)] are included in w_F, viz., those denoted as "saturation potential" by Brandow, Ref. 27. See Ref. 27 for a discussion of the remaining, "proper" rearrangement energy.

[34] The factor $\frac{3}{4}$ in (2.26) arises because only even states contribute. For the other basis of (7.2), see Sec. 6. Equation (7.2) is of course based on many assumptions; in particular, the form of the integral in (2.26) assumes that only the direct, long-range forces at r_1 depend on the density at other points, r_2, while all other forces are determined by the local density at r_1. However, this is easy to modify: e.g., long-range exchange forces can be included in D, (2.27), except that they depend of course on k_m, so that in the end an average has to be taken. Deviations of the short-range and tensor forces from LDA are treated below, in (2.37)–(2.39).

[34a] In making this comparison, we assume that $U(a)$ and $U(b)$ are both zero.

e.g., G_s. From the general formula, Ref. 16, Eq. (A14),

$$G=G_{\mathrm{TF}}+G_{\mathrm{TF}}(Q/e_{\mathrm{TF}}-Q/e)G, \tag{2.30}$$

where the operators without symbol refer to the "correct" situation in the finite nucleus. Now the defect function is

$$(Q/e)G=\zeta\approx\zeta_{\mathrm{TF}}, \tag{2.31}$$

so we have (since $Q^2=Q$)

$$G-G_{\mathrm{TF}}=\langle\zeta_{\mathrm{TF}}|e-e_{\mathrm{TF}}|\zeta\rangle. \tag{2.32}$$

Thanks to our assumption of vanishing U in intermediate states, $e-e_{\mathrm{TF}}$ is independent of the intermediate state k'; thus we can effect closure and obtain

$$\begin{aligned}G-G_{\mathrm{TF}}&=(e-e_{\mathrm{TF}})\langle k_0|\zeta_{\mathrm{TF}}\zeta|k_0\rangle\\&\approx-2D_l(R)\int\zeta_{\mathrm{TF}}{}^2d\tau.\end{aligned} \tag{2.33}$$

The result (2.33) is extremely simple. The two chief components of G which depend on the energy denominators are (1) the short-range force G_s and (2) the effect of the tensor force (cf. Sec. 3) $G_{T\,\mathrm{eff}}$. Their sum is therefore to be corrected, as compared to the LD approximation, by

$$G-G_{\mathrm{TF}}=-2D_l(R)\int(\zeta_s{}^2+w^2)d\tau, \tag{2.34}$$

where w is the D-state admixture into the ^3S state caused by tensor forces. The effect on the potential energy of an occupied state is obtained by[19] multiplying G by $\frac{3}{4}\rho$ so that

$$\Delta U=-\tfrac{3}{2}D_l\rho\int(\zeta_s{}^2+w^2)d\tau\equiv-2D_l\kappa. \tag{2.35}$$

The integral is the "wound" in the wave function. For a soft-core potential and normal nuclear-matter density, the contribution of ζ_s to κ is about 0.05, that of the D function w is about twice as great. Dahlblom finds,[11] for Reid's[35] soft-core potential, and for $k_F=1.36$ F^{-1},

$$\kappa=0.14. \tag{2.36}$$

This is the value we shall use. For the hard-core Reid potential, Dahll, Østgaard, and Brandow[18] find for nuclear matter[36] $\kappa=0.17\pm0.01$.

Equation (2.35), then, is the error made in estimating the hard-core and tensor contribution from the LDA. However, we should remember that in (2.27) we assumed that the potential energy of a nucleon is changed from its TF value only by the long-range force. In reality, there is the additional change ΔU, (2.35), so that (2.27) should be replaced by

$$E_m=E_{\mathrm{TF}}+D_l+\Delta U. \tag{2.37}$$

Accordingly, (2.35) now becomes

$$\Delta U=-(D_l+\Delta U)2\kappa=-D_l(2\kappa/(1+2\kappa)), \tag{2.38}$$

$$E_m=E_{\mathrm{TF}}+D_l/(1+2\kappa). \tag{2.39}$$

The effect is therefore that D_l is reduced by the factor $1+2\kappa$, i.e., to about 78% [cf. (2.36)]. This will be used in Sec. 8.

Equation (2.38) indicates that the potential energy, due to short-range and tensor forces, is decreased by about $0.22D_l$. Since D_l is mostly positive, this gives added binding. This is the effect found by Köhler[29] and Wong.[28] They point our that the use of the self-consistent particle energies E_m+E_n in (2.10) increases the binding of O^{16} by 1 to 2.5 MeV per particle. We shall similarly show in Sec. 10 that the effect of the correction factor $1+2\kappa$ in (2.39) decreases the surface energy of a large but finite nucleus by about 12% which, for O^{16}, gives about 1.0 MeV per particle. We have thus shown that the LDA must indeed be modified for the Köhler-Wong effect, but also how this modification can be achieved without deviating from the spirit of the LDA. The factor $(1+2\kappa)^{-1}$ is just the damping effect due to the self-consistency of G and U. A similar result was obtained by Brandow[12] for the case of an infinite system.

E. Pauli Operator

It remains to discuss the Pauli operator Q, or better $1-Q$. This is the projection on the occupied states. Here again we make the TF, or Slater,[25] approximation of replacing the occupied states by local plane waves. This is justified because the function standing to the right of $1-Q$,

$$e^{-1}G\phi_m\phi_n, \tag{2.40}$$

is well localized because it is equal to ζ^R. We thus exclude, as in nuclear matter, all Fourier components of ζ in which the momentum of one (or both) nucleons falls within two spheres of radius $k_F(R)$ whose centers are displaced by $2P$.

In the surface, where plane waves are not a good approximation, the Pauli operator $1-Q$ is small. Moreover, the important thing in this limit of low density is the volume in momentum space which is excluded; the exact shape of the functions in $1-Q$ is not important [they all are reasonably constant over the region in which (2.40) is large]. Thus also in this region the result is essentially the same as that of the LDA.

The Pauli aspect of the LDA has recently been investigated by Wong,[32] who compares it for O^{18} and O^{16} with a "global method" in which $1-Q$ is the projection on the actual HF functions, assumed to be oscillator

[35] R. Reid, thesis, Cornell University, 1967 (unpublished).

[36] For O^{18}, Wong (Ref. 28) finds $\kappa=0.25$. This might be connected with the lower average density of this nucleus. Indeed, Dahlblom (Ref. 11) finds that κ has a minimum of 0.12 at about $k_F=1.0$, and then increases again to 0.14 at $k_F=0.7$, but he does not find any k as high as 0.25.

functions. In general, the LDA gives too much binding, by about 0.4 MeV per particle[37] for O^{16}. The sign of this effect can perhaps be understood as follows: The operator (2.40) has a certain spread in r, so that we should actually take the Pauli operator at $\mathbf{R}+\mathbf{r}$ and $\mathbf{R}-\mathbf{r}$, rather than at $\mathbf{R}$. Since the effect of $1-Q$ on the energy per particle is more than linear in ρ, the average result from $\mathbf{R}+\mathbf{r}$ and $\mathbf{R}-\mathbf{r}$ is likely to be greater than the result by taking the density $\rho(\mathbf{R})$.

The global Pauli effect is seen to have the opposite sign of the effect discussed in Sec. 2 D, viz., of taking the "global" (i.e., correct) E_m in the energy denominator rather than the TF approximation. The global Pauli effects seems less important than that of the energy denominators. We have not found any simple method to approximate the global Pauli effect.

F. Comparison with Brueckner *et al.*

The principle of the LDA was previously proposed and used by Brueckner, Gammel, and Weitzner.[3] It has been repeatedly attacked because the results[4] obtained by Brueckner *et al.* were only mediocre. These disappointing results we attribute to the following factors:

(1) Brueckner *et al.* did not take into account the strong density dependence of the equivalent central force in the ^3S state which is generated by the tensor force. This was first pointed out by Köhler[38] and will be discussed in our Sec. 3. The need for stronger density dependence was also pointed out by Kumar *et al.*[6]

(2) They assumed that nuclear matter has a density determined by $r_0=1.00$ F, while a more correct value is 1.12 F.[27] As a consequence, their finite nuclei had too high central density.

(3) The "saturation potential"[27,30] was not sufficiently understood.

(4) One of their results, the very gradual decrease of density from center to surface, was probably correct (see Sec. 7 of this paper) but was considered wrong at the time.

(5) Some approximations had to be made in the application of the theory.

We do not believe, therefore, that any lack of success in the Brueckner theory of finite nuclei should be held against the LDA as such. The salient point in our justification of LDA is that the part of the nuclear interaction which cannot be treated by Born approximation is of short range and therefore very suitable for a local approximation. This being so, we wish to push the LDA much beyond that of Brueckner *et al.* These authors transformed the G matrix of nuclear-matter theory at density ρ to coordinate space, obtaining

$$\langle \mathbf{r}_1,\mathbf{r}_2|G(\rho)|\mathbf{r}_1',\mathbf{r}_2'\rangle, \tag{2.41}$$

and then let this operate on the two-body wave function $\phi_m\phi_n$. Expression (2.41) is a nonlocal potential which embodies the velocity dependence of nuclear-matter theory, i.e., the fact that $U(m)$ depends on the momentum k_m. This is a complicated procedure. Since the important effects (at least for central forces) are of short range, it should be satisfactory to replace them instead by a δ-function interaction. That is, we replace the entire short-range interaction[39] between nucleons m and n by

$$G_s(\rho(\mathbf{r}_1))|\phi_m(\mathbf{r}_1)|^2|\phi_n(\mathbf{r}_2)|^2\delta(\mathbf{r}_1-\mathbf{r}_2)d\tau_1 d\tau_2. \tag{2.42}$$

This has the correct dimension (energy). Expression (2.42) can of course be integrated over r_2 and summed over n, and the result may be written

$$W_s=\int d\tau G_s(\rho(\mathbf{r}))|\phi_m(\mathbf{r})|^2\rho(\mathbf{r})$$

$$\equiv\int d\tau|\phi_m(\mathbf{r})|^2U_s(r). \tag{2.43}$$

This defines a potential $U_s(\mathbf{r})$ due to short-range forces. This is, in our approximation, independent of m; in other words, not velocity-dependent: This is reasonable since the momenta involved in the short-range forces are of order $1/c$, much larger than the momenta of the nucleons in the Fermi sea.[40] Equation (2.43) may finally be summed over m and divided by 2 to take each pair into account only once; this gives

$$W_s=\frac{1}{2}\int d\tau G_s(\rho(\mathbf{r}))\rho^2(\mathbf{r}) \tag{2.44}$$

for the contribution to the energy from short-range forces.

In practice, $G_s(\rho)$ is rather ill-defined because the MS separation distance is somewhat arbitrary, e.g., it may be questioned whether the same d should be used for all momenta and all states LSJ. More important, the use of just the G_s of MS in (2.44) is inaccurate because there is also the MS interference term, and the second-order Born term for long-range forces. Instead of this, we may use the following procedure: We consider the separation distance d as well as $G_s(^1S_0)$ as free parameters, to be chosen so as to get a best fit to $G(^1S_0,\rho)$ for all relevant densities which is both simple and accurate. The simplicity consists in using the Born approximation for $r>d$ and having a simple dependence of G_s on ρ. The accuracy is obtained by fitting an explicit nuclear-matter calculation, such as that of Sprung and Bhargava[10] or Dahlblom.[11] Similar procedures may be used for the ^3S state (see Sec. 3).

[37] C. W. Wong (private communication).
[38] S. Köhler, Phys. Rev. **137**, B1145 (1965).
[39] This contains all the terms enumerated below (2.8a).
[40] The origin of the main velocity-dependent forces is discussed in Sec. 4. There it is also shown that the velocity dependence of the short-range force is indeed very small.

G. Summary

In this way, we have greatly simplified the LDA of Brueckner *et al.* without, in our opinion, impairing its accuracy. On the contrary, we believe that we can achieve greater accuracy by using, to the greatest possible extent, results already obtained in nuclear-matter theory. Numerical inaccuracies are minimized by calculating a series of simple problems, well within the capacity of computers, such as the nuclear-matter problem, rather than one problem of enormous complication, like the solution of the Hartree-Fock equations for a complicated nucleus with a nonlocal effective potential (2.41), which varies very rapidly with the arguments. By going back to the nuclear-matter results, we have much better insight into the possible errors, and we keep the effect of the non-Born term small; in fact, we find that W_s is only about 20% of the total potential energy. Without the intermediate step of nuclear matter, the enormous repulsions and attractions inside and just ourside the repulsive core can easily make large errors in the result, just by the numerical procedure.

3. SPECIAL COMPONENTS OF THE NUCLEAR FORCE

A. Tensor Force

The smallness of the second-order Born approximation, pointed out in Sec. 2, does not apply to the tensor force. In fact, it contributes nothing in first Born approximation, and a lot in second. Hence, Scott and Moszkowski[41] concluded that their separation method was poor for tensor forces. The modified separation method[16] (MMS) must be used instead of the original MS method, but even this is numerically in error by about 20%.[18]

However, without using MS separation, it is possible to define an effective central force to replace the tensor force. This has been done by Brandow.[27,30] The resulting effective central force depends on the state, e.g., for infinite nuclear matter on k, P, and k_F. A useful approximation, based on Brandow's theory and somewhat on MS separation, has been proposed by Brown and Kuo.[31] They point out that a long-range tensor force

$$v_T(r)S_{12} \tag{3.1}$$

gives rise, in second-order Born approximation, to an interaction which is approximately

$$v_{\text{eff}}=-[v_T^2(r)/e_{\text{av}}](8-2S_{12}). \tag{3.2a}$$

Here e_{av}, according to their calculation, is about 220 MeV for nuclear-matter density. Only the effective central-force part, the 8 in (3.2a), is important for nuclear binding.

[41] B. L. Scott and S. A. Moszkowski, Ann. Phys. (N. Y.) 14, 107 (1961).

Equation (3.2a) is an oversimplification of Brown and Kuo's result: The *short*-range tensor force mixes $^3D_1$ state into an unperturbed ^3S state, and there is therefore added to (3.2a)

$$v_{\text{eff}}'=-w_s(r)v_T(r)(8-2S_{12}), \tag{3.2b}$$

where $w_s(r)$ is the ^3D-state wave-function admixture arising from the short-range forces (range less than MS separation distance d); of course, only the tail of w_s for $r>d$ contributes to (3.2b). The term (3.2b), according to Brown and Kuo, is a fraction (perhaps $\frac{1}{4}$) of (3.2a) and does not change the general behavior.

The effective central force in the ^3S state is then the sum of the central parts of (3.2a) and (3.2b), plus the basic long-range central force between free nucleons,

$$v_{\text{eff}\,c}=-8(w_sv_T+v_T^2/e_{\text{av}})+v_c(r). \tag{3.3}$$

Brown and Kuo point out that this is of the same order of magnitude as the long-range, central force in the singlet even state, and has a similar shape (r dependence). Therefore, just like the singlet force, $v_{\text{eff}\,c}$ acts essentially only in first Born approximation, higher orders being small.

We shall adopt these ideas with one important modification: The energy denominator in (3.3) represents the average difference between the energy of an intermediate state and that of an occupied one. With increasing density, the Pauli principle forbids the low-lying intermediate states; therefore e_{av} will increase. Another influence in the same direction is the difference in potential energy between intermediate and occupied states which becomes increasingly positive with increasing density. If e_{av} increases, the effective attraction (3.3) will decrease; in other words, there is saturation of the tensor-force contribution. We may then write

$$e_{\text{av}}=e_{\text{av}\,0}(1+\alpha k_F^n)/(1+\alpha k_{F0}^n), \tag{3.4}$$

where $e_{\text{av}\,0}$ is Brown and Kuo's 220 MeV, and α and n are two parameters which may be determined; k_F is the Fermi momentum, and k_{F0} is k_F for nuclear-matter density (to which Brown and Kuo normalized).

