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Transmission of Electromagnetic Waves through a Conducting Slab. II. The Peak at Cyclotron Resonance

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This is the second in a series of three papers concerned with the propagation of electromagnetic waves through a metallic slab of finite thickness. In the situation studied here, the relation between the current in the slab and the electric field which drives it is highly nonlocal because of the finite electron mean free path. We calculate the transmission amplitude under these conditions, assuming that there is a steady magnetic field normal to the faces, and assuming also that electrons within the slab scatter diffusely at the surfaces. A transmission peak is predicted at cyclotron resonance which, in the limit of slab width much larger than an electron mean free path, has a line shape which approaches the square root of a Lorentzian and a phase which shifts by π across the line. All of this is superposed on the Gantmakher-Kaner oscillations which are also present. This resonant behavior, which is absent when the electrons are assumed to be scattered specularly from the surface of the slab, can be understood in terms of electron transport from Fresnel zones on the Fermi sphere which expand at resonance so that the first zone covers the entire hemisphere. The absence of this resonance when the electrons are scattered specularly is tentatively ascribed to a fortuitous cancellation which arises because the equatorial electrons, whose response gives rise to currents which shield the interior of the metal, themselves undergo a resonance which excludes the field from the metal. In carrying out these calculations using the two-sided Wiener-Hopf technique for the nonlocal wave equation, we found that the technique itself can be simplified considerably if the quantity of interest is the transmission amplitude (the field at the emergent face of the slab) rather than the full-field amplitude in the slab. We show that this simplification arises when certain short-range parts of the field are eliminated.

I. INTRODUCTION

THIS is the second in a series of three papers concerned with the propagation of electromagnetic waves through a metallic slab of finite thickness. In the situation studied here, the relation between the current in the slab and the electric field which drives it is highly nonlocal because of the finite electron mean free path. We calculate the transmission amplitude under these conditions assuming that there is a steady magnetic field normal to the faces, and assuming also that electrons within the slab scatter diffusely at the surfaces of the slab. A transmission peak is predicted at cyclotron resonance which, in the limit of slab width much larger than an electron mean free path, has a line shape which approaches the square root of a Lorentzian and a phase which shifts by π across the line. All of this is superposed on the Gantmakher-Kaner[1] oscillations which are also present. The resonant behavior, which is absent when the electrons are assumed to be scattered specularly from the surface of the slab, can be understood in terms of electron transport from Fresnel zones on the Fermi sphere which expand at resonance so that the first zone covers the entire hemisphere. The absence of this resonance when the electrons are scattered specularly is tentatively ascribed to a fortuitous cancellation which arises because the equatorial electrons, whose response gives rise to currents which shield the interior of the metal, themselves undergo a resonance which excludes the field from the metal.

In carrying out these calculations using the two-sided Wiener-Hopf technique[2] for the nonlocal wave equation, we found that the technique itself can be simplified considerably if the quantity of interest is the transmission amplitude (the field at the emergent face of the slab) rather than the full-field amplitude in the slab. We show that this simplification arises when certain short-range parts of the field are eliminated. Thus, this paper has two objectives: One is to present the physics described above, and the other is to describe the mathematical simplification (which is connected more with the two-sided Wiener-Hopf technique than with the specific electromagnetic problem). Although we chose the logical order of presenting the method before using it, the reader who is interested only in the physical results and discussion is invited to read the next few paragraphs and then to skip to Sec. IV, and read that section in its entirety. The rest of the paper will serve him only if he is interested in how the results were obtained.

In an earlier paper,[2] we sketched out the derivation of the equations which govern the electromagnetic fields in a metal slab under the following simple conditions. A plane, circularly polarized, transverse electromagnetic wave is normally incident on a metallic slab of thickness L. The frequency of the wave is low enough (below infrared) so that the metal can be represented as a degenerate, zero-temperature Fermi gas composed of the conduction electrons, which are treated as free. Electrons reaching the face of the slab are assumed to be scattered diffusely. Using the standard assumptions of the Boltzmann equation in the relaxation-time approximation,[3] we found that the

[1] V. F. Gantmakher and E. A. Kaner, Zh. Eksperim. i Teor. Fiz. **48**, 1572 (1965) [English transl.: Soviet Phys.—JETP **21**, 1053 (1965)].
[2] G. A. Baraff, J. Math. Phys. (to be published).
[3] G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) **A195**, 336 (1948).

field satisfied the following nonlocal wave equation

$$\left(\frac{d^2}{dz^2}+k_0^2\right)e(z)=\int_0^L K(|z-z'|)e(z')dz' \qquad 0\le z\le L, \quad (1.1a)$$

where $K(|z-z'|)=-i\omega\mu_0\sigma(|z-z'|)$, σ being the ordinary conductivity which relates current and electric field *in an infinite medium* by

$$j(z)=\int\sigma(|z-z'|)e(z')dz'.$$

This equation differs from that given by Reuter and Sondheimer[3] for diffuse-boundary scatter only in that the thickness of the slab L, which appears as the upper limit on the integral, is in their work infinite, and that the conductivity σ which appears here must be calculated assuming the presence of a static magnetic field.