Rather than pursuing this line of reasoning in detail, we merely take its general features: The effective central force v_{eff} has a fixed shape (dependence on r), but its magnitude depends on k_F. Leaving aside the shape for the moment, we wish to determine the density dependence as much as possible in accord with nuclear-matter theory. Noting that v_{eff} is to be used in Born approximation, we wish to make its expectation value for an unperturbed two-particle wave function equal to the G matrix. We can do this, of course, only on the average over all pairs of nucleons, i.e., we want $v_{\text{eff}}(\rho)$ to reproduce the contribution of a triplet state to the binding energy of nuclear matter at given density ρ.

We consider the effect of the tensor force on the ^3S state. Recent results of Dahlblom,[11] using Reid's soft-

TABLE I. Ratio of triplet to singlet even interactions, and comparison with empirical formula (according to Dahlblom[a]).

k_F	0.7	0.9	1.1	1.3	1.5	1.7
$^3S/^1S$	1.41	1.25	1.12	0.98_5	0.85	0.68
$1.85-0.67k_F$	1.38	1.25	1.11	0.98	0.85	0.74
$1,46-0.28k_F^2$	1.32	1.23	1.12	0.98_5	0.83	0.65
$^3D_1/^1D$	−0.66	-0.64_5	−0.62	-0.59_5	−0.56	−0.52
All $^3D/^1D$	1.43	1.43	1.41	1.39	1.35	1.29
$1.48-0.05_5k_F^2$	1.45	1.43_5	1.41	1.39	1.35_5	1.32
All $L\geq 1/^1D$	1.41	1.48	1.32	1.08	0.78	0.56

[a] See Ref. 11.

TABLE II. Contribution of states to potential energy (according to Dahlblom[a]) (MeV).

k_F	0.7	0.9	1.1	1.3	1.5	1.7
S	−8.81	−14.70	−21.5	−27.2	−34.0	−37.8
^1P	0.11	0.33	0.86	1.98	4.03	7.24
^3P	−0.05	− 0.15	− 0.27	− 0.23	+ 0.27	1.43
D	−0.24	− 0.89	− 2.32	− 4.88	− 8.97	−15.0
$L\geq 3$	0.04	0.17	0.45	0.93	1.68	2.7
$L\geq 1$	−0.14	− 0.54	− 1.28	− 2.20	− 2.99	− 3.6
$\%L\geq 1$	1.6	3.5	5.6	7.2	8.1	8.7

[a] See Ref. 11.

core potential of April, 1967, may be represented by[42]

$$\mathcal{R}=U(^3S)/U(^1S)=1.85-0.67k_F. \tag{3.5}$$

Nearly equally good is the interpolation formula

$$\mathcal{R}'=1.46-0.28k_F^2. \tag{3.6}$$

Table I shows that (3.5) agrees within 0.01 with Dahlblom's results between $k_F=0.9$ and 1.5, the important range for finite nuclei, while (3.6) agrees within 0.02. The density dependence of (3.6) is more reasonable than (3.5) on the basis of the energy denominator (3.4); also, it gives a more reasonable (smaller) value for $k_F=0$.

Tensor forces work, of course, also in the ^3D and the ^3P states. The total contribution of ^3P states to nuclear-matter binding is, however, very small (see Table II); it will be discussed later on in this section. The contribution of $^3D_3$ is also very small. In the $^3D_2$ state, the tensor force contributes strongly, but there is no coupling to another state; thus we have in effect a central force with radial dependence different from ^1D. The $^3D_1$ state is complicated by its coupling to ^3S, but the ratio of its contribution to ^1D varies slowly with density, both according to Dahlblom (see Table II) and to Sprung and Bhargava.[10] The same is true for the $^3D_2$ and $^3D_3$ states. Table I gives the ratio of the contribution of all ^3D states to ^1D; it varies only from 1.43 to 1.29 and may be well represented by

$$\mathcal{R}_2=U(^3D)/U(^1D)=1.48-0.05_5k_F^2. \tag{3.7a}$$

The effective interaction in D states may then be taken as

$$v_{\text{eff}}(D)=v(^1D)(1+\mathcal{R}_2). \tag{3.7b}$$

Equations (3.5) and (3.6) represent concisely the effect of saturation in the tensor force for the ^3S state. This is a very important effect; in a subsequent paper, with Nemeth,[43] we shall show that it is at least as important as the effect of the repulsive short-range forces, (2.27). If we use (3.6), the tensor saturation depends on k_F^2, the same as the short-range forces (2.27); this is another advantage of (3.6) over (3.5). From Table I it is seen that at $k_F=1.28$ the binding from the ^3S state has decreased to being equal to the singlet contribution; at normal density ($k_F=1.36$), it is less than ^1S. This result depends somewhat on the nucleon forces used, and especially on the potential energy in intermediate states: In the calculations of Sprung and Bhargava,[10,42] in which that potential is strongly negative, the ^3S interaction becomes equal to ^1S only at $k_F\approx 1.6$ F^{-1}.

The tensor saturation is probably the main contributor to the density dependence which Kumar *et al.*[6] required to get agreement between their statistical theory and observations. Brueckner *et al.*[3,4] failed to get a strong density dependence of their long-range forces.

It should be noted that (3.6) is purely empirical, and only valid in the density region $k_F=0.9$ to 1.5 F^{-1}. Accidentally, extrapolation to $k_F=0$ gives the fairly reasonable ratio 1.46; the ratio of the Blatt-Jackson[44] strength functions for triplet and singlet is about 1.3 to 1.4. Most other interpolation formulas, like (3.5), give too high a ratio[45] at $k_F=0$.

We have compared the triplet to the singlet interaction, in terms of their contribution to the binding energy of nuclear matter. This is the most important quantity, and the shape of the interactions is relatively less important. Therefore, for a first approximation, it should be permissible to use the shape of the long-range ^1S force also for the triplet state, thus writing

$$v_t(^3S)=\mathcal{R}v_t(^1S,r). \tag{3.8}$$

A better approximation would be the following: We solve the nuclear-matter equation for the coupled triplet states ^3S, $^3D_1$. In the reference spectrum approximation (which may not be sufficient for these states, according to Sprung and Bhargava,[20] and to Dahll, Østgaard, and Brandow[18]), the equation for the ^3S-state defect function is, in the notation of BBP [see their Eq. (6.9)],

$$(\gamma^2-d^2/dr^2)\chi_{01}=v_c u_{01}+v_T u_{21}, \tag{3.9}$$

where v_c and v_T are the central and tensor potential,

[42] The results of Sprung and Bhargava for Reid's hard-core potential could be represented by $\mathcal{R}=2.56-0.96k_F$. This leads to larger $\mathcal{R}$ than Dahlblom's.

[43] J. Nemeth and H. A. Bethe (to be published).

[44] J. D. Jackson and J. M. Blatt, Rev. Mod. Phys. **22**, 77(1950).

[45] A possible explanation for the excessive contribution from ^3S states at low k_F (which also appears at $k_F=0.7$ in Table I) may be the behavior of the term $v(Q/e)G$ in the integral equation. For scattering, the Q/e is replaced by $1/e_0$, which has a pole for $k'<k_F$. In nuclear matter, the operator Q removes this pole so that all contributions are positive, while for free-particle scattering positive and negative contributions partly compensate.

and u_{01}, u_{21} the S- and D-state components of the wave function. Asymptotically,

$$u_{01} \to \mathcal{J}_0(kr) \equiv \sin kr \quad r \to \infty\,, \qquad u_{21} \to 0. \tag{3.10}$$

We postulate the existence of a MMS separation[16] distance d; and for $r > d$, we define the effective, long-range potential by setting

$$v_c u_{01} + v_T u_{21} = v_{\text{eff}}(r)\mathcal{J}_0(kr)\,. \tag{3.11}$$

The separation distance d itself is determined by making the G matrix agree with that from a more exact calculation, such as that of Dahll *et al.*[18] A slight difficulty arises in (3.11) because of the zeros of $\mathcal{J}_0$ which in general do not coincide with those of the left-hand side. The first zero occurs for $kr = \pi$. If $k = (0.3)^{1/2} k_F$ and $k_F = 1.36$ F^{-1}, this is at $r \approx 4$ F. At this large distance, $u_{01} \approx \mathcal{J}_0$ and $v_T u_{21}$ is negligible in comparison, so that we may put

$$v_{\text{eff}} \approx v_c \tag{3.12}$$

for large r. (Siemens[46] has suggested a procedure of averaging over k which avoids the zero more elegantly.) The effective potential as defined by (3.11) is identical with the proposal of Brandow.[30] A similar procedure may be used for the state in which $^3D_1$ is dominant but this is probably not warranted because its contribution to the energy is so small.

This v_{eff} automatically gives for the long-range G matrix

$$\langle k | G_l(^3S) | k \rangle = \int \mathcal{J}_0^2(kr) v_{\text{eff}}(r) dr\,, \tag{3.13}$$

which is the desired behavior. It is probably sufficient to consider an average of v_{eff} over the momenta k in the Fermi sea. We may then calculate v_{eff} from (3.1) as a function of k_F and represent it in the form

$$v_{\text{eff}}(k_F, r) = v_{\text{eff}\,1}(r) - k_F^2 v_{\text{eff}\,2}(r)\,, \tag{3.14}$$

where the dependence on k_F is modeled after (3.6). This calculation has not yet been carried out.

B. Even and Odd States

In Table II and Fig. 1, we show the results of Dahlblom[11] on the contributions of various two-particle states to the binding energy of nuclear matter, using the Reid hard-core potential of April, 1967. Clearly the main contribution comes from the S states. Indeed, the second-to-last line in Table II gives the contribution of all states other than S, and the last line gives the same as a percentage of the total potential energy. This quantity never exceeds 9% up to $k_F = 1.7$. This small contribution of the higher-L states results from a substantial cancellation between the attractive D states and the repulsive ^1P and $L \geq 3$ states. Somewhat surprisingly, the repulsion in ^1P increases rather faster with k_F than the attraction in the D states; the states $L \geq 3$ are about 18% of the D states (repulsive) at all k_F; the ^3P states change from very slight attraction to repulsion; therefore, all states $L \geq 1$ together give an attraction which is a decreasing fraction of the D states and of the ^1D state (last line of Table I). This result is welcome because it ensures saturation.

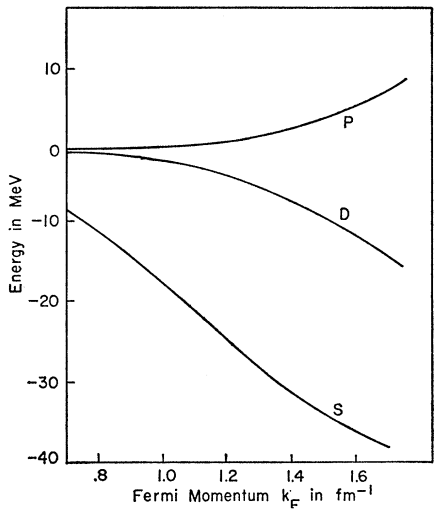

FIG. 1. Contributions to nuclear-matter energy from S, P, and D states, according to Dahlblom. Energy in MeV/particle versus k_F in F^{-1}.

To some extent, the results bear out the old idea that the forces are nearly of the Serber type,

$$v = \tfrac{1}{2} v_e (1 + P_M)\,, \tag{3.15}$$

where P_M is the Majorana exchange operator. In other words, the forces act essentially only in even states. In order to include the ^1P state (and the states $L \geq 3$), we only need to choose the exchange force a bit larger than the ordinary one.

C. S-State Forces

Noyes[47] pointed out already in 1959 that the potential in S states is stronger than in D (and presumably other even) states. This fact is embodied in the "quadratic spin-orbit term" in the Hamada-Johnston and the Breit potentials. It was confirmed again by the recent work of Reid[35] who found the attraction in S states [apart from the one-pion-exchange potential, (OPEP)] nearly twice as strong as in D.

To take this fact into account, we may consider the force as made up of the Serber force (3.13), which acts generally in even states, and an additional force

$$v_s P_s\,, \tag{3.16}$$

which acts on S states only. Here P_s is a projection

[46] P. J. Siemens (private communication).

[47] P. Noyes, *Proceedings of the Conference on Nuclear Forces and the Few-Nucleon Problem, 1959*, edited by T. C. Griffin and E. A. Power (Pergamon Press, Inc., New York, 1960).

operator which selects the S component of the wave function. Writing the unperturbed wave function of two nucleons as

$$\Phi=\phi_m(\mathbf{r}_1)\phi_n(\mathbf{r}_2)=\phi_m(\mathbf{R}+\tfrac{1}{2}\mathbf{r})\phi_n(\mathbf{R}-\tfrac{1}{2}\mathbf{r}), \quad (3.17)$$

we have

$$P_s\Phi=(4\pi)^{-1}\int d\Omega\Phi, \quad (3.18)$$

where $d\Omega$ means an integral over the direction of the relative coordinate vector $\mathbf{r}$, keeping its magnitude r and the center-of-mass coordinate $\mathbf{R}$ fixed. This prescription is somewhat cumbersome in practice; it will be applied to simplified wave functions Φ in the forthcoming paper by the author and Nemeth.[43]

We may even go further: As we showed in Table II, the S states contribute over 90% of the total potential energy. This suggests that, for approximate calculations in finite nuclei, the *entire* force be chosen to be an S-state interaction[15,48] [(3.16) and (3.18)] without any Serber force at all. The ^1S force may then be taken to be a realistic (e.g., Reid) potential, and the ^3S of the form (3.8). A correction factor may be applied to give the observed binding energy.

4. EXCHANGE FORCES, SELF-ADJOINT HAMILTONIAN

We shall assume in the remainder of this paper that the forces between two nucleons in the nucleus may be written

$$v_l(r)(1+P_M)+G_s(\rho)\delta(\mathbf{r}), \quad (4.1)$$

where $\mathbf{r}$ is the relative coordinate, v_l the long-range force in even states, and P_M the Majorana operator. The last term was justified in Sec. 2, with ρ the density at the center of mass of the two nucleons. The expression for the long-range force (first term) is obviously not the best in the light of Sec. 3, but is chosen for simplicity. A better treatment will be given by Nemeth and Bethe.[43]

The term P_M in (4.1) implies an exchange force of the same sign (attractive) as the ordinary force $v(r)$. An exchange force is essentially nonlocal, like the exchange term in the Hartree-Fock (HF) equation. This implies a momentum dependence of the effective force: If the two interacting particles have large relative momentum k (i.e., $kr_0\gg1$, where r_0 is the effective range of the forces), the exchange term P_M is small; for small relative momentum, it is as large as the "direct" term, 1 in (4.1). Thus (4.1) automatically contains momentum dependence of the potential energy of a nucleon, which is one of the prominent features of nuclear-matter theory.

We wish to show in this section that (4.1) is a better way to include momentum dependence than the effective-mass approximation of nuclear-matter theory. In that theory, the total energy for momentum p is written[16,49]

$$E(p)=p^2/2m^*, \quad (4.2)$$

where m^* is the effective mass and the factor $\hbar^2/M$ has been omitted. The potential energy is then

$$U(p)=\tfrac{1}{2}p^2(1/m^*-1). \quad (4.3)$$

If we wish to use this in coordinate space for a finite nucleus, this gives a term

$$-\tfrac{1}{2}(1/m^*-1)\nabla^2\psi. \quad (4.4)$$

Such a term would be easily acceptable if m^* were constant. Actually, for full nuclear matter density, the calculation of $U(p)$ for states in the Fermi sea by Dahlblom[11] gives

$$1/m^*-1\approx0.6, \quad m^*\approx\tfrac{5}{8}, \quad (4.5)$$

while for zero density, of course, $1/m^*-1=0$. Therefore, m^* is a rapid function of density and varies with r, the distance from the origin, in a finite nucleus. Then a term of the type (4.4) in the Schrödinger equation is not self-adjoint but must be replaced by[26]

$$-\tfrac{1}{2}\nabla\cdot[(1/m^*(r)-1)\nabla\psi], \quad (4.6)$$

which is more complicated.

A further complication arises from the fact that the potential energy of occupied states is not always well represented by (4.3). It is true that in Dahlblom's calculations (4.3) is a good approximation to $U(p)$. But in some other calculations (for different potentials), and especially for large k_F, $U(p)$ increases rapidly for low p, but for p approaching k_F, the increase becomes much slower. Therefore a more general function $U(\rho,p)$ should be introduced, and this is very difficult to express as a differential, self-adjoint operator on ψ. We have not been able to find a manageable expression of such a general $U(\rho,p)$ operator.

The chief reason for the deviations from (4.3) is the exchange force. As was pointed out before, it is large for low relative momentum k, small for high k. But in the limit $k\to\infty$ the exchange goes to zero; so the difference in potential energy between $k=0$ and high k tends to a limit, rather than increasing indefinitely with k as (4.3) implies. The potential energy of a hole state p increases with p because a particle of high p tends to have higher *relative* momentum k relative to an average

[48] This suggestion was first made by K. A. Brueckner [Phys. Rev. **97**, 1353 (1955)]. Moszkowski and Scott (Refs. 15 and 41), who investigated this problem most thoroughly, have suggested that the saturation of the ^3S force may be compensated by the attraction in D states. Table II shows that the total contribution of states $L>1$ is weaker than the ^3S saturation. It is therefore not possible to consider the total potential energy as just a constant coefficient times the ^1S interaction. However, it might be possible to write the total as

$$V(^1S)(A-Bk_F^2). \quad (3.19)$$

But we are not sure whether this will reproduce the radial and exchange dependence of the force sufficiently well.

[49] We use p for the momentum of a particle, reserving k for relative momentum, (4.7).

particle in the Fermi sea; but this effect should saturate as p increases, thus making (4.3) a poor approximation.

The exchange force is also the main cause of the dependence of U on momentum altogether. With our simple assumption (4.1), the ordinary long-range force $v_l \cdot 1$ gives no momentum dependence. The momentum dependence of the short-range force, G_s in (4.1), can be deduced by a modification of effective-range theory. We shall calculate the scattering length for two interacting *free* nucleons as a function of their relative momentum, and show later that the modification in nuclear matter is not important.

Consider two nucleons of relative momentum, k interacting with potential V_s in which the factor $M/\hbar^2$ has been absorbed; then

$$u''+(k^2-V_s)u=0. \tag{4.7a}$$

Assume that V_s is a short-range force such that

$$V_s=0 \quad \text{for} \quad r>d. \tag{4.7b}$$

Consider also the Schrödinger equation for a standard momentum k_0,

$$u_0''+(k_0^2-V_s)u_0=0. \tag{4.7c}$$

Multiply (4.7a) by u_0, (4.7b) by u, subtract and integrate from 0 to d:

$$u_0u'-uu_0'\,|_d=-(k^2-k_0^2)\int_0^d uu_0dr. \tag{4.8}$$

Let ϕ and ϕ_0 be the wave functions of free nucleons, normalized by

$$\phi=(\sin kr)/k, \quad \phi_0=(\sin k_0r)/k_0. \tag{4.9a}$$

Then for these we have the equation corresponding to (4.8):

$$-\phi_0\phi'+\phi\phi_0'\,|_d=(k^2-k^{02})\int_0^d \phi\phi_0dr. \tag{4.9b}$$

Now normalize the u functions by setting

$$u(d)=\phi(d), \quad u_0(d)=\phi_0(d). \tag{4.10}$$

Add (4.8) and (4.9b) and divide by $\phi(d)\phi_0(d)$:

$$\left(\frac{u'}{u}-\frac{\phi'}{\phi}\right)-\left(\frac{u_0'}{u_0}-\frac{\phi_0'}{\phi_0}\right)\Bigg|_d = \frac{k^2-k_0^2}{\phi(d)\phi_0(d)}\int_0^d(\phi\phi_0-uu_0)dr. \tag{4.11}$$

Now we use the MS idea that d is the separation distance for the momentum k_0, i.e., that

$$u_0'/u_0=\phi_0'/\phi_0 \quad \text{at} \quad r=d. \tag{4.12}$$

For $r>d$, by assumption, $V_s=0$ so that

$$u=[\sin(kr+\delta)]/k, \qquad u'(d)/u(d)=k\cot(kd+\delta) \tag{4.13}$$

and, assuming δ to be small,

$$u'/u-\phi'/\phi\,|_d=-k\delta/\sin^2kd. \tag{4.14}$$

We may define the scattered amplitude by

$$-f(k)=-\frac{\delta}{k}=\frac{\sin^2kd}{k^2}\left(\frac{u'}{u}-\frac{\phi'}{\phi}\right)_d=\phi^2(d)\left(\frac{u'}{u}-\frac{\phi'}{\phi}\right)_d, \tag{4.15}$$

where we have used (4.9a). Now inserting (4.11), we get

$$-f(k)=(\phi/\phi_0)_d(k^2-k_0^2)\int_0^d(\phi\phi_0-uu_0)dr. \tag{4.16}$$

From the definition (4.9a), ϕ and ϕ_0 are obviously very close for $r\leq d$, so (4.16) may be simplified to

$$-f(k)\approx(k^2-k_0^2)\int(\phi_0^2-u_0^2)dr, \tag{4.17}$$

a result very similar to the ordinary effective-range relation.[50] The G matrix is related to f by[51]

$$\langle k|G_s|k\rangle=-4\pi(\hbar^2/M)f = (\hbar^2/M)(k^2-k_0^2)\int 4\pi(\phi_0^2-u_0^2)dr. \tag{4.18}$$

The integral in (4.18) is related to, but is larger than the familiar expression

$$\int\zeta^2d\tau=4\pi\int(\phi_0-u_0)^2dr, \tag{4.19}$$

which represents the probability of finding the nucleon outside the Fermi sea. While (4.19) is about twice the core volume, (4.18) is roughly four times that volume. An estimate is given in Appendix A. A fair approximation is

$$C\equiv\int 4\pi(\phi_0^2-u_0^2)dr\approx\tfrac{2}{3}\pi cd(c+d), \tag{4.20a}$$

where c is the core radius and d the separation distance. In a similar approximation,

$$\int\zeta^2d\tau=\tfrac{2}{3}\pi c^2(c+d). \tag{4.20b}$$

Further, it is reasonable to choose k_0^2 to be the mean-square relative momentum,

$$k_0^2=0.3k_F^2. \tag{4.21}$$

We may now calculate the contribution from short-range forces (exclusive of the dispersion effect) to the

[50] H. A. Bethe, Phys. Rev. **76**, 38 (1949).

[51] The first equality in (4.18) is most easily verified in the Born approximation when $G=\int v d\tau$.

potential energy of a nucleon; this is

$$U_s(k_1)=\tfrac{3}{4}\rho\langle k_1k_2|G_s|k_1k_2\rangle_{\rm av}, \quad (4.22)$$

where the average is taken over k_2, at fixed k_1. Now

$$\mathbf{k}=\tfrac{1}{2}(\mathbf{k}_1-\mathbf{k}_2), \quad (k^2-k_0^2)_{\rm av}=\tfrac{1}{4}(k_1^2-0.6k_F^2), \quad (4.23)$$

therefore, using (4.18), (4.20a), and (4.22),

$$U_s(k_1)=\tfrac{3}{16}\tfrac{2}{3}\pi cd(c+d)\rho(\hbar^2/M)(k_1^2-0.6k_F^2). \quad (4.24)$$

Using $c=0.4$, $d=1.0$ F, $\rho=0.170$ F^{-3}, this gives

$$U_s(k_1)=0.037(\hbar^2/M)(k_1^2-0.6k_F^2). \quad (4.25)$$

On the other hand, the calculations of Dahlblom[11] give, according to (4.5),

$$U(k_1)=0.30(\hbar^2/M)(k_1^2+\text{const}). \quad (4.26)$$

Thus the *short-range force* accounts *only* for about *12% of the total momentum dependence* of the single-particle potential in nuclear matter. This proves the earlier claim that the *chief cause of momentum dependence is the exchange force.*

This situation is not changed by the "dispersion" term of MS. This term is approximately equal to

$$\langle k_1k_2|G_s{}^D|k_1k_2\rangle=\langle e-e_0\rangle_{\rm av}\int\zeta^2d\tau$$
$$=[U(a)+U(b)-U(k_1)-U(k_2)]_{\rm av}\int\zeta^2d\tau, \quad (4.27)$$

where the average is taken over the "particle states," a and b. This average is essentially independent of k_1, k_2. Averaging also over k_2, as in (4.23), we find the dispersion contribution to $U(k_1)$:

$$U_D(k_1)=-\tfrac{3}{4}\rho\left(\int\zeta^2d\tau\right)U(k_1)+\text{const}. \quad (4.28)$$

Using the same constants as in (4.25), this is

$$U_D(k_1)\approx-0.060(U(k_1)+\text{const}). \quad (4.29)$$

We may use this result in three ways, viz.:

(1) If the short-range forces were the *only* contribution to the dependence of U, the dispersion term would only change (4.25) by 6%, so that the agreement with (4.26) is not improved.

(2) For *any* contributions to U, (e.g., those from long-range, exchange forces), the dispersion effect simply reduces the momentum dependence by 6%.

(3) Since (4.26) represents the total momentum dependence, the dispersion effect contributes in fact about

$$U_D(k_1)\approx-0.018(\hbar^2/M)k_1^2. \quad (4.30)$$

This wipes out one-half of the U_s in (4.25). Thus altogether the short-range forces give a negligible contribution to the momentum dependence. They do, on the other hand, give a major contribution to saturation.

We conclude that, in the force model represented by (4.1), the momentum dependence is almost entirely provided by the exchange term. This can be treated in a self-adjoint manner by the standard Hartree-Fock procedure. If there are forces acting only in S states, as was discussed in Sec. 3, these also imply a momentum dependence, and also can be incorporated in a self-adjoint Hamiltonian.

5. TOTAL ENERGY, HARTREE-FOCK EQUATION

The total energy of the nucleus may be written in terms of densities[52]:

$$W=\frac{3}{8}\int v_l(\mathbf{r}_1-\mathbf{r}_2)[\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)+|\rho(\mathbf{r}_1,\mathbf{r}_2)|^2]d\tau_1d\tau_2$$
$$+\frac{1}{2}\int G_s(\rho(\mathbf{r}))\rho^2(\mathbf{r})d\tau+\int T(\mathbf{r})d\tau, \quad (5.1)$$

where

$$\rho(\mathbf{r}_1,\mathbf{r}_2)=\sum_n\phi_n(\mathbf{r}_1)\phi_n{}^*(\mathbf{r}_2) \quad (5.2)$$

is Dirac's mixed-density matrix, the sum going over all occupied states n. Further,

$$\rho(\mathbf{r})=\rho(\mathbf{r},\mathbf{r}) \quad (5.3)$$

is the ordinary density, and

$$T(r)=-(\hbar^2/2M)\phi_n{}^*\nabla^2\phi_n \quad (5.4)$$

is the kinetic-energy density. The first term in (5.1), with $\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)$, is the effect of the "ordinary" long-range interaction [the term 1 in (4.1)], the second term that of the exchange interaction [term P_M in (4.1)]; the factor $\frac{3}{8}$ is made up of a factor $\frac{1}{2}$ which ensures that each interacting pair is counted only once, and a factor $\frac{3}{4}$ for the probability of having either spin or isospin (or both) of the two nucleons different; if they are the same, the antisymmetry of the wave function makes the interaction (4.1) vanish. It should be noted that, according to (4.1), the interaction in even states is $2v_l(r)$. The third term in (5.1) is the short-range contribution[53] (2.2).

We may take the variation of (5.1) with respect to the wave function $\phi_n{}^*(r_1)$ and thus obtain the Hartree-Fock equation,

$$-(\hbar^2/2M)\nabla^2\phi_n(\mathbf{r}_1)+U(\mathbf{r}_1)\phi_n(\mathbf{r}_1)+\int\langle\mathbf{r}_2|U|\mathbf{r}_1\rangle$$
$$\times\phi_n(\mathbf{r}_2)d\tau_2+F(\rho)\phi_n(\mathbf{r}_1)=E_n\phi_n(\mathbf{r}_1), \quad (5.5)$$

[52] Equation (5.1) is the simple LDA, without the modification introduced in Sec. 2(D). We use this for simplicity in Secs. 5 and 6 and then, in Sec. 7, introduce the factor $(1+2\kappa)^{-1}$ derived in (2.40).

[53] This term does not have the factor $\frac{3}{4}$ which appears in the first term of (5.1) which arises from even-state interactions. In the case of short-range forces, there are contributions from both even and odd states, and (2.44) is the complete result.

where

$$U(\mathbf{r}_1)=\frac{3}{4}\int v_l(|\mathbf{r}_1-\mathbf{r}_2|)\rho(\mathbf{r}_2)d\tau_2, \tag{5.6}$$

$$\langle\mathbf{r}_2|U|\mathbf{r}_1\rangle=\tfrac{3}{4}v_l(|\mathbf{r}_1-\mathbf{r}_2|)\rho(\mathbf{r}_1,\mathbf{r}_2)d\tau_2 \tag{5.7}$$

are, respectively, the Hartree (direct) and Fock (exchange) potential. $F(\rho)$ is given by

$$F(\rho)=\frac{1}{2}\frac{d}{d\rho}(\rho^2 G_s(\rho)) \tag{5.8}$$

and depends only on the local density $\rho(\mathbf{r}_1)$. Finally, E_n is the usual Lagrange parameter coming from the condition that the total number of particles be held fixed; E_n is of course the Hartree-Fock energy.

Equation (5.5) is clearly very simple since v_l is a known function of r. It differs from the usual Hartree-Fock equation only by the short-range term (5.8). If we make the assumption (2.6), $G_s\sim\rho^\lambda$, then

$$F(\rho)=(1+\tfrac{1}{2}\lambda)\rho G_s. \tag{5.9}$$

Thus the contribution of the repulsive short-range forces to the effective HF potential is larger than the simple interaction of one nucleon with all others which would be only ρG_s. The extra term $\frac{1}{2}\lambda\rho G_s$ is similar to the results of Brueckner and Goldman,[33] who chose to call this a "rearrangement potential." The entire term (5.9) has the effect of keeping the density at the center of the nucleus from becoming too high; if it is omitted, a large nucleus will collapse.[54]

Of course, (5.5) is as good or as bad as the approximations made in obtaining (4.1). In our opinion, the most serious of these is the neglect of the density dependence of the effective central force which results from the tensor force; this could be included as a factor depending on density[43] which multiplies the two potentials U.

Another important omission is that of the spin-orbit force. Bhargava[55] has shown how an effective "global" spin-orbit force arises from the elementary two-nucleon spin-orbit force, and may be represented by a term

$$C(\rho)(d\rho/dr)\mathbf{l}\cdot\mathbf{s}\psi_n. \tag{5.10}$$

Unfortunately, the interaction $C(\rho)$ so calculated seems too small[28] by about a factor of 2.

6. STATISTICAL THEORY

In this paper, we shall use a statistical approximation to (5.1), rather than the (more elaborate) Hartree-Fock approximation. The aim of the statistical treatment is to get general results for the density distribution, valid for all nuclei of sufficient size, rather than for specific ones.

The first and the third term in (5.1) are straightforward. For the kinetic-energy (last) term, we make the usual Thomas-Fermi approximation,

$$T(r)=\tfrac{3}{5}(\hbar^2/2M)k_F^2(r)\rho(r), \tag{6.1}$$

where the Fermi momentum is related to the density as usual,

$$k_F=(3\pi^2/2)^{1/3}\rho^{1/3}. \tag{6.2}$$

For the mixed density, we make the corresponding approximation which has been used successfully by Slater[25] for a long time in treating the exchange term in the HF theory of atoms. Namely, we consider the wave functions in the neighborhood of the center-of-mass point

$$\mathbf{R}=\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2) \tag{6.3}$$

as locally plane waves. Then

$$\rho(\mathbf{r}_1,\mathbf{r}_2)=\pi^2(\sin k_F r-k_F r\cos k_F r)/r^3, \tag{6.4}$$

where k_F is the Fermi momentum appropriate to R.