The field at the faces of the slab must satisfy the boundary conditions appropriate to a wave of unit amplitude incident from negative z. These are

$$\left(1+\frac{1}{ik_0}\frac{d}{dz}\right)e(z)=2, \quad \text{at} \quad z=0, \qquad (1.1b)$$

$$\left(1-\frac{1}{ik_0}\frac{d}{dz}\right)e(z)=0, \quad \text{at} \quad z=L. \qquad (1.1c)$$

In the earlier paper[2] we presented a method for the solution of Eq. (1.1). Our method resulted in a Fredholm integral equation for the Fourier transform of $e(z)$, whose iterative solution converged as $\exp(L/l)$ where l is the range of the kernel. (For the conductivity kernel, this range is equal to the electron mean free path.) This method is most useful when the thickness of the slab is of the order of, or greater than, the range. The zeroth iterate contains both the ordinary semi-infinite-medium Wiener-Hopf solution and the Fabry-Perot resonances arising from multiple internal reflection of the wave. The higher iterates represent the multiple reflections of the single-particle excitations, i.e., of that part of $e(z)$ which results from branch-cut singularities in the Fourier transform of the kernel.

In the remainder of this introduction, we review the solution to (1.1) and discuss when and why a simplification is to be expected. This simplification results from the elimination of some short-range parts of the field and can be carried out when the kernel K has a transform whose only upper-half-plane singularity is a *single* branch cut.[4] In Sec. II we show how this can be done. The results at the end of Sec. II are only a reformulation of the solution given in Ref. 2. However, in some cases where the techniques of Ref. 2 would require a high-speed digital computer, the reformulation given here is amenable to pencil and paper calculation. In Sec. III, we apply the method to the problem for which it was developed, namely, a calculation of the transmission amplitude for an electromagnetic wave across a metallic slab under the conditions listed earlier. We confine our attention to the simplest free-electron model of a metal, and, within that model, examine only the "extreme anomalous limit",[3] where the nonlocality of the conductivity is most pronounced. We find a transmission peak at cyclotron resonance, and a phase shift of π across the peak. Our tentative physical understanding of this is described in Sec. IV.

To solve (1.1), by the methods of Ref. 2, we introduce the dispersion function or characteristic function, call it $\psi(k)$, which is the Fourier transform of the operators in Eq. (1.1a):

$$\psi(k)\equiv k^2-k_0^2+\int_{-\infty}^{\infty}K(|z|)e^{-ikz}dz. \qquad (1.2)$$

It turns out that the function $e(z)$ in the slab has Fourier components at each singularity and branch cut of $1/\psi$. The singularities can be poles and branch points which, because $\psi(k)$ is even, must occur in pairs. If there are $2N$ poles, their location is given by solving the dispersion equation

$$\psi(k_n)=0, \quad \text{Im}k_n>0, \quad n=1,2,\cdots N. \qquad (1.3)$$

(We have chosen $\text{Im}k_n>0$ as the convention to fix which of the two roots will be designated as $+k_n$.) The function ψ in the electromagnetic problem has a single pair of branch points $k=\pm\beta$ (with $\text{Im}\beta>0$ fixing the sign of β). We let branch cuts run radially outward from the branch points to infinity along the lines $k=\pm\beta u$, $1\le u\le\infty$. Then the field in the slab has the form

$$e(z)=\sum_{n=1}^{N}(a_ne^{ik_nz}+b_ne^{-ik_nz}) +\int_1^{\infty}[a(u)e^{i\beta uz}+b(u)e^{-i\beta uz}]du. \qquad (1.4)$$