We have tested both (6.1) and (6.4) in conditions of varying density. To test (6.1), we chose a one-dimensional potential linear in x,

$$V=Fx. \tag{6.5}$$

The wave functions are then simply Bessel functions of order $\frac{1}{3}$. We assume that all states up to a certain energy, $E=0$, are filled. The density and kinetic-energy density can then be calculated as functions of x, and (6.1) is found to be a good approximation, provided the density is sufficiently high, compared with F [see (6.5)]. If we use the value of the slope of the potential F appropriate for the nuclear surface, we find that (6.1) and (6.2) are correct within about 10%, provided

$$\rho>0.028\ \mathrm{F}^{-3}=0.17\rho_0, \tag{6.6}$$

with ρ_0 the density of nuclear matter. This will be sufficient for our purposes.

This theory is developed in Appendix B. In particular, we show that the density is given by (6.2) with k_F^2 given by $2M\hbar^{-2}(E-V)$. We also develop a WKB theory in the presence of a nonlocal potential, such as the HF potential (5.5).

For the test of (6.4), we calculated[56] numerically the wave functions in a more elaborate and more realistic, but still one-dimensional, potential, viz.,

$$\begin{aligned} V&=-V_0(1-e^{x/a})^2, && x>0\\ V&=0, && x<0. \end{aligned} \tag{6.7}$$

The wave functions are plane waves in the y and z direction, but more complicated in the x direction,

$$\psi=e^{ik_y y+ik_z z}\varphi(x). \tag{6.8}$$

We chose $V_0=40$ MeV, $a=1.2$ F, and the HF energy of

[54] This has been particularly well demonstrated by R. K. Badhuri and E. L. Tomusiak, Nucl. Phys. 88, 353 (1966).

[55] P. Bhargava, thesis, McMaster University, 1966 (unpublished).

[56] This calculation was done by Y. C. Lin, to whom I am very much obliged (see Appendix C).

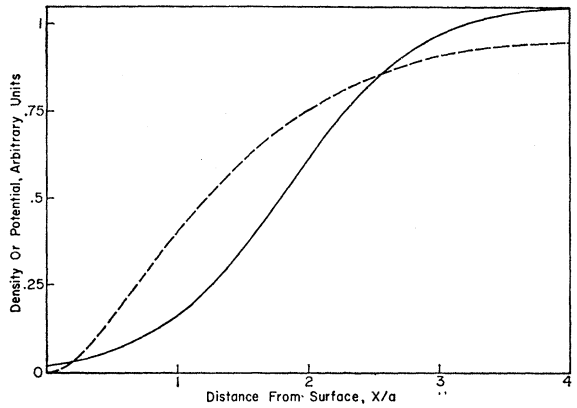

FIG. 2. Density (solid line) versus position x (distance from surface) according to Lin. The particles are assumed to move in the potential given by the dashed line. For details, see Appendix C.

the most energetic nucleon, $E_F=-8$ MeV. Figure 2 shows the density $\rho(x)$ for the potential (6.7). The potential itself is also shown in the figure; the comparison bears out the finding of Berg and Wilets[8] that the potential extends farther out than the density (by about $0.6a$) and has less slope. The mixed density was calculated; we may write

$$\rho(\mathbf{r}_1,\mathbf{r}_2)=\rho(X/a)F(X/a,x/a,t/a)\,, \tag{6.9}$$

where

$$X=\tfrac{1}{2}(x_1+x_2)\,,\quad x=x_1-x_2\,,$$
$$t=[(y_1-y_2)^2+(z_1-z_2)^2]^{1/2}, \tag{6.10}$$

and $\rho(X/a)$ is the ordinary density at X. For given X, the results for F were plotted against r/a, with

$$r=(x^2+t^2)^{1/2}=|\mathbf{r}_1-\mathbf{r}_2|\,. \tag{6.11}$$

Two such plots are shown in Figs. 3(a) and 3(b). It is very satisfactory that the points fall very nearly on a single curve, regardless of the values of x and t: in other words, the mixed density depends essentially only on the distance between $\mathbf{r}_1$ and $\mathbf{r}_2$, independently of the direction of the vector $\mathbf{r}_1-\mathbf{r}_2$, in spite of the large anisotropy of the potential as seen from the center-of-mass point X.

The results are then compared with the Slater theory (6.4), in which

$$F_S=\frac{\rho(\mathbf{r}_1,\mathbf{r}_2)}{\rho(R)}=3\frac{\sin y-y\cos y}{y^3}\,;\quad y=k_F r\,, \tag{6.12}$$

with k_F given by $\rho(R)$. The curves in Fig. 3 give the Slater theory. The agreement with the calculated points is good in Fig. 3(b), which refers to a density $\rho=0.35\rho_0$, where ρ_0 is the nuclear-matter density; the maximum deviation F_S-F is 0.07. In Fig. 3(a), for a density $\rho=0.16\rho_0$, the agreement is only fair (maximum deviation F_S-F about 0.2). As might be expected, the agreement becomes excellent for higher density, and poor for $\rho<0.16\rho_0$. Generally, the Slater formula overestimates the mixed density—except at very large r which, however, are unimportant because both F and v_l are very small.

The calculations show that the Slater approximation is quite good in spite of the rapid variation of the density: As Fig. 2 shows, this varies by $0.45\rho_0$ from $x/a=1$ to 2, i.e., by more than the density at the midpoint, $x/a=1.5$. Nevertheless, Fig. 3(b) shows quite good agreement up to $r/a=5$, perhaps surprising in view of the rapid variation of ρ. We believe this justifies the use of the Slater approximation for practical calculations.

Details of Lin's calculations of F are given in Appendix C.

Accepting then (6.9), (6.12) for the mixed density, the exchange term in (5.1) may be written [$\mathbf{r}=|\mathbf{r}_1-\mathbf{r}_2|$, $\mathbf{R}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$]

$$\frac{3}{8}\int v_l(r)\rho^2(\mathbf{R})F_S^2(\rho(\mathbf{R}),r)d^3Rd^3r$$
$$=\int\rho^2(\mathbf{R})\Phi(\rho(\mathbf{R}))d^3R\,, \tag{6.13}$$

with

$$\Phi(\rho)=\frac{3}{8}\int v_l(r)F_S^2(\rho,r)d^3r\,. \tag{6.14}$$

From (6.12) it is clear that Φ decreases with increasing density; for realistic nuclear potentials v_l, $\rho\Phi$ is approximately constant for ρ between about $\frac{1}{4}\rho_0$ and ρ_0. The important point about (6.13) is that it depends only on *one* coordinate[57] $\mathbf{R}$: The function Φ is a unique function of the density, for any given potential v_l.

Inserting (6.14) into (5.1), and changing the integration variable $\mathbf{R}$ to $\mathbf{r}$, we get

$$W=\frac{3}{8}\int v_l(\mathbf{r}_1-\mathbf{r}_2)\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)d\tau_1d\tau_2$$
$$+\int\{\rho^2(\mathbf{r})[\tfrac{1}{2}G_s(\rho)+\Phi(\rho)]+T(\rho(\mathbf{r}))\}d\tau\,. \tag{6.15}$$

Only the first term depends on the density at two different points, all the others just depend on one ρ. If we assume ρ to be very slowly varying, we may replace $\rho(\mathbf{r}_2)$ in the first integral by $\rho(\mathbf{r}_1)$ and get

$$-V_0\int\rho^2(\mathbf{r})d\tau\,, \tag{6.16}$$

$$V_0=-\frac{3}{8}\int v_l(r)d\tau\,, \tag{6.17}$$

so that

$$W=\int\{\rho^2(\mathbf{r})[-V_0+\tfrac{1}{2}G_s(\rho)+\Phi(\rho)]+T(\rho(\mathbf{r}))\}d\tau\,. \tag{6.18}$$

[57] This was pointed out to me by Mrs. Judith Nemeth.

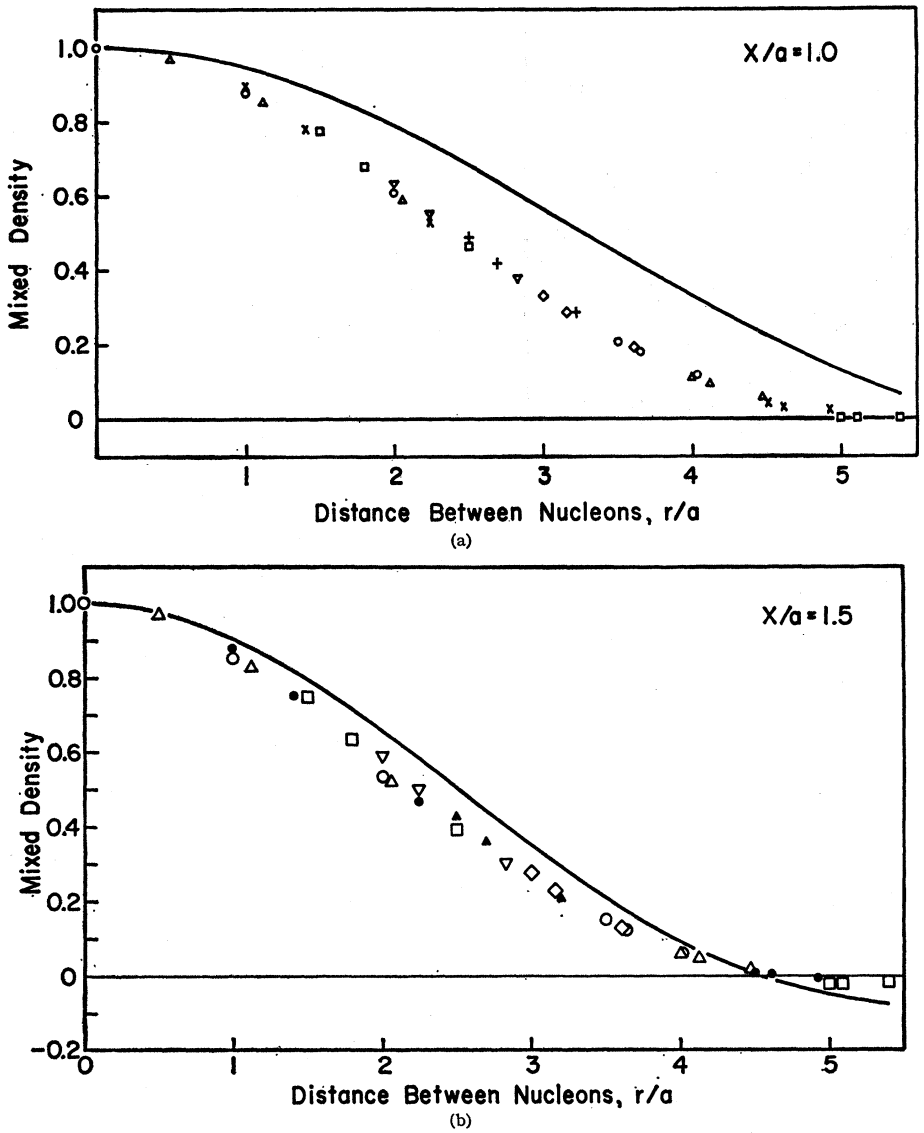

FIG. 3. Mixed density $\rho(\mathbf{r}_1,\mathbf{r}_2)$ versus distance $r=|\mathbf{r}_1-\mathbf{r}_2|$, for fixed center-of-mass position $X=\frac{1}{2}(x_1+x_2)$. Solid line = Slater formula (6.12). Points derived from wave functions in the potential of Fig. 2 by Lin. Symbols correspond to different values of $x=|x_1-x_2|/a$, viz., ○=0.0 or 3.5, △=0.5 or 4.0, ×=1.0 or 4.5, □=1.5 or 5.0, ▽=2.0, +=2.5, ◇=3.0. (a) For density $\rho/\rho_0=0.16$, (b) for density $\rho/\rho_0=0.35$.

Now the curly bracket is the total energy per unit volume for (almost) constant density; we write

$$\{\ \}=\rho W(\rho), \tag{6.19}$$

where $W(\rho)$ is the energy *per particle* at constant density ρ. Using this, (6.15) may be rewritten

$$W=\int W(\rho)\rho(\mathbf{r})d\tau+\frac{3}{8}\int d\tau_1\rho(\mathbf{r}_1)d\tau_2 v_l(\mathbf{r}_1-\mathbf{r}_2)\times[\rho(\mathbf{r}_2)-\rho(\mathbf{r}_1)]. \tag{6.20}$$

The first term in this formula contains $W(\rho)$, the energy at constant density ρ. This may be considered as known from nuclear-matter calculations. In this way, we make maximum use of our knowledge of nuclear matter. The second term contains the variations of density and is therefore only important near the surface. In addition, it contains the long-range potential v_l, also presumed known. It arises entirely from the simplest term in W, the Born-approximation long-range interaction. *All* the complicated parts of the interaction, including the saturation feature, are contained in the nuclear-matter energy $W(\rho)$.

In the form (6.20), we can now finally insert the modification (2.39), viz.,

$$W=\int W(\rho)\rho(\mathbf{r})d\tau+(1+2\kappa)^{-1}\frac{3}{8}\int d\tau_1\rho(\mathbf{r}_1)d\tau_2 v_l(\mathbf{r}_1-\mathbf{r}_2)\times[\rho(\mathbf{r}_2)-\rho(\mathbf{r}_1)]. \tag{6.21}$$

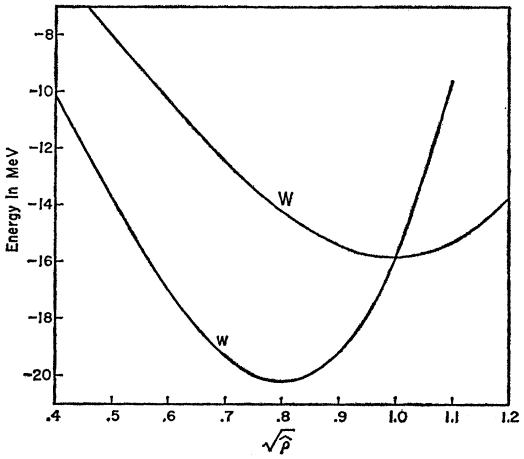

FIG. 4. The energy per particle, W, and the Fermi energy at constant density, w, as functions of the density $\hat{\rho}=\rho/\rho_0$ for nuclear matter, according to Dahlblom's calculations, with his potential energy increased by 18.5%. The two curves intersect at $\hat{\rho}=1$ and at $\hat{\rho}=0$ (not shown). In the region $\frac{1}{4}<\hat{\rho}<1$, which is important for finite nuclei, w is much lower than W.

7. VARIATION, INTEGRAL EQUATION

We now vary $\rho(\mathbf{r})$ in (6.20), with the subsidiary condition

$$\int \rho(\mathbf{r})d\tau = A = \text{const}. \tag{7.1}$$

This yields immediately

$$\frac{d}{d\rho}(\rho W(\rho))_{r_1} + \frac{3}{4}\int d\tau_2 v_l(\mathbf{r}_1-\mathbf{r}_2)[\rho(\mathbf{r}_2)-\rho(\mathbf{r}_1)] = E_F. \tag{7.2}$$

The first term is obviously the change of energy when one particle is added to a piece of nuclear matter of constant density ρ. By definition, this is the "local Fermi energy" at constant density, $w_F(\rho)$, which we introduced in (2.2); so we have

$$w_F(\rho) = \frac{d}{d\rho}(\rho W(\rho)) = W(\rho) + \rho\frac{dW}{d\rho}. \tag{7.3}$$

In the following, we shall drop the subscript F in w_F and we call $w(\rho)$ the "one-particle energy." The second term in (7.2) is the only remaining deviation from constant-density nuclear matter. As was shown in Sec. 2 D, this should be reduced by a factor $(1+2\kappa)^{-1}$ so that (7.2) is replaced by

$$w(\rho(r_1)) + \frac{3}{4}\frac{1}{1+2\kappa}\int d\tau_2 v_l(\mathbf{r}_1-\mathbf{r}_2) \times[\rho(\mathbf{r}_2)-\rho(\mathbf{r}_1)] = E_F. \tag{7.4}$$

The right-hand side of (7.2) and (7.4) is, by definition of the Lagrange parameter, the change of energy when one nucleon is added to the nucleus, and hence the energy of the most energetic nucleon, i.e., the Fermi energy E_F. For a given nucleus, E_F is a constant, independent of $\mathbf{r}_1$.

The one-particle energy $w(\rho)$ is not identical with the energy per nucleon $W(\rho)$, as is clear from (7.3). Near the equilibrium density of nuclear matter, ρ_0, we may write

$$W(\rho) = W_0 + \tfrac{1}{2}C(\hat{\rho}-1)^2 + O(\hat{\rho}-1)^3, \tag{7.5a}$$

where

$$\hat{\rho} = \rho/\rho_0 \tag{7.5b}$$

will be a useful abbreviation for the following. C is related to the usual compression modulus K,

$$K = r_0^2(d^2W/dr_0^2) = 9C. \tag{7.6}$$

According to the calculations of Dahlblom,[11] K is about 157 MeV, hence $C\approx 17.5$ MeV. Using (7.3) and (7.4),

$$w(\rho) = W_0 + C(\hat{\rho}-1) + O(\hat{\rho}-1)^2. \tag{7.7}$$

Hence w is linear in $\hat{\rho}$ near nuclear-matter density, $\hat{\rho}=1$, and $w<W_0$ for $\hat{\rho}<1$, which is the normal case in finite nuclei. The minimum of w is at a density less than nuclear-matter density, and is lower than W_0. This difference between w and W will be essential in Sec. 8.

We have used Dahlblom's nuclear-matter calculations.[11] In order to get the correct binding energy (15.85 MeV), we have arbitrarily multiplied his potential energies by 1.185. Then the minimum of total energy occurs at $k_F=1.44$ F^{-1}, as compared to the semiempirical value[27] of 1.36 F^{-1}. The binding energy is represented very well by the analytic formula

$$W = -15.85 + 47.5y^2 + 32.5y^3, \tag{7.8a}$$

$$y = \hat{\rho}^{1/2} - 1. \tag{7.8b}$$

This fits Dahlblom's values, corrected by the factor 1.185, within 0.1 MeV over the range from $k_F=0.7$ to 1.7. The use of $\rho^{1/2}$ gives a much better fit than a similarly simple formula in ρ or in $k_F\sim\rho^{1/3}$. Moreover, (7.8a) is more convenient for integration of the differential equation in Sec. 8. For a rough approximation (to 0.5 MeV from $k_F=0.9$ to 1.7) we may use

$$W = -15.85 + 33y^2 \equiv W_0 + 2Cy^2, \tag{7.9}$$

which agrees with (7.8a) at $\hat{\rho}=\frac{1}{3}$ and 1. Inserting (7.8a) in (7.3) gives

$$w = -15.85 + 47.5y + 144y^2 + 81y^3. \tag{7.10}$$

Figure 4 compares w and W. It shows that w has a minimum of about -20.2 MeV at $\rho=0.64$, and w is again equal to $W_0=-15.85$ at $\rho_1=0.31$. If W is given by (7.9) the minimum of w is at $\hat{\rho}=0.56$ and is -20.0 MeV; $w=W_0$ at $\hat{\rho}_1=0.25$.

Near the center of an extremely large nucleus, the density will be constant, and equal to ρ_0. Hence the second term in (7.2) is zero, and the first term is W_0 so that

$$E_F = W_0 \tag{7.11}$$

if A is very large. We may then consider small deviations of ρ from ρ_0 which will occur near the surface of a very large nucleus. Such a surface may be considered as plane so that ρ will be a function of x only, where x is the coordinate perpendicular to the surface. Inserting then (7.7) into (7.2) we get the integral equation

$$C(\hat{\rho}-1)+\frac{3}{4}\frac{\rho_0}{1+2\kappa}\int d\tau_2 v(r_{12})[\hat{\rho}(x_2)-\hat{\rho}(x_1)]=0, \quad (7.12)$$

$$C(\hat{\rho}(x_1)-1)+\int dx_2 K(|x_1-x_2|) \times[\hat{\rho}(x_2)-\hat{\rho}(x_1)]=0, \quad (7.13)$$

with

$$K(s)=\frac{3\pi}{2}\frac{\rho_0}{1+2\kappa}\int_s^\infty v(r)rdr. \quad (7.14)$$

Equation (7.13) is a linear, homogeneous integral equation for $\hat{\rho}-1$ with a displacement kernel $K(x_1-x_2)$. Any such equation has as a solution an exponential,[58,59]

$$1-\hat{\rho}(x)=e^{-\alpha x}, \quad (7.15)$$

where the constant α can be determined from the kernel K and the constant C. Mrs. Nemeth has determined $\alpha=0.64$ F^{-1}, with certain assumptions on the nuclear forces, using Sprung and Bhargava's[10] nuclear-matter calculations, and neglecting κ. Including κ, the value of α is likely to be larger, about 0.7 to 0.75 F^{-1}.

The solution (7.15) is of course only valid as long as $1-\hat{\rho}\ll 1$. It shows, however, that (in this Thomas-Fermi model) the nuclear-matter density is approached by a "saturation function", $1-e^{-\alpha x}$, as we go into the nucleus. This behavior is *qualitatively* the same as that of the so-called "Fermi density distribution"

$$\rho_F=[1+e^{(r-R)/s}]^{-1}\approx 1-e^{(r-R)/s}, \quad (7.16)$$

but here experimentally $s\approx 0.6$ F, much smaller than $1/\alpha$. This will be discussed in more detail in Sec. 8.

For x approaching 0, (7.15) will cease to be valid because $U(\rho)$ can no longer be represented by $W_0+C(\hat{\rho}-1)$. Then the integral equation (7.13) must be solved explicitly. We have not considered this worthwhile because of the very crude assumptions about the potential. Better assumptions are made in the paper by Nemeth and Bethe.[43]

At still smaller x, ρ becomes smaller than 0.15, the satistical theory ceases to be valid, and the density must be derived from explicit calculation of the wave functions.

8. DIFFERENTIAL EQUATION

A. Approximations Made

To simplify the problem further, we approximate the integral equation (7.13) by a differential equation. We consider the density ρ as slowly variable compared with the range of the nuclear force. This is not a very good assumption because the thickness of the nuclear surface is of the same order as the range of the nuclear force, but we shall see that we have some numerical factors in our favor. However, we must be prepared for considerable errors in this approximation.

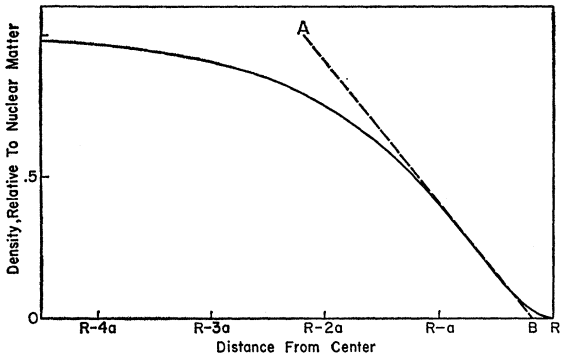

FIG. 5. Theoretical density distribution from the differential theory of Sec. 8 versus r. AB is the tangent drawn at the point of steepest slope, the distance r_B-r_A is $b=2a$.

If ρ is slowly variable we expand $\rho(\mathbf{r}_2)$ in (7.4) in the neighborhood of r. The linear term in $\mathbf{r}_2-\mathbf{r}_1$ obviously contributes nothing, and using (7.3) and (7.5a), Eq. (7.4) becomes

$$B\nabla^2\hat{\rho}=w(\hat{\rho})-E_F, \quad (8.1)$$

with

$$B=-\frac{1}{6}\frac{3}{4}\frac{\rho_0}{1+2\kappa}\int v_l(r)r^2d\tau. \quad (8.2)$$

B has been defined so as to be positive; it is a constant characterizing the long-range potential. Since

$$\tfrac{3}{8}\rho_0\int v_l(r)d\tau\equiv V_l \quad (8.3)$$

is the long-range contribution to the potential energy per nucleon at normal density ρ_0, we may write

$$B=-V_l\langle r^2\rangle/3(1+2\kappa). \quad (8.4)$$

We shall now again assume a plane surface, with x the coordinate perpendicular to it; thus (8.1) is

$$Bd^2\hat{\rho}/dx^2=w(\hat{\rho})-E_F\approx w(\hat{\rho})-W_0, \quad (8.5)$$

where in the last member we have assumed that the nucleus is very large so that $E_F\approx W_0$. Equation (8.5) shows immediately that the curvature of ρ versus x is negative if $w(\rho)<W_0$. A negative curvature is required to make $\hat{\rho}$ drop from the central value $\hat{\rho}=1$ to lower values at the surface (cf. Fig. 5). It is therefore very important that w indeed falls below W_0 when $\hat{\rho}<1$, as shown by (7.7) and Fig. 4. If w were replaced by W, it would always be $>W_0$, Eq. (7.4), and we could never get a negative curvature of $\rho(x)$. Thus it is essential that w and W are not identical, Eq. (7.3).

[58] No constant factor is needed on the right-hand side of (7.15) because the zero of x can be adjusted suitably.

[59] A similar solution was found in the Thomas-Fermi theory of Seyler and Blanchard, Ref. 9.

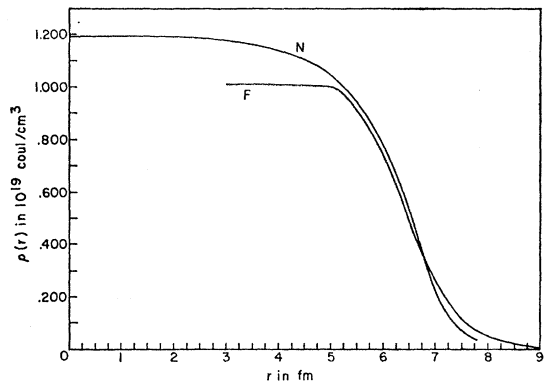

FIG. 6. Density versus radius for Au. Density in units of 10^{19} coulomb/cm^3, radius in fermis. F=Fermi distribution, N=our distribution, Eq. (8.32). For discussion, see Sec. 9 C.

Equation (8.5) can immediately be integrated once to give, using (7.3),

$$\tfrac{1}{2}B\left(\frac{d\hat{\rho}}{dx}\right)^2=\int_1^{\hat{\rho}} d\hat{\rho}[w(\hat{\rho})-W_0]=\hat{\rho}(W(\hat{\rho})-W_0). \quad (8.6)$$

Thus the slope of the $\hat{\rho}(x)$ curve is directly related to the energy per nucleon, $W(\rho)$, which follows from nuclear-matter theory. The steepest slope occurs of course at the inflexion point $\hat{\rho}_1$, which is defined by [cf. (8.5)]

$$w(\hat{\rho}_1)=W_0. \quad (8.7a)$$

As discussed below (7.10), with the approximate formula (7.8a), we have

$$\hat{\rho}_1=0.31. \quad (8.7b)$$

This is quite different from the Fermi-type distribution (7.16) which has its inflextion point at $\hat{\rho}_{F1}=0.5$. The distributions themselves are compared in Fig. 6.

B. Steepest Slope

The steepest slope is given by (8.6), with $\hat{\rho}=\hat{\rho}_1$. It may be used to define an effective surface thickness,

$$b=(d\hat{\rho}/dx)^{-1}. \quad (8.8)$$

To interpret this, draw the tangent to the $\rho(x)$ curve at the inflexion point (Fig. 5); its intersections with the lines $\hat{\rho}=1$ and $\rho=0$ have a distance $\Delta x=b$. From (8.6),

$$b=[B/2\hat{\rho}_1(W(\hat{\rho}_1)-W_0)]^{1/2} \quad (8.9)$$

and using (8.4),

$$b=[|V_l|/6(1+2\kappa)\hat{\rho}_1(W(\hat{\rho}_1)-W_0)]^{1/2}\langle r^2\rangle^{1/2}. \quad (8.10)$$

Obviously, $W(\hat{\rho}_1)-W_0$ is not very large; for W as given by (7.8a),

$$W(\hat{\rho}_1)-W_0=6.4 \text{ MeV}. \quad (8.11)$$

By contrast, $|V_l|$ is the main part of the potential energy per nucleon in nuclear matter of full density. The total potential energy per nucleon at density ρ_0 is[60] 42 MeV; therefore $|V_l|$ should be at least[61] 31 MeV. Using also (8.7b),

$$b=1.6\langle r^2\rangle^{1/2}(1+2\kappa)^{-1/2}. \quad (8.12)$$

Thus b is considerably larger than $\langle r^2\rangle^{1/2}$, which in turn is a rather large measure of the range of the nuclear forces. This confirms our earlier statement that numerical factors make the thickness of the nuclear surface rather larger than the range of nuclear forces, hence giving some justification for the use of the differential equation. The reason for this is that $U(\hat{\rho})-W_0$ in (8.5) is never very large, thus keeping down the curvature of $\hat{\rho}(x)$.

To estimate $\langle r^2\rangle$ we use the Reid hard-core potential[35,62] of 1965, in the ^1S state; this is

$$V=-(10.46 \text{ MeV}/x)(e^{-x}+39.63e^{-3x}) \quad \text{for } x>0.296 \quad (8.13)$$
$$x=\mu r, \quad \mu=0.70 \text{ F}^{-1}.$$

Here $1/\mu$ is the pion Compton wavelength. The first term in the parenthesis of (8.13) is the OPEP, the second is an empirical approximation to the exchange of heavier particles. For $x<0.296$, there is a repulsive core. We take only the long-range part of (8.13), for $r>d$, and assume the separation distance to be $d=1.0$ F so that

$$x_0=\mu d=0.70. \quad (8.14)$$

The we get

$$\mu^2\langle r^2\rangle = \frac{e^{2x_0}(x_0^3+3x_0^2+6x_0+6)+13.21(x_0^3+x_0^2+\frac{2}{3}x_0+2/9)}{e^{2x_0}(x_0+1)\qquad\qquad +13.21(x_0+\frac{1}{3})}. \quad (8.15)$$

Using the entire expression, we get $\langle r^2\rangle=6.84$ F^2. Inserting this and $\kappa=0.14$ into (8.12),

$$b=3.7 \text{ F}. \quad (8.16)$$

This is much too large; the observed "surface thickness" (Sec. 9) is about $b=2.4$ F.

We believe, however, that the OPEP contribution should be omitted. As is well known, OPEP is attractive in even-, repulsive in odd-parity states, and zero on the average; it therefore should be considered an exchange force only, and does not contribute to the ordinary force. Accordingly, we should omit the first

[60] We take $k_F=1.44$ F^{-1}, the Dahlblom equilibrium value, hence average kinetic energy=25.8 MeV, and the binding energy 15.9 MeV.

[61] We assumed in (5.1) that there are ordinary and Majorana attractive forces, and short-range repulsive ones. The repulsive energy may be about 5 MeV, the total *attractive* potential is then 47 MeV, and we estimate that the ordinary forces contribute twice as much to this as the Majorana ones.

[62] R. V. Reid, Ref. 34. We should really have used his soft-core potential, but the general conclusions are the same.

term in both numerator and denominator of (8.15), which gives

$$\mu^2\langle r^2\rangle=\tfrac{2}{3}+x_0^2(x_0+1)/(x_0+\tfrac{1}{3}),$$
$$\langle r^2\rangle=3.007\ \mathrm{F}^2, \qquad (8.17)$$
$$b=2.47\ \mathrm{F}.$$

This is in very good agreement with the observed $b=2.4$ F. This agreement, however, is probably accidental. Our assumptions about the nuclear forces are very rough. They will be improved by Nemeth and Bethe[43]; unfortunately, improved forces seem to increase b again. On the other hand, solution of the integral equation (7.12) instead of the differential equation (8.5) helps to decrease b.

Equation (8.12) shows that b is proportional to $(1+2\kappa)^{-1/2}$ where κ, given by (2.36), is the probability of finding a nucleon out of its model state. In accord with Sec. 2, we have assumed $\kappa=0.14$; this reduces b by about 12%. Since b tends to be too large, this reduction improves agreement with observation. This then is an important consequence of the modification of the LDA which we introduced in Sec. 2 D. The long-range forces are effectively diminished in strength, and this makes the surface of the nucleus thinner.