In Ref. 2, we wrote $e(z)$ as the sum of a part decaying to the right [the a_n and $a(u)$ terms] and a part decaying to the left [the b_n and $b(u)$ terms] and treated the field within the slab (which arises from a wave incident from the left) as the sum of the field produced by symmetric waves incident left and right and antisymmetric waves incident left and right. By this means, the field was given as a sum of four terms:

$$e(z)=f^+(z)+f^-(z)+f^+(L-z)-f^-(L-z). \qquad (1.5)$$

We wrote each of the four terms as

$$f^{\pm}(z)=i\sum_{n=1}^{N}\varphi_n^{\pm}\left(e^{ik_nz}-\beta\int_1^{\infty}f_n^{\pm}(t)e^{i\beta tz}dt\right) \qquad (1.6)$$

[4] It may well be that these simplifications occur for kernels with multiple-branch cuts. If so, it should be trivial to demonstrate, but we have given absolutely no thought to this matter.

and found that each $f_n^{\pm}(t)$ had to satisfy the equation

$$f_n^{\pm}(t)=\frac{X^+(\beta t)-X^-(\beta t)}{2\pi i}\left\{\frac{1/X(k_n)}{\beta t-k_n}\pm\frac{e^{ik_nL}/X(-k_n)}{\beta t+k_n}\mp\int_1^\infty\frac{f_n^{\pm}(u)e^{i\beta uL}du}{(u+t)X(-\beta u)}\right\}. \quad (1.7)$$

The function $X(k)$, which appears in the computation of f_n, is a complicated but explicit functional of the function $\psi(k)$, namely

$$X(k)\equiv(\beta-k)^{N-1}\exp\frac{1}{2\pi i}\int_1^\infty\frac{\beta du\ln G(\beta u)}{\beta u-k}, \quad (1.8a)$$

where

$$G(\beta u)\equiv\psi^-(\beta u)/\psi^+(\beta u). \quad (1.8b)$$

Here, $\psi^{\pm}(\beta u)$ are limiting values of $\psi(k)$ as it approaches the point $k=\beta u$ from one side of the branch cut or the other:

$$\psi^{\pm}(\beta u)\equiv\lim_{\epsilon\to 0^+}\psi[\beta(u\pm i\epsilon)]. \quad (1.8c)$$

Similarly, $X^{\pm}(\beta t)$ are limiting values of $X(k)$ on either side of the cut

$$X^{\pm}(\beta t)=\lim_{\epsilon\to 0^+}X[k(t\pm i\epsilon)]. \quad (1.8d)$$

Once the $f_n^{\pm}(t)$ have been determined by iterating (1.7), the coefficients $\varphi_n^{\pm}$ can be calculated by solving the set of algebraic equations

$$\sum_{n=1}^{N}C_{ln}^{\pm}\varphi_n^{\pm}=0,\quad l=0,1,\cdots N-2 \quad (1.9a)$$

$$i\sum_{n=1}^{N}B_n^{\pm}\varphi_n^{\pm}=1, \quad (1.9b)$$

where

$$C_{ln}^{\pm}\equiv\frac{k_n^l}{X(k_n)}\pm\frac{(-k_n)^le^{ik_nL}}{X(-k_n)}\mp\beta\int_1^\infty\frac{(-\beta u)^lf_n^{\pm}(u)e^{i\beta uL}}{X(-\beta u)}du,$$

$$B_n^{\pm}\equiv(1+k_n/k_0)\pm(1-k_n/k_0)e^{ik_nL}-\beta\int_1^\infty dt\,f_n^{\pm}(t)\times[(1+\beta t/k_0)\pm(1-\beta t/k_0)e^{i\beta tL}]. \quad (1.10)$$

Condition (1.9b) represents the imposition of the spatial-boundary conditions (1.1b) and (1.1c). The other conditions (1.9a) arose in a rather subtle way from use of Muskhelishvili methods[5] in obtaining the solution, a procedure inspired by Case's pioneering work in transport theory.[6]

[5] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953).

[6] A series of papers by Case and his co-workers appears in Ann. Phys. (N. Y.) from 1959 to 1963. Two important examples are: K. M. Case, Ann. Phys. (N. Y.) **9**, 1 (1960); R. Zelazny, A. Kuszell, and J. Mika, Ann. Phys. (N. Y.) **16**, 69 (1961).

Consider Eq. (1.6) for a moment. We shall call that part of the field described by the integral the "single-particle excitations." The range of those excitations whose amplitude is $f_n^{\pm}(t_0)$ is clearly $1/t_0\,\mathrm{Im}\beta$. The largest values of $f_n^{\pm}(t)$ describe the excitations with short range, those t values nearest unity the excitations of longest range. It is reasonable that only excitations with a range longer than the thickness of the slab should suffer multiple reflection in the slab. The structure of Eq. (1.7), whose iterative solution generates the multiple-reflection series for the single-particle excitations, confirms that this is so. In that equation, $f_n^{\pm}(u)$ under the integral sign appears multiplied by $\exp(i\beta uL)$, which cuts the integral off at about $u=1/L\,\mathrm{Im}\beta$, discarding those excitations with range shorter than L.