C. Radial Distribution

Until now, we did not need to make any assumptions about the form of the function $W(\rho)$ in (8.6); we merely needed its value at ρ_1, the inflexion point of the curve $\rho(x)$. To determine this curve itself, we must integrate (8.6) once more. This is a simple quadrature which can still be done for arbitrary $W(\rho)$; however, to get a simple analytical answer, we now use the simple formula (7.9) for W, and (8.6) becomes

$$B(dy/dx)^2=Cy^2. \qquad (8.18)$$

This integrates immediately to give

$$y=-e^{-x/a}, \qquad (8.19)$$

$$a=(B/C)^{1/2}. \qquad (8.20)$$

The negative sign in (8.19) is supplied because by its definition (7.8b), $y<0$, when the density is less than nuclear-matter density. The minus sign in the exponent is arbitrary; it means that the nucleus is in the half-space $x>0$ while

$$\rho=0 \quad \text{for} \quad x<0; \qquad (8.21)$$

this joins continuously to (8.19).

The length scale a may be compared with b. Inserting (7.9) into (8.9), and remembering that with (7.9), we have $\hat{\rho}_1=\frac{1}{4}$, hence $y(\hat{\rho}_1)=\frac{1}{2}$, we get

$$b=2a. \qquad (8.22)$$

The empirical value for b, mentioned above, thus means $a=1.2$ F. Inserting (8.19) into (7.8b) gives the density distribution

$$\rho/\rho_0=\hat{\rho}=(1-e^{-x/a})^2. \qquad (8.23)$$

This is plotted in Fig. 5. In a spherical nucleus, we may identify $x=R-r$, with R some definition of the nuclear radius, then[63]

$$\rho/\rho_0=1-\exp[(r-R)/a]^2 \quad r<R$$
$$=0 \quad r>R. \qquad (8.24)$$

It can be directly verified that the steepest slope occurs at the point $\hat{\rho}_1=\frac{1}{4}$, and that at this point $d\hat{\rho}/dx=1/b=1/2a$. The tangent at the point of steepest slope is shown in Fig. 5.

Equation (8.24) is our final result, except for a tail on the distribution, which we shall discuss later. As (8.24) stands, the density terminates at $r=R$, but it does so with zero slope. As distinct from the "Fermi distribution" (7.16), our result (8.24) is not symmetrical about the half-density point $\hat{\rho}=\frac{1}{2}$. Indeed, as we have pointed out, the steepest slope occurs at a lower density, $\hat{\rho}_1=\frac{1}{4}$. This is directly related to the behavior of $W(\rho)$ and $w(\rho)$ as discussed in Sec. 7. Since we believe this behavior to be (qualitatively) well established, we are confident that our "unsymmetrical" density distribution should be closer to the truth than the symmetrical Fermi type. Use of the integral equation (7.4) instead of (8.5) shifts the point of maximum slope to lower density[43] than $\hat{\rho}_1$. On the other hand, with the more accurate expression (7.8a) for W, we have $\hat{\rho}_1=0.31$. We believe that the effect of the integral equation is stronger, and that the steepest slope occurs at $\hat{\rho}<0.25$.

In line with its "asymmetry," (8.24) approaches full density much more slowly than the Fermi distribution (7.16). Comparison with electron experiments (Sec. 9) shows that the steepest slope of (7.16) should be chosen to be about the same as for (8.24), in order to get agreement. This requires $s=b/4$; therefore for $R-r\gg s$ (deep in the nucleus), (7.16) becomes

$$\rho_F\approx 1-e^{-4(R_F-r)/b}, \qquad (8.25)$$

while (8.24) is

$$\rho_B=1-2e^{-2(R_B-r)/b}. \qquad (8.26)$$

The notation R_F, R_B is to indicate that the nuclear radius must be chosen differentialy for the two distributions, in order to fit experiment. Disregarding this difference and the factor 2 in front in (8.26), we see that the Fermi distribution approaches unity twice as fast as ours. Figure 6 illustrates this point. The very gradual approach to saturation density (as we go toward the center of the nucleus) is in accord with the results of Brueckner *et al.*[3] for finite nuclei.

The radius R introduced in (8.24) is of course larger than any reasonable definition of a nuclear radius. We

[63] A distribution of the same shape has been discussed by W. J. Swiatecki and W. D. Myers (private communication); see also Nucl. Phys. **81**, 1 (1966).

may, e.g., define an effective radius by

$$\tfrac{4}{3}\pi R_{\rm eff}{}^3 = A/\rho_0 , \tag{8.27}$$

where A is the total number of particles. Then

$$\tfrac{1}{3}R_{\rm eff}{}^3 = \tfrac{1}{3}R^3 - \tfrac{3}{2}R^2 a + \tfrac{1}{2}Ra^2 - (15/4)a^3 + 4a^3 e^{-R/a} - \tfrac{1}{4}a^3 e^{-2R/a} \tag{8.28}$$

in which the last two terms are normally negligible. Approximately,

$$R_{\rm eff} \approx R - \tfrac{3}{2}a . \tag{8.29}$$

D. Exponential Tail

Beyond R, the TF theory is not valid. Instead, all the nucleon wave functions have exponential tails which are discussed in Appendix B. The asymptotic behavior of the density is given by the exponential tail; thus

$$\rho = Ce^{-\alpha(r-R)} , \tag{8.30}$$

where C is a constant and

$$\alpha = (8M\epsilon)^{2/2}/h , \tag{8.31}$$

with ϵ some suitable average binding energy. Taking[64] $\epsilon = 16$ MeV,

$$\alpha = 1.76\ \text{F}^{-1} . \tag{8.32}$$

As a simple approximation, we have assumed

$$\begin{aligned} \rho &= (8.24) \quad \text{for} \quad r < R - a\ln 2 , \\ \rho &= (8.30) \quad \text{for} \quad r > R - a\ln 2 , \end{aligned} \tag{8.33}$$

with C fixed by continuity of ρ. The derivative $d\rho/dr$ is then almost continuous.

E. Comparison with the Theory of Wilets

Our theory has much similarity with the phenomenological theory of Wilets.[8] He started from the observed surface thickness and surface energy, and then derived a suitable phenomenological potential. This potential turns out very similar to ours; e.g., compression modulus $K = 175$ MeV (our value 157 MeV), radius $r_0 = 1.07$ F (Dahlblom's calculation, after increase of the potential energy by 18.5%, gives $r_0 = 1.06$ F) [cf. Sec. 7 of our paper, near Eq. (7.7)].

To our gradient term in the energy [cf. (10.14) and (10.15)]

$$-\tfrac{1}{2}B\rho_0 \int \hat{\rho}\nabla^2\hat{\rho}\, d\tau = \tfrac{1}{2}B\rho_0 \int (\nabla\hat{\rho})^2 d\tau \tag{8.34}$$

corresponds in Wilets' theory:

$$\xi\frac{\hbar^2}{8M}\int \frac{(\nabla\rho)^2}{\rho} d\tau = \xi\frac{\hbar^2}{8M}\rho_0 \int \frac{(\nabla\hat{\rho})^2}{\hat{\rho}} d\tau . \tag{8.35}$$

Wilets (private communication) finds that an average of $\hat{\rho} = \frac{1}{3}$ in (8.35) is appropriate, and $\xi = 0.65$. Then our B should correspond to

$$\xi(3\hbar^2/4M) = 20\ \text{MeV} , \tag{8.36}$$

whereas our B, from (8.9) or (8.20), is 24 MeV. The agreement is very satisfactory.

Wilets obtains a radial distribution

$$\hat{\rho} = (\hat{\rho}_F)^n , \tag{8.37}$$

where $\hat{\rho}_F$ is the "Fermi" distribution (7.16) and $n = 1.28$. Thus his distribution is unsymmetrical, but not as much as ours.

9. COMPARISON WITH ELECTRON SCATTERING

The best way to determine the thickness of the nuclear surface is from the scattering of electrons of 100–300 MeV.[65] Accordingly, we have initiated two calculations of the electron scattering by heavy elements (Au,Pb) by the density distribution calculated in Sec. 8.

(*a*) *Approximate calculation.* An approximate calculation was carried out by Lin.[66] He used the density distribution *with* an exponential tail described in (8.33) and calculated the scattering of 236-MeV electrons by Au. For the calculation, he used the method of Yennie, Boos, and Ravenhall,[67] which is an adaptation of WKB to relativistic electron scattering, and avoids the tedious summation of many partial waves which is needed in the standard method[65] and which requires extremely high accuracy of computation. Following Dr. Yennie's advice, Lin did not compare his results directly with experiment but with the theoretical result for the standard Fermi-type distribution,

$$\rho = (1 + e^{(r-R')/a'})^{-1} , \tag{9.1}$$

with $R' = 6.29$ F, $a' = 0.55$ F. In this way, errors inherent in the YBR theory were thought to be minimized.

The result is shown in Fig. 7, for 3 choices of the parameters

$$\begin{aligned} &(1) \quad R = 7.56\ \text{F}, \quad a = 1.20 , \\ &(2) \quad R = 7.45\ \text{F}, \quad a = 1.20 , \\ &(3) \quad R = 7.56\ \text{F}, \quad a = 1.0 . \end{aligned} \tag{9.2}$$

In order to bring out the maxima and minima more

[64] At *very* large distances, we should take ϵ to be the binding energy of the least-bound nucleon, which is 8 MeV, for an average nucleus. At intermediate distances, say $\alpha(r-R) = 1$ to 2, the more strongly bound nucleons also contribute so that 16 MeV may be a reasonable average. For protons (which are measured by electron scattering), the Coulomb potential further increases ϵ. These points will be further investigated. $\epsilon = 16$ MeV is probably not a bad average.

[65] B. Hahn, D. G. Ravenhall, and R. Hofstadter, Phys. Rev. **101**, 1131 (1956).

[66] Y. C. Lin (private communication).

[67] D. R. Yennie, F. L. Boos, and D. G. Ravenhall, Phys. Rev. **137**, B882 (1965).

clearly, we have plotted

$$Q=\theta^{11}d\sigma/d\Omega, \tag{9.3}$$

with θ measured in radians; this just happens to make the order of magnitude of Q reasonably independent of θ, for our particular energy and Z. Choice 1 of (9.2) is obviously closest to the Fermi curve and is, in fact, quite close up to $\theta=110°$. Experiments extend only to about 80°. The sensitivity to the parameters can be seen from the other two curves: The smaller radius, choice (2), displaces maxima and minima too far to large angles. The thinner surface, choice (3), makes the maxima too high, especially the two beyond 80°. In deciding about the parameters, we[68] considered the lower angles (35°–60°) as most significant; this rules out parameters (2). The behavior between 60° and 80° favors (1) as against (3), although an intermediate value, like $a=1.15$, might do even better. The last maximum, at 130°, even with our favored choice (1), comes out higher than with the Fermi distribution: This difference is probably significant, and could be used to discriminate between the distributions once experiments at these very low cross sections (about 10^{-33} cm^2) become available.

(*b*) *Exact calculation.* An exact calculation was kindly carried out by Elton.[69] He used exact phase shifts, but he assumed the distribution to be given by (8.24) for *all* r, i.e., he did not include an exponential tail. The best agreement with the scattering of electrons of 153 and 183 MeV by Au was obtained (Ref. 69, Fig. 1) for

$$R=7.56, \quad a=1.0 \text{ F}. \tag{9.4}$$

The results for $a=1.2$ were definitely less good. On the other hand, in a later analysis of new experiments on Pb, Elton[70] obtained $a=1.2$ F.

(*c*) *Resulting density distribution.* The density distribution corresponding to $R=7.56$ F, $a=1.20$ F is plotted in Fig. 6, together with that for the Fermi-type distribution (9.1). The two curves are seen to coincide closely from about 5.5 to 7 F. In particular, the steepest slope is nearly the same, and the radius where it occurs is also. However, there are distinct differences. The most important of these is that our curve I approaches full density much more slowly than the Fermi curve, as already pointed out in (8.26). Consequently, since the behavior near R is similar in the two curves, and since both distributions are normalized to give the same total number of protons in the nucleus, our curve goes to a higher density in the center than the Fermi curve. Further, the mean square radius will be less in our theory than with the Fermi distribution [see (*d*), below].

[68] I am indebted to Dr. D. Yennie for advice on this point.

[69] L. R. B. Elton, in Proceedings of the Conference on High-Energy Physics and Nuclear Structure, Rehovoth, 1967 (unpublished).

[70] L. R. B. Elton, Proceedings of the International Conference on Electromagnetic Sizes of Nuclei (unpublished).

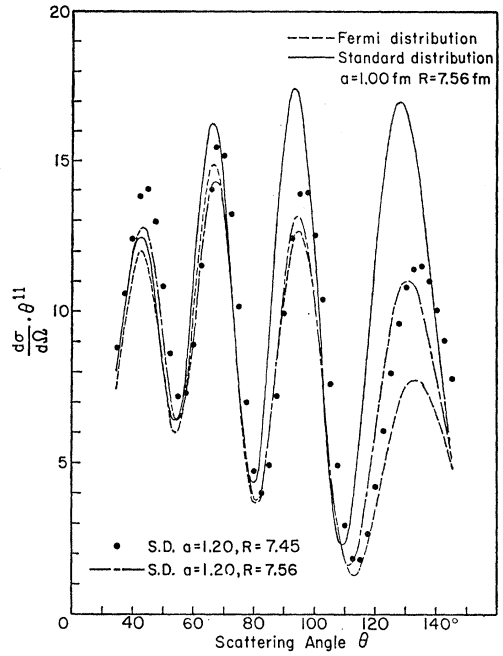

FIG. 7. Scattering of electrons of 236 MeV by Au. Ordinate is $\theta^{11}\,d\sigma/d\Omega$, where θ^{11} is the scattering angle in radians, and $d\sigma/d\Omega$ the differential cross section in μb/sr. Calculations by Lin, using the theory of Yennie, Boos, and Ravenhall, for various radial proton distributions of type (8.32). Solid line (1) $a=1.2$, $R=7.56$ F; dots (2) $a=1.2$, $R=7.45$; long- and-short-dashed line (3) $a=1.0$, $R=7.56$ F; dashed line shows the conventional Fermi distribution.

It has been popular[65] to define the "thickness" as the distance in which the density rises from 10 to 90% of the central density. This is definitely *not* a good measure for our distribution which approaches saturation so slowly. If we use (8.33) with $a=1.2$, we get

$$t_B=r_{10}-r_{90}=2.27\alpha+0.91/\alpha=3.24 \text{ F} \tag{9.5}$$

as compared with the Fermi-distribution result

$$t_F=1.10b=2.64 \text{ F}, \tag{9.6a}$$

assuming[71] $b=2.4$ F; the value usually quoted is only

$$t_F=2.4 \text{ F}. \tag{9.6b}$$

The comparison of calculated electron scattering shows that the steepest slope of $\rho(x)$ should be about the same for our and the Fermi distribution, and that the 10–90% thickness is not a good measure.

(*d*) *Muonic x rays.* Elton[69] has also calculated the "atomic" energy levels of muons in the Coulomb field of a Au^{197} nucleus. The agreement is very poor (Ref. 70, Fig. 3): The $K\alpha_1$ line, corresponding to the transition $2p_{3/2}-1s_{1/2}$, is

experimentally	5.76 MeV,
theoretically for $a=1.2$	6.11 MeV.

[71] As in (8.8), b is defined as the reciprocal of the steepest slope of $\rho(x)$. For the Fermi distribution (7.16), $t=4s$.

Thus our nuclear charge distribution makes the nucleus too close to a point charge; it gives too small an $\langle r^2\rangle$. The Fermi distribution does much better (5.87 MeV) because it gives a larger $\langle r^2\rangle$, as pointed out above. Elton has shown, in agreement with previous authors, that the muonic x-ray energies measure essentially $\langle r^2\rangle$.

This situation will undoubtedly be improved when the exponential tail (8.30) is included, which Elton did not do. Unfortunately, this seems to explain at best one-half of the discrepancy. Another effect in the right direction is presumably the "wine-bottle shape" of the proton distribution, i.e., greater density of protons near the surface, due to Coulomb interaction. Treatment of this effect requires simultaneous treatment of the "symmetry energy," i.e., of the preferential interaction between unlike nucleons.

(e) *Comparison of surface thickness with theory.* The analysis of electron-scattering experiments has given $a=1.0$ to 1.2 F. In (8.17) we calculated $b=2a=2.47$ F, in good agreement with the larger experimental value. Unfortunately, attempts to improve on the assumed nuclear forces[43] have so far given a larger value for b than 2.7. Moreover, the experimental surface thickness refers to the charge, and when allowance is made for the finite radius of the proton (0.8 F), the experimental thickness should be corrected down, to perhaps 2.0–2.2 F.

Thus the theory tends to give a somewhat too large surface thickness. A similar result was found by Hara,[5] but some other, earlier attempts have given too small a thickness. Our large result is partly due to the rather slow increase of nuclear-matter energy $W(\rho)$ when the density falls below normal nuclear-matter density ρ_0, i.e., due to the relatively small compression modulus, $K=150$ MeV, which corresponds to (7.9). [In the original Brueckner calculations, K was over 200 MeV. Our more accurate Eq. (7.8a) corresponds to $K=210$ MeV but still gives $b=2.47$ F.]

10. SURFACE ENERGY

Equation (6.21) gives the total energy of the nucleus. We separate the surface from the volume energy by subtracting (7.1) multiplied by W_0, giving for the surface energy

$$W_s\equiv W-AW_0=\int(W(\rho)-W_0)\rho d\tau + \tfrac{3}{8}(1+2\kappa)^{-1}\int d\tau_1\rho(\mathbf{r}_1)\int d\tau_2 v_t(\mathbf{r}_1-\mathbf{r}_2) \times[\rho(\mathbf{r}_2)-\rho(\mathbf{r}_1)]. \quad (10.1)$$

Here we use (7.4) and find

$$W_s=\int[W(\rho)-W_0+\tfrac{1}{2}E_F-\tfrac{1}{2}w(\rho)]\rho(r)d\tau. \quad (10.2)$$

We write this

$$W_s=\tfrac{1}{2}(E_F-W_0)A+\int[W(\rho)-\tfrac{1}{2}W_0-\tfrac{1}{2}w(\rho)]\rho d\tau. \quad (10.3)$$

For a very large nucleus, the surface may be considered as plane, of area S, which reduces the integral to a one-dimensional one over x, the coordinate perpendicular to the surface:

$$W_s=S\int[W(\rho)-\tfrac{1}{2}W_0-\tfrac{1}{2}w(\rho)] \times\rho(x)dx+\tfrac{1}{2}A(E_F-W_0). \quad (10.4)$$

The surface energy is thus proportional to S, i.e., to $A^{2/3}$, and we may write

$$W=AW_0+\gamma A^{2/3}. \quad (10.5)$$

From its definition as a Lagrange parameter,

$$E_F=\partial W/\partial A=W_0+\tfrac{2}{3}\gamma A^{-1/3}. \quad (10.6)$$

The last term in (10.4) is therefore $\frac{1}{3}\gamma A^{2/3}=\frac{1}{3}W_s$, and W_s is $\frac{3}{2}$ times the first term in (10.4). To evaluate this we need to know $\rho(x)$. The simplest expression for this in integral theory is (7.15), from which

$$dx=\alpha^{-1}d\hat{\rho}/(1-\hat{\rho}), \quad (10.7)$$

so that (10.4) becomes

$$\frac{W_s}{S}=\frac{3}{2}\frac{\rho_0}{\alpha}\int[W(\rho)-\tfrac{1}{2}W_0-\tfrac{1}{2}w(\rho)]\frac{\hat{\rho}}{1-\hat{\rho}}d\hat{\rho}. \quad (10.8)$$

Of course, (7.15) is known to fail for low $\hat{\rho}$ but they do not contribute much.

From (7.8a) and (7.10) we get after some algebra

$$W(\rho)-\tfrac{1}{2}W_0-\tfrac{1}{2}w(\rho)=\tfrac{1}{2}(47.5-32.5)(1-\hat{\rho}^{1/2}) +\tfrac{1}{4}32.5\hat{\rho}^{1/2}(1-\hat{\rho}). \quad (10.9)$$

Inserting this into (10.8) yields

$$W_s=\tfrac{3}{2}(S\rho_0/\alpha)[7.5(5/6-\ln 2)+8.12\cdot\tfrac{1}{4}] =4.62\text{ MeV}(S\rho_0/\alpha). \quad (10.10)$$

For a spherical nucleus of radius $R=r_0A^{1/3}$,

$$S\rho_0=3A/R=3A^{2/3}/r_0. \quad (10.11)$$

Then the constant in the Weizsäcker formula is

$$W_sA^{-2/3}=3\cdot 4.62\text{ MeV}/\alpha r_0=17\text{ MeV}, \quad (10.12)$$

using $r_0=1.12$ and an estimated $\alpha=0.73$ [cf. below (7.15)]. This is in embarrassingly good agreement with the empirical value[72] of

$$W_sA^{-2/3}=18.0\text{ MeV}. \quad (10.13)$$

It should be remembered that (7.15) is only correct for $1-\hat{\rho}$ small, and our estimate of α is a pure guess.

[72] A. E. S. Green, Rev. Mod. Phys. **30**, 569 (1958), Table III.

An alternative calculation of W_s uses the differential theory of Sec. 8. This has the advantage that $\rho(x)$ can be determined analytically, at least within the framework of the theory. In differential theory, we expand $\rho(\mathbf{r}_2)-\rho(\mathbf{r}_1)$ in (10.1) in powers of $\mathbf{r}_2-\mathbf{r}_1$; then using the definition (8.2), (10.1) becomes[73]

$$W_s/\rho_0=\int(W(\rho)-W_0)\hat{\rho}d\tau-\tfrac{1}{2}B\int\hat{\rho}\nabla^2\hat{\rho}d\tau. \quad (10.14)$$

The second term "looks like" a kinetic energy but of course it is not: It arises from the finite range of the nuclear forces v_l. We now assume again a plane surface of area S, and integrate the second term of (10.14) by parts with respect to x; this gives

$$W_s/S\rho_0=\int(W(\rho)-W_0)\hat{\rho}dx+\tfrac{1}{2}B\int(d\hat{\rho}/dx)^2dx. \quad (10.15)$$

Use of (8.6) shows that the second term is equal to the first. In this picture, then, half the surface energy may be considered to arise from the fact that the "local" nuclear-matter energy $W(\rho)$ is greater than W_0, the other half from the nonlocality of the interaction, i.e., the second term in (10.1).

Next we eliminate dx in favor of $d\rho$, using (8.6), and obtain

$$W/S\rho_0=(2B)^{1/2}\int\hat{\rho}^{1/2}(W(\rho)-W_0)^{1/2}d\hat{\rho}. \quad (10.16)$$

Obviously, neither very small $\hat{\rho}$ nor $\hat{\rho}$ near 1 contribute strongly; the main contribution comes from the neighborhood of the steepest slope of $\hat{\rho}(x)$. Using (7.9) for $W(\rho)$ we find

$$W/S\rho_0=(2B)^{1/2}(33\text{ MeV})^{1/2}/6. \quad (10.17)$$

Since B is not well known, we evaluate it from (8.9) in terms of the surface thickness b, noting that with (7.9), $\hat{\rho}_1=\frac{1}{4}$ and $W(\rho_1)-W_0=\frac{1}{4}\times33$ MeV. Inserting also (10.11) we get

$$W_sA^{-2/3}=(33/4)\text{ MeV}(b/r_0). \quad (10.18)$$

With $r_0=1.12$ F and the "theoretical" value (8.17), $b=2.47$ F, this gives

$$W_sA^{-2/3}=18.2\text{ MeV}, \quad (10.19)$$

again in close agreement with (10.10). Use of the "empirical" surface thickness $b=2.4$ F gives the equally good result

$$W_sA^{-2/3}=17.7\text{ MeV}. \quad (10.20)$$

Thus the result for the surface energy, on any of the three calculations, is surprisingly good. We can only hope that this agreement will not be spoiled when better nuclear forces are used.[74]

[73] The form (10.14) is quite similar to the theory of Wilets, Ref. 8.

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APPENDIX A: EFFECTIVE-RANGE THEORY FOR MOSZKOWSKI-SCOTT

In the effective-range theory of Sec. 4, we require two intergrals, (4.18), and (4.19), viz.,

$$C'=\int(\phi_0^2-u_0^2)dr, \quad (A1)$$

$$A=\int\chi^2dr=\int\zeta^2d\tau/4\pi=\int(\phi_0-u_0)^2dr. \quad (A2)$$

We evaluate these, assuming

$$\phi=r, \quad (A3)$$

$$\begin{aligned}&\chi=\phi_0-u_0=r && \text{for } r<c,\\ &\chi=[(d-r)/(d-c)]^2 && \text{for } c<r<d,\\ &\chi=0 && \text{for } d<r.\end{aligned} \quad (A4)$$

This leads to

$$A=(c^2/15)(3d+2c), \quad (A5)$$

$$C'=(c/30)(5d^2+4cd+c^2). \quad (A6)$$

For c not too different from d, these may be approximated by

$$A=(c^2/6)(d+c), \quad (A7)$$

$$C'=(cd/6)(d+c). \quad (A8)$$

[74] Tabakin and Amos [Nucl. Phys. **100A**, 574 (1967)] also calculated surface energy and thickness, using various interactions between two nucleons. They find that an interaction (velocity-dependent) which gives the correct ^1S phase shifts from 0 to 310 MeV, gives too much surface energy (30 MeV) and a surface thickness $t=1.8$ F. A velocity-dependent potential which saturates at the correct binding energy gives 27 MeV and $t=3.8$ F. Only one of the potentials chosen by them gives approximately the correct results, 22 MeV and 2.1 F. We believe that the large uncertainty indicated by their results can be narrowed down by choosing a potential which agrees with two-body scattering data and at the same time gives the correct binding energy and density for nuclear matter. We believe we have used such a potential in this paper and obtained satisfactory results.

These expressions have been used in (4.20a) and (4.20b), For $d/c=2$ or 2.5, (A8) is too large by about 3.5%, (A7) too small by 7–9%.

APPENDIX B: THOMAS-FERMI APPROXIMATION FOR KINETIC ENERGY

In this Appendix, we shall study the wave functions and the density in an arbitrary, one-dimensional potential $V(x)$. We assume

$$V(x)\rightarrow -V_0 \tag{B1}$$

for sufficiently large, positive x,

$$dV/dx\leq 0 \quad \text{for all} \quad x \tag{B2}$$

$$V(0)=0. \tag{B3}$$

Then, as usual, in a neighborhood of $x=0$ we take

$$V(x)=-Fx \quad \text{for} \quad x \text{ small} \tag{B4}$$

with $F>0$. When calculating the density, we assume that all states $E<0$ are occupied, all states $E>0$ empty. This corresponds to a Fermi energy

$$E_F=0. \tag{B5}$$

By adding a constant E_F to both E and V, an arbitrary Fermi energy E_F can be treated similarly.

Our aim is to find the relation between the local density $\rho(x)$ and the local kinetic energy of the most energetic particle, $E_F-V(x)$. If Fermi statistical theory is valid,

$$\rho=\tfrac{2}{3}\pi^{-2}k_F^{\,2}=2^{5/2}(3\pi^2)^{-1}M^{3/2}\hbar^{-3}(E_F-V)^{3/2}. \tag{B6}$$

We are interested in the deviations from this relation.

We shall use the WKB method to treat the wave functions over most of the range of x, where $V(x)$ is an arbitrary function. Near the turning point we use the explicit solution of the Schrödinger equation in the potential (B4), and the study of (B6) will be made with the help of this solution.

The wave functions in the potential (B1) are

$$\psi=\phi(x)\,\exp i\mathbf{k}'\cdot\mathbf{r}, \tag{B7}$$

where $\mathbf{k}'$ is a vector in the y, z plane, the "perpendicular momentum." Then ϕ satisfies

$$\phi''+(2E-2V(x)-k'^2)\phi=0, \tag{B8}$$

where we have put $\hbar^2/M=1$. We define k_x and k_{x0} by

$$k_x^{\,2}(x)=2E-2V(x)-k'^2, \tag{B9}$$

$$k_{x0}^{\,2}(x)=2E-2V_0-k'^2. \tag{B10}$$

For any given energy E, k' is limited by the condition

$$k_{x0}^{\,2}>0, \quad k'^2<2(E-V_0). \tag{B11}$$

The wave function $\phi(x)$ is of course a standing wave which we normalize by requiring

$$\phi\rightarrow 2^{1/2}\cos(k_{x0}x+\alpha) \quad \text{as} \quad x\rightarrow\infty; \tag{B12}$$

this means that the *average* density inside the nucleus goes to unity.

In the region of space where $k_x^{\,2}>0$ we may use the ordinary WKB formula

$$\phi(x)=[2k_{x0}/k_x(x)]^{1/2}\cos\left(\int^x k_x(x')dx'+\beta\right)$$

$$\equiv(2k_{x0})^{1/2}\phi_1(k_{x0},x), \tag{B13}$$

which is normalized in accord with (B12); ϕ_1 then contains no normalizing factor referring to the interior of the nucleus. The total density is then

$$\rho(x)=4(2\pi)^{-3}\int^{k_{F0}} d^3k_0|\psi(k_{x0},k_y,k_z;x)|^2 \tag{B14}$$

$$=(2\pi^3)^{-1}\int_0^{k_{F0}} 2dk_{x0}\int_0^{k_{F0}^2-k_{x0}^2}\pi d(k'^2)\,|\psi(k_{x0},x)|^2$$

$$=\pi^{-2}\int_0^{k_{F0}^2} d(k_{x0}^{\,2})(k_{F0}^{\,2}-k_{x0}^{\,2})\phi_1^{\,2}(k_{x0},x). \tag{B15}$$

In (B14), the integral is over all values of the vector k_0, up to k_{F0}. In the next equation, we have taken into account that $|\psi|$ depends only on k_{x0} and x which permits, in (B15), integration over the perpendicular momentum k'.

To recover the usual TF result, we use the "ordinary WKB" (B13) and average the $\cos^2$ factor in $\phi_1^{\,2}$ to give $\frac{1}{2}$ as long as $k_x^{\,2}(x)>0$; for $k_x^{\,2}<0$, where ϕ_1 decreases exponentially, we set $\phi_1=0$. Now from (B9) and (B10) for fixed x, $k_x^{\,2}=k_{x0}^{\,2}-\text{const}$; therefore (B15) becomes

$$\rho(x)=\pi^{-2}\int_0^{k_F^2(x)} d(k_x^{\,2})[k_F^{\,2}(x)-k_x^{\,2}(x)](2k_x)^{-1}$$

$$=(\tfrac{2}{3}\pi^2)^{-1}k_F^{\,3}, \tag{B.16}$$

which is the ordinary Thomas-Fermi relation (B6).