If we observe the field at $z=L$, the short-range excitations from the incident face will not be present. However, they will influence the reflection of the incident wave, and thereby the amount of energy entering the slab. Short-range excitations will also play a role at the emergent face of the slab where they will influence the internal reflection of waves within the slab. These two effects of the short-range excitations are contained in the mathematics in (1.10), where knowledge of $f_n^{\pm}(t)$ is required at all t, and in (1.5) and (1.6) where evaluating $f^{\pm}(0)$ again requires knowledge of $f_n^{\pm}(t)$ at all t.

We showed in Ref. 2, that our equations can be solved explicitly when the width of the slab is infinite. One suspects that the role of the short-range excitations in determining the reflections should be the same whether the slab is finite or infinite. This suggests that it might be possible, in some way, to use the explicit solution for the infinite medium in order to bypass the task of calculating the short-range part of the excitations. This suggestion motivates the present work. It does turn out to be possible to recast Eqs. (1.5)–(1.10) into a form where only the long-range part of the single-particle excitations appear explicitly. This will be demonstrated in the next section.

II. REMOVAL OF THE SHORT-RANGE EXCITATIONS

Consider the coefficient $B_n^{\pm}$ defined by Eq. (1.10). Part of this coefficient seems to depend on knowledge of the entire excitation spectrum, namely

$$\mathcal{I}\equiv\int_1^\infty\beta dt\,f_n^{\pm}(t)(1+\beta t/k_0). \quad (2.1)$$

It is this term which we must try to reexpress. Using the integral equation (1.7) to evaluate $f_n^{\pm}(t)$ gives

$$\mathcal{I}=\frac{1}{2\pi i}\int_1^\infty\beta dt[X^+(\beta t)-X^-(\beta t)](1+\beta t/k_0)\left[\frac{1/X(k_n)}{\beta t-k_n}\pm\frac{e^{ik_nL}/X(-k_n)}{\beta t+k_n}\mp\int_1^\infty\frac{f_n^{\pm}(u)e^{i\beta uL}\beta du}{(\beta t+\beta u)X(-\beta u)}\right]. \quad (2.2)$$

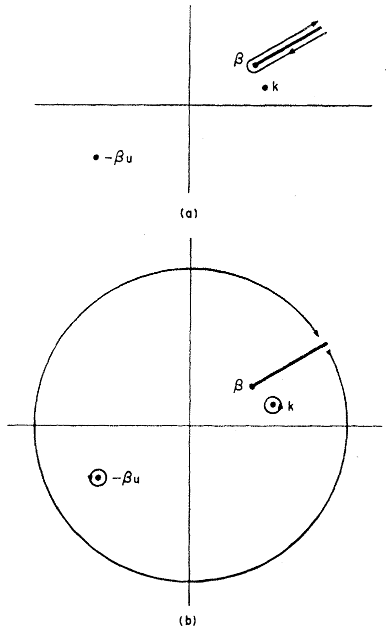

FIG. 1. (a) Original path of integration in the complex-η plane; (b) deformed path.

The t integral here can be regarded as an integral in the complex η plane where the path of integration is inwards from $t=\infty$ to $t=1$ along the line $\eta=\beta(t-i\epsilon)$ and outwards from $t=1$ to $t=\infty$ along the line $\eta=\beta(t+i\epsilon)$. The function $X(\eta)$ has been constructed with just those properties[5,2] so that its integral along the infinitesimal semicircle about the branch point $\eta=\beta$ will vanish in the limit that the radius of the semicircle goes to zero. Thus (2.2) can be written

$$\mathcal{g}=\frac{1}{2\pi i}\int_C d\eta\, X(\eta)(1+\eta/k_0)\left[\frac{1/X(k_n)}{\eta-k_n}\pm\frac{e^{ik_nL}/X(-k_n)}{\eta+k_n}\mp\int_1^\infty\frac{f_n^\pm(u)e^{i\beta uL}\beta du}{(\eta+\beta u)X(-\beta u)}\right], \quad (2.3)$$

where the contour C starts at infinity, comes in just below the cut, circles the branch point and goes off again to infinity just above the cut, as in Fig. 1(a).