We now investigate the behavior of (B15) near the surface, where $k_x^{\,2}$ is near zero and where ϕ_1 can no longer be represented by (B13). As in the derivation of the WKB connection formulas, we assume that V is linear in x near the point $k_x^{\,2}=0$. We are particularly interested in the most energetic nucleons, i.e., the case $k_{x0}^{\,2}=k_{F0}^{\,2}$, because their wave function contributes most to the density in the surface. Therefore we set

$$k_F^{\,2}(x)\equiv k_{F0}^{\,2}+2V_0-2V(x)=Fx, \tag{B17}$$

$$k_x^{\,2}(x)=k_{x0}^{\,2}-k_{F0}^{\,2}+Fx\equiv F(x-x_1), \tag{B18}$$

so that the Schrödinger equation for ϕ_1 is

$$\phi_1''+F(x-x_1)\phi_1=0. \tag{B19}$$

As is well known, this can be solved in terms of Bessel functions of order $\frac{1}{3}$. Normalizing to conform with (B13), we get

$$\phi_1(x)=\tfrac{1}{3}(\pi)^{1/2}F^{-1/6}\xi^{1/2}[J_{-1/3}(\tfrac{2}{3}\xi^{3/2})+J_{1/3}(\tfrac{2}{3}\xi^{3/2})], \quad \xi>0 \quad \text{(B20a)}$$

$$\phi_1(x)=\tfrac{1}{2}(\pi/3)^{1/2}i^{4/3}F^{-1/6}|\xi|^{1/2}H_{1/3}^{(1)}(\tfrac{2}{3}i|\xi|^{3/2}), \quad \xi<0 \quad \text{(B20b)}$$

where

$$\xi=F^{1/3}(x-x_1), \quad \text{(B21a)}$$

$$x_1=(k_{F0}^2-k_{x0}^2)/F. \quad \text{(B21b)}$$

If we use the asymptotic formula for $J_{\pm 1/3}$ for large argument, (B20a), becomes identical with (B13) if k_x in that formula is replaced by (B18). For large negative ξ, (B20b) gives

$$\phi_1(x)=\tfrac{1}{2}|k_x|^{-1/2}\exp-\int_x^{x_1}|k_x(x')|dx', \quad \text{(B22)}$$

with k_x given by (B18), again conforming with the WKB. In terms of Watson's definitions,[75]

$$i^{4/3}H_{1/3}^{(1)}(iz)=(2/\pi)K_{1/3}(z), \quad \text{(B23a)}$$

$$(2/\sqrt{3})(J_{-1/3}(z)+J_{1/3}(z))=\sqrt{3}J_{1/3}(z)-Y_{1/3}(z). \quad \text{(B23b)}$$

In terms of ξ, (B21a), we may write (B15)

$$\rho=\pi^{-2}F\int_{-\infty}^{\xi_0}f(\xi)(\xi_0-\xi)d\xi, \quad \text{(B24a)}$$

where

$$\xi_0=F^{1/3}x, \quad \text{(B24b)}$$

$$f(\xi)=F^{1/3}\phi_1^2(x). \quad \text{(B24c)}$$

We have defined $f(\xi)$ so as to be independent of F. The lower limit of the integral in (B24a) should actually be the value of ξ corresponding to $k_{x0}=0$, i.e.,

$$\xi_m=F^{1/3}(x-k_{F0}^2F^{-1}), \quad \text{(B25a)}$$

but k_{F0}^2 is so large that, for the interesting values of x, ξ_m is very large and negative and hence $f(\xi_m)$ is negligible small, see (B22); hence the integral can be extended to $-\infty$. The integral in (B24a) has been calculated numerically on the basis of Watson's tables[75]; the result for the integral in (B24a) is given in Table III. We may compare this with the result of the Thomas-Fermi approximation which results when we use (B13), (B18), and (B21a), and replace $\cos^2$ in (B13) by $\frac{1}{2}$, giving

$$f_0(\xi)=\tfrac{1}{2}F^{1/3}k_x^{-1}=\tfrac{1}{2}\xi^{-1/2}, \quad \text{(B25b)}$$

$$\int_{-\infty}^{\xi_0}f_0(\xi)(\xi_0-\xi)d\xi=\tfrac{2}{3}\xi_0^{3/2} \quad \text{if} \quad \xi_0>0$$
$$=0 \quad \text{if} \quad \xi_0<0. \quad \text{(B26a)}$$

TABLE III. The integral in Eq. (B24a).

ξ_0	Exact	Thomas-Fermi	ξ_0	Exact	Thomas-Fermi
−1.0	0.0078	0	0.6	0.316	0.310
−0.8	0.0137	0	0.7	0.376	0.391
−0.6	0.0230	0	0.8	0.444	0.477
−0.4	0.0381	0	1.0	0.607	0.667
−0.2	0.0613	0	1.2	0.805	0.876
0	0.0987	0	1.4	1.038	1.103
0.1	0.1194	0.0211	1.6	1.301	1.349
0.2	0.1471	0.0597	1.8	1.586	1.610
0.3	0.1799	0.1096	2.0	1.887	1.885
0.4	0.2186	0.1686	2.35	2.442	2.400
0.5	0.2638	0.2356			

The agreement is seen to be quite good for

$$\xi_0>0.5. \quad \text{(B26b)}$$

In fact, the deviation is always less than 10% from there on. For $\xi_0<0.6$, the TF method gives (of course) too low a density; from 0.6 to 2.0, the TF density is too high; then it becomes too low again, etc. These fluctuations of the correct density around the TF value were noticed and discussed by Kohn and Sham.[76] But the fluctuations are small and may be ignored for our purposes.

To extract the physical significance of these results, we must obtain the value of the "force." Wilets[8] has given evidence that the potential falls more slowly at the surface than the density. We have confirmed this by obtaining the potential from our particle density and the nuclear forces, but these calculations are incomplete and we therefore take Wilets' result. He represents the potential by the Fermi (Woods-Saxon) formula

$$V=-V_0[1+e^{(r-R)/s}]^{-1}, \quad \text{(B27)}$$

with $s=0.65$ F. We choose V_0 to be the potential energy of a nucleon at the top of the Fermi distribution at nuclear-matter density,[11] $V_0=57$ MeV$=1.37$ F^{-2}. We identify F with the steepest slope of[77] $2V$, viz.,

$$F=2V/4s=10.6 \text{ F}^{-3}. \quad \text{(B28)}$$

Using this in (B24), and taking $\rho_0=0.170$ F^{-3}, we find

$$\text{At} \quad \xi_0=0, \quad \rho=\rho_3=0.0106 \text{ F}^{-3}=0.062\,\rho_0, \quad \text{(B29)}$$

$$\text{At} \quad \xi_0=0.5, \quad \rho=\rho_2=0.0280 \text{ F}^{-3}=0.165\rho_0. \quad \text{(B30)}$$

As we discussed on the basis of Table III, the TF approximation is valid (within 10%) for $\rho>\rho_2$, i.e., whenever the density is more than 17% of nuclear-matter density. This is a very satisfactory result. The other result, ρ_3, is the actual density at the point where

[75] G. N. Watson, *Bessel Functions* (Cambridge University Press, Cambridge, England, 1952).

[76] W. Kohn and L. J. Sham, Phys. Rev. **137**, A1687 (1963).

[77] See (B17) for factor 2.

the TF density goes to zero, i.e., at the classical turning point of the fastest nucleon; this ρ_3 is 6% of nuclear-matter density. These rather low values of ρ_2 and ρ_3 are due to the low value of the "force" F, i.e., to the relatively gentle drop of the potential at the surface.[78] It is this which makes the TF approximation valid.

Velocity-Dependent Forces

So far, we have assumed an ordinary potential. We shall now consider a velocity-dependent potential. For simplicity,[79] we take one which permits definition of an effective mass, which seems to be a fair approximation for the "hole states" in nuclear matter, according to Sprung[10] and Dahlblom.[11] Thus we assume that the potential of a particle of momentum k at position x is

$$U(k,x)=V(x)+\tfrac{1}{2}\lambda(x)k^2. \tag{B31}$$

Then, setting[79] $\hbar^2/2M=1$,

$$\nabla^2\psi+[2(E-V(x))+\lambda(x)\nabla^2]\psi=0, \tag{B32}$$

where we have replaced k^2 by $-\nabla^2$. To make this equation self-adjoint, we must replace it by[26]

$$\nabla\cdot[(1+\lambda)\nabla\psi]+2(E-V)\psi=0. \tag{B33}$$

We may introduce the effective mass by $1+\lambda=1/m^*$. It is easy to show that (B33) is self-adjoint. We use (B7) and have

$$\frac{d}{dx}\left(\frac{1}{m^*}\frac{d\phi}{dx}\right)+\left(2E-2V-\frac{k'^2}{m^*}\right)\varphi=0. \tag{B34}$$

We now solve (B34) by the WKB method, setting

$$\phi=F(x)e^{i\alpha(x)}, \tag{B35}$$

with F and α real. Inserting into (B34) gives from the real part, neglecting $F''/F\phi'^2$,

$$\alpha'^2=2m^*(E-V)-k'^2, \tag{B36}$$

which is a familiar result (prime$=d/dx$, except in k'). From the imaginary part,

$$2F'/F=-d\ln\alpha'/dx+d\ln m^*/dx, \tag{B37}$$

so that

$$F=\text{const}\times(m^*/\alpha')^{1/2}=\text{const}\times v_x^{-1/2}, \tag{B38}$$

where v_x is the velocity in the x direction.

This result follows also from the continuity equation: The current is

$$j_x=\frac{1}{2im^*}\left(\psi^*\frac{d\psi}{dx}-\psi\frac{d\psi^*}{dx}\right)=v_x|\phi|^2, \tag{B39}$$

and this should be conserved. This argument is entirely general. We therefore conclude that generally (even if the effective-mass approximation is not correct) the WKB solution (B13) may be replaced by

$$\phi(x)=(2v_{x0})^{1/2}\phi_2(E,k',x),$$

$$\phi_2(x)=[v_x(x)]^{-1/2}\cos\left[\int^x k_x(x')dx'+\beta\right], \tag{B40}$$

where from Hamilton's equations the group velocity v_x is given by

$$v_x(x)=(1/\hbar)(\partial E/\partial k_x)_{x,k'}. \tag{B41}$$

Inserting into (B15) (and setting $\hbar=1$)

$$\rho(x)=2\pi^{-2}\int v_{x0}dk_{x0}d(k'^2)\phi_2^2(k_{x0},k',x)$$

$$=2\pi^{-2}\int_0^{k_F}d(k'^2)\int_{E(k')}^{E_F}dE\phi_2^2(E,k',x), \tag{B42}$$

where $E(k')$ is the energy corresponding to a *total* wave number $k=k'$ in the interior of the nucleus. The expression (B42) is very satisfactory because E and k' are the parameters which determine the wave function ϕ_2.

In the region where the wave function ϕ_2 is given by the simple WKB formula (B40), the formula (B42) yields easily the normal Thomas-Fermi result (B16).

We now consider the nuclear surface, i.e., the neighborhood of the classical turning points. Obviously, only the states of high E and low k' will have significantly large ϕ_2; the others will have decayed exponentially. We are therefore dealing with only a narrow range of energies, and it is therefore permissible to expand the local wave number,

$$k^2(E,x)=2E-2U(k,x), \tag{B43}$$

in powers of $E-E_F$. Moreover we assume, as in (B17), that near the turning point, k^2 is linear in x. Finally, also in analogy with (B17), we choose $x=0$ to be the point where $k^2(E_F)=0$. Then we may write[80]

$$k^2(E,x)=2m^*(E-E_F)+Fx, \tag{B44}$$

where m^* is the effective mass near $x=0$ and near

[78] Already many years ago, this was pointed out by Swiatecki. He remarked incidentally that the force on a nucleon at the surface of a nucleus is about the same as the force of the earth's gravity on a man (75 kg weight).

[79] We have also considered the case of exchange forces, as in Sec. 4, and obtained a similar result. But the argument is complicated, and some approximations had to be made.

[80] It would be more accurate to expand around a somewhat lower energy than E_F, to represent the mean energy [or mean k_x^2, see (B9)] of the states which contribute to ρ. This slightly changes the best definition of m^* and F but nothing else.

$E=E_F$, defined in the usual way. We introduce

$$k_x^2(E,k',x)=k^2(E,x)-k'^2, \tag{B45a}$$

$$k_{x1}^2=k_x^2(E,k',0)=2m^*(E-E_F)-k'^2, \tag{B45b}$$

$$k_x^2=k_{x1}^2+Fx. \tag{B45c}$$

Then we may introduce k_1 instead of E as the second integration variable in (B42), thus

$$\rho(x)=\pi^{-2}(m^*)^{-1}\int_0^{k_F} d(k'^2)\int_{k_m^2}^{-k'^2} d(k_{x1}^2)\phi_2^2 \times(k_{x1},k',x), \tag{B46a}$$

with

$$k_m^2=2m^*(E(k'^2)-E_F)-k'^2. \tag{B46b}$$

This lower limit is unimportant. Now ϕ_2 must be normalized by (B40) in the region of sufficiently large x where the "ordinary WKB" is valid. Since $v_x=k_x/m^*$, this normalization differs from (B13) exactly by the factor $(m^*)^{1/2}$; thus we have

$$\phi_2=(m^*)^{1/2}\phi_1. \tag{B47}$$

This exactly eliminates the factor $(m^*)^{-1}$ in (B46a). ϕ_1 we have the same normalization and differential equation as in (B13) ff. so that we may use the old result (B20). Interchanging then the order of integration in (B46a),

$$\rho(x)=\pi^{-2}\int_{-k_F^2}^{0} d(k_{x1}^2)\int_0^{-k_{x1}} d(k'^2)\phi_1^2(k_{x1},x)$$

$$=\pi^{-2}\int_{-k_F^2}^{0} d(k_{x1}^2)(-k_{x1}^2)\phi_1^2(k_{x1},x), \tag{B48a}$$

which is identical with (B15) if we set

$$k_{x1}^2=k_{x0}^2-k_{F0}^2. \tag{B48b}$$

Hence the case of a velocity-dependent force is reduced to the earlier case of an ordinary, velocity-independent force, if only we accept the approximation (B44) in the surface region. The variation of m^* with x in the interior does not matter, only its value near $x=0$—and even this drops out in the final answer (B48a). Moreover, the effective-mass approximation (B44) itself needs to be valid only in the neighborhood of E_F and $x=0$.

The earlier conclusions about the validity of the WKB for $\rho>0.17\rho_0$ are thereby justified in this very general case.

APPENDIX C: CALCULATION OF MIXED DENSITY

Lin has calculated numerically the mixed density (5.2). He use a potential

$$\begin{aligned} V&=-V_0(1-e^{x/a})^2, && x>0\\ V&=0, && x<0 \end{aligned} \tag{C1}$$

which is patterned after the density distribution (8.23). He used

$$\begin{aligned} V_0&=40\text{ MeV},\\ a&=1.2\text{ F},\\ E_F&=-8\text{ MeV}. \end{aligned} \tag{C2}$$

The value for a is that derived by Lin from electron scattering, Eq. (9.2); perhaps a larger value (by about 20%) should be used according to Wilets' remark [see below Eq. (B26b)]. E_F was taken to correspond to an average real nucleus, rather than nuclear matter. V_0 might have been chosen larger, about 50–60 MeV, to correspond to the potential energy of the most energetic nucleon inside a large nucleus [cf. above (B27)].

Lin calculated the wave functions $\phi(x)$, Eq. (6.8), by direct solution of the Schrödinger equation in the potential (C1), for various values of

$$E_x=E-(\hbar^2/2M)(k_y^2+k_z^2) \tag{C3}$$

between V_0 and E_F, in intervals of 4 MeV. The calculation of $\rho(\mathbf{r}_1,\mathbf{r}_2)$ from (5.2) is then straightforward. As stated in (6.9), the mixed density is a function of the x coordinate of the center of mass, $X=\frac{1}{2}(x_1+x_2)$, and of the components of the relative coordinate $\mathbf{r}_1-\mathbf{r}_2$ perpendicular and parallel to the surface. All lengths are on the scale of a in (C1). The results are plotted in Figs. 3(a) and 3(b), and discussed in Sec. 6 below Eq. (6.11). It is most remarkable that, for given X, $\rho(\mathbf{r}_1,\mathbf{r}_2)$ in fact depends almost exclusively on the distance $|\mathbf{r}_1-\mathbf{r}_2|$ and not on the direction of the vector $\mathbf{r}_1-\mathbf{r}_2$, in spite of the fact that the wave functions (6.8) depend on x and y, z in entirely different ways. Points corresponding to different values of $x=x_1-xf_2$ are marked by different symbols on Figs. 3(a) and 3(b), and it is seen that all the different symbols form one continuous curve. In fact, this statement is more accurate than the agreement with the Slater curve (6.4). It would, accordingly, be possible to define a more accurate formula for ρ as a function of r, but this would depend on the special potential (C1) assumed, and the simplicity of the Slater formula would be lost.