Now consider the contour C to be swept outwards from the cut until it becomes the circle at infinity, traversed in the negative sense, Fig. 1(b). That is, $\eta\rightarrow Re^{i(\varphi+\xi)}$ and $d\eta=-i\eta d\varphi$ where $0<\varphi<2\pi$ and where ξ is the angle between the cut and zero. As the contour is swept outwards, it will cross the singularities of the integrand. There are poles at $\eta=k_n$, $\eta=-k_n$, and $\eta=-\beta u$,[7] but, because $X(\eta)$ is everywhere analytic in the cut plane, there are no other singularities. The pole singularities contribute residues which are easily evaluated and the result is now

$$\mathcal{g}=(1+k_n/k_0)\pm(1-k_n/k_0)e^{ik_nL}\mp\int_1^\infty f_n^\pm(u)e^{i\beta uL}\beta du$$
$$-\lim_{R\rightarrow\infty}\frac{1}{2\pi i}\int_0^{2\pi} id\varphi\;\eta X(\eta)(1+\eta/k_0)\left[\frac{1/X(k_n)}{\eta-k_n}\pm\frac{e^{ik_nL}/X(-k_n)}{\eta+k_n}\mp\int_1^\infty\frac{f_n^\pm(u)e^{i\beta uL}\beta du}{(\eta+\beta u)X(-\beta u)}\right]. \quad (2.4)$$

When this is inserted into (1.10), the residue terms here cancel the other terms in $B_n^\pm$, so that $B_n^\pm$ is equal to the φ integral in (2.4). The φ integral can be carried out by expressing the integrand as a power series in η (choosing the form of the power series appropriate to large η). The φ integral then projects out the constant term of this series.

To facilitate this expansion, let us write the power-series expansions for various factors of the integrand: for example,

$$\frac{1/X(k_n)}{\eta-k_n}\pm\frac{e^{ik_nL}/X(-k_n)}{\eta+k_n}\mp\int_1^\infty\frac{f_n^\pm(u)e^{i\beta uL}\beta du}{(\eta+\beta u)X(-\beta u)}=\frac{1}{\eta}\sum_{l=0}^{\infty}y_l^\pm(n)/\eta^l. \quad (2.5)$$

Note that the coefficients of this series, the $y_l^\pm(n)$, are identical to the coefficients $C_{ln}^\pm$ defined in (1.10). Therefore if the coefficients $\varphi_n^\pm$ satisfy (1.9), it will follow that

$$\sum_{n=1}^{N} y_l^\pm(n)\varphi_n^\pm=0, \quad l=0,1,\cdots,N-2. \quad (2.6)$$

Similarly, we can write a power-series expansion for $X(\eta)$. Using (1.8), we write

$$X(\eta)=(-\eta)^{N-1}\sum_{p=0}^{\infty}x_p/\eta^p, \quad (2.7a)$$

where, in particular,

$$x_0=1, \quad (2.7b)$$

$$x_1=-(N-1)\beta-\frac{1}{2\pi i}\int_1^\infty \beta du\ln G(\beta u)\equiv -J. \quad (2.7c)$$

Using (2.5) and (2.7), the integrand in (2.4) is

$$(-1)^{N-1}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}x_p y_l^\pm(n)\eta^{p+l+1-N}(1+\eta/k_0)$$

and the constant term, which is projected out by the φ integration, gives simply

$$B_n^\pm=(-1)^{N-1}\left[\sum_{l=0}^{N-1}x_{N-1-l}y_l^\pm(n)+\frac{1}{k_0}\sum_{l=0}^{N}x_{n-l}y_l^\pm(n)\right].$$

[7] Alternatively, instead of reversing the order of integration and picking up a pole at $\eta=-\beta u$, one could have considered the singularity in the η plane to be another cut which the deformed contour would surround. Evaluation of the u integral would then be performed *first* by using the Plemelj formulas, giving the same result.

When we use this expression in (1.9b), the condition (2.6) removes many of the terms of the n summation. Those that are left yield

$$\sum_{n=1}^{N}\left[(k_0-J)y_{N-1}^{\pm}(n)+y_N^{\pm}(n)\right]\varphi_n^{\pm} = -i(-1)^{N-1}k_0. \quad (2.8)$$

This is to be used instead of (1.9b) for calculating the coefficients $\varphi_n^{\pm}$. Significantly, each appearance of $f_n^{\pm}(t)$ in (2.8) is as a factor of the product $f_n^{\pm}(t)\exp i\beta tL$, which makes knowledge of the large-t behavior of $f_n^{\pm}$ unnecessary. The effects of the short-range excitations must be contained within the factor J, which also plays an important part in setting the normalization of the semi-infinite medium solution.

We now consider the emergent field $e(L)$ as given by (1.5):

$$e(L)=f^{+}(L)+f^{-}(L)+f^{+}(0)-f^{-}(0). \quad (2.9)$$

The terms $f^{\pm}(0)$ defined in (1.6) again appear to require knowledge of $f_n^{\pm}(t)$ at all t. The cure for this apparent difficulty is the same as before: use the integral equation for $f_n^{\pm}(t)$, convert to a path integral, expand the path outwards to pick up residues and a contribution from the circle at infinity. The residue terms arising from the calculation of $f_n^{\pm}(0)$ this time turn out to equal $\mp f_n^{\pm}(L)$. The contribution from the circle at infinity is again evaluated using (2.5), (2.6), and (2.7). We finally obtain

$$f^{\pm}(0)=\mp f^{\pm}(L)+i(-1)^{N-1}\sum_{n=1}^{N}y_{N-1}^{\pm}(n)\varphi_n^{\pm} \quad (2.10)$$

which substituted into (3.1) yields

$$e(L)=i(-1)^{N-1}\sum_{n=1}^{N}\left[y_{N-1}^{+}(n)\varphi_n^{+}-y_{N-1}^{-}(n)\varphi_n^{-}\right]. \quad (2.11)$$

By using (2.11) and (2.8) instead of (1.5) and (1.9b) to evaluate $e(L)$, we have obtained a form where each appearance of $f_n^{\pm}(t)$ is accompanied by the factor $e^{i\beta tL}$, which discards all excitations whose range is less than the slab width. This is the desired reformulation.

III. ELECTROMAGNETIC WAVE IN THE DIFFUSE SLAB

In the case of a circularly polarized electromagnetic wave propagating through a fully degenerate free-electron gas, use of the standard scattering-time approximation leads to a dispersion function[8,9,2]

$$\psi(k)=k^2-k_0^2-\frac{3}{2}\frac{\mu_0 nq^2\omega}{kp_F}\left[\frac{1}{2}\left(1-\frac{\beta^2}{k^2}\right)\ln\frac{\beta-k}{\beta+k}-\frac{\beta}{k}\right], \quad (3.1a)$$

where

$$\beta=\frac{\pm 1+\omega/\omega_c}{R}+\frac{i}{l}. \quad (3.1b)$$

For metallic densities and frequencies below the infrared of interest here, neglect of k_0^2, the displacement-current contribution, is justified. The $\pm$ sign in β refers to the two possible senses of circular polarization for the transverse electromagnetic wave whose time dependence is $e^{-i\omega t}$. Here, n is the number density of the gas; p_F is its Fermi momentum; $\omega_c=qB/m$, the cyclotron frequency; $R=V_F/\omega_c$, the cyclotron radius; and $l=V_F\tau$ the mean free path. The branch of the logarithm to be used is the one which vanishes at $k=0$.

It becomes convenient to introduce several parameters related to those already listed. First, polar coordinates for β are useful

$$\beta=e^{i\xi}/r \quad (3.2a)$$

so that

$$\xi=\tan^{-1}[1/(\pm\omega_c+\omega)\tau] \quad (3.2b)$$

is the angle between the cut and the real axis, and the distance from the branch point to the origin is given by

$$1/r=[(\pm\omega_c+\omega)^2+1/\tau^2]^{1/2}/v_F. \quad (3.2c)$$

In the limit of low frequency and large $\omega_c\tau$, r becomes the cyclotron radius. At cyclotron resonance, it increases and becomes the mean free path. We define η (a real number, not the complex variable used briefly in Sec. II) as

$$\eta\equiv\frac{3}{2}\frac{\mu_0 nq^2\omega r^3}{p_F}\equiv\frac{3}{2}\left(\frac{V_F}{C}\right)^2\frac{\omega\omega_p^2}{[(\pm\omega_c+\omega)^2+1/\tau^2]^{3/2}} \quad (3.3)$$

and

$$\rho\equiv(\tfrac{1}{2}\pi\eta)^{1/3}. \quad (3.4)$$

[Here ω_p is the plasma frequency $(nq^2/m\epsilon_0)^{1/2}$]. In the limit of long mean free path, the value $\eta=1$ describes the onset of Doppler-shifted cyclotron resonance.[9] At small values of η, the helicon wave is a well-defined excitation and the electron gas behaves as if its conductivity were local. At large values of η, the nonlocality is severe. In the absence of a magnetic field, large values of η correspond to the so-called extreme anomalous limit.[3] The large-η limit (with a magnetic field present) will be our concern here.

At large values of η, $1/\rho$ is small enough so that useful results can be obtained by working to lowest order in $1/\rho$ throughout. This we shall do.

In terms of these parameters, the dispersion relation is

$$\psi(k)=\beta^2\left\{(k/\beta)^2-\frac{\eta e^{-3i\xi}}{(k/\beta)}\left[\frac{1}{2}\left(1-\frac{\beta^2}{k^2}\right)\ln\frac{\beta-k}{\beta+k}-\frac{\beta}{k}\right]\right\}. \quad (3.5)$$

[8] P. M. Platzman and S. J. Buchsbaum, Phys. Rev. **132**, 2 (1963).

[9] P. B. Miller and R. R. Haering, Phys. Rev. **128**, 126 (1962).

A. Location of Roots

The number of roots k_n can be determined by following the change in the phase of ψ along its branch cuts.[2] At large η, one will find either one or two roots in the upper half-plane, depending on the value of ξ. Assuming that the roots are at large k/β, one can expand the logarithm and, applying some care to maintain the correct branch of the logarithm, can seek roots by setting

$$k\psi(k)/\beta^3 \approx (k/\beta)^3 + \eta e^{-3i\xi} \times i\pi/2T = 0,$$

where $T = \mathrm{sgn}\,\mathrm{Im}(k/\beta)$. This cubic equation would have three roots were it not for the factor T which arises from discontinuities across the branch cut of the logarithm. By taking $T = \pm 1$ without considering its k dependence, one has three possible roots in the upper half-plane,

$$k_l = (\beta\rho) e^{i(\varphi_l - \xi)}, \tag{3.6a}$$

where

$$\varphi_1 = \tfrac{1}{6}\pi, \quad \varphi_2 = \tfrac{1}{2}\pi, \quad \varphi_3 = \tfrac{5}{6}\pi. \tag{3.6b}$$

Only one or two of these are actually solutions for a given ξ because T depends on k. The values of l, designating the actual roots, depend on ξ as follows:

$$\begin{array}{lll} 0<\xi<\frac{1}{6}\pi & l=2 & N=1 \\ \frac{1}{6}\pi<\xi<\frac{1}{2}\pi & l=2,1 & N=2 \\ \frac{1}{2}\pi<\xi<\frac{5}{6}\pi & l=1 & N=1 \\ \frac{5}{6}\pi<\xi<\pi & l=1,3 & N=2. \end{array} \tag{3.7}$$

Since N, the number of roots found here, agrees with the number independently determined by the phase arguments, there are no roots which we have overlooked by assuming large k/β.

B. Integral Equation for f_n

In Sec. II, we established that we need $f_n^{\pm}(t)$ only for t of order unity. Since k_n is of size $\beta\rho$, we can, to lowest order in $1/\rho$, neglect βt compared to k_n in Eq. (1.7). The term e^{ik_nL} is of order $e^{-\rho}$; we neglect it also, and (1.7) becomes

$$f_n^{\pm}(t) = [X^-(\beta t) - X^+(\beta t)/2\pi i k_n X(k_n)] \times F^{\pm}(t), \tag{3.8a}$$

where $F(t)$ satisfies the equation

$$F^{\pm}(t) = 1 \pm \int_1^{\infty} \frac{W(u)e^{i\beta uL}F^{\pm}(u)du}{(u+t)}, \tag{3.8b}$$

and where

$$W(u) \equiv [X^-(\beta u) - X^+(\beta u)]/2\pi i X(-\beta u). \tag{3.8c}$$

If we iterate this equation (3.8b) to convergence, we develop a series of the form

$$F^{\pm}(t) = g(t) \pm h(t), \tag{3.9}$$

where $g(t)$ is the sum of all even iterates (the zeroth, second$\cdots$) and $h(t)$ is the sum of the odd iterates.

C. Equations for φ_n and $e(L)$

Inserting (3.8) and (3.9) into (2.5) gives

$$y_l^{\pm}(n) = \frac{k_n^l}{X(k_n)} \pm \frac{\beta}{k_n X(k_n)} \int_1^{\infty} (-\beta u)^l W(u) e^{i\beta uL} \times [g(u) \pm h(u)] du$$

$$= \frac{k_n^{l+1} + w_l \pm z_l}{k_n X(k_n)}, \tag{3.10a}$$

where

$$w_l = -\beta^{l+1} \int_1^{\infty} (-u)^l W(u) h(u) e^{i\beta uL} du, \tag{3.10b}$$

$$z_l = -\beta^{l+1} \int_1^{\infty} (-u)^l W(u) g(u) e^{i\beta uL} du. \tag{3.10c}$$

It is convenient to change the normalization of the coefficients $\varphi_n^{\pm}$ by introducing

$$\theta_n^{\pm} \equiv i(-1)^{N-1} \varphi_n^{\pm}/k_n X(k_n). \tag{3.11}$$

Using (3.10a) and (3.11) into (2.10) gives

$$e(L) = \sum_{n=1}^{N} [(k_n^N + w_{N-1} + z_{N-1})\theta_n^+ - (k_n^N + w_{N-1} - z_{N-1})\theta_n^-] \tag{3.12}$$

while inserting (3.10a) and (3.11) into (2.6) and (2.8) gives the algebraic equations which must be solved for the $\theta_n^{\pm}$ coefficients:

$$\sum_{n=1}^{N} (k_n^{l+1} + w_l \pm z_l)\theta_n^{\pm} = 0, \quad l = 0, \cdots N-2, \tag{3.13a}$$

$$\sum_{n=1}^{N} [(k_n^N + w_{N-1} \pm z_{N-1})(k_0 - J) + (k_n^{N+1} + w_n \pm z_n)]\theta_n^{\pm} = k_0. \tag{3.13b}$$

Equations (3.13) are to be solved for $\theta_n^{\pm}$ and the result used to evaluate $e(L)$ in (3.12). Although the details of solution for the two cases $N=1$ and $N=2$ differ, the final forms do not. To lowest order in $1/\rho$, we obtain[10]

$$e(L) = 2k_0 z_0/(k_0 + \sum_{n=1}^{N} k_n - J)^2$$

or, making use of (3.10c) and (2.7c),

$$e(L) = -2k_0\beta \int_1^{\infty} W(u) g(u) e^{i\beta uL} du \Big/ \left[k_0 + \sum_{n=1}^{N} k_n - (N-1)\beta - \frac{1}{2\pi i}\int_1^{\infty} \beta du \ln G(\beta u)\right]^2. \tag{3.14}$$

[10] We shall see that in the range near $u=1$, the function $W(u)$, which is the same in (3.10) as in (3.8), is of order unity and that therefore $w_l/k_n^{l+1} \approx 1/\rho^{l+1}$ and that $z_l/k_n^{l+1} \approx 1/\rho^{l+1}$.

This result is correct to lowest order in $1/\rho$ and deserves some comment. In the numerator of this expression for the emergent fields is the function $W(u)g(u)$ which consists of W times the sum of even iterates of (3.8b). The first term of $g(u)$ is 1 and each higher term is smaller than the one before it by $\exp(-2L\,\text{Im}\beta)$. The interpretation of these higher terms as partial amplitudes which have been returned to the emergent face after additional round trips through the slab is clear, especially since the function $W(u)\exp i\beta uL$, which is the first partial amplitude (the first term in the series), is the same function as appears in (3.8b), the equation which generates the iterates. If the numerator integral describes the attenuation in the slab on the first and successive traverses of the slab, then the other terms in (3.14) must describe the effect of the incident and emergent faces in allowing the wave into and out of the slab, i.e., must describe reflection at the faces. This interpretation is consistent with the absence of any thickness dependence in these other terms.

D. Evaluation of the Emergent Fields

In Appendix A, we show that, to within an accuracy of $1/\rho$,

$$\frac{1}{2\pi i}\ln G(\beta u)=N-\tfrac{3}{2}+q(u)\,,\quad 1\leq u<\rho^{1/2},$$

where N is the number of upper-half-plane roots which depends on ξ as given by (3.7) and where $q(u)$ is a rather smooth looking real function of u which depends on none of the physical parameters of the problem.[11] At $u=1$, $q(1)=\frac{1}{2}$, and at large u, $q(u)$ goes as $4/\pi^2u$. The term $N-\frac{3}{2}$ persists up to $u=\rho$. In addition, $(\frac{1}{2}\pi i)\ln G(\beta u)$ develops a peak of area ρ at $u=\rho$. The height and width of this peak depends on ξ, the peak being sharpest at those values of ξ which swing the cut alongside one of the roots. We show, to accuracy $\rho^{-1}\ln\rho$, that

$$\frac{1}{2\pi i}\int_1^\infty \beta du\,\ln G(\beta u)=\beta\rho[N-\tfrac{3}{2}+F(\xi)]\,,\qquad(3.15\text{a})$$

where $F(\xi)$, which arises from the peak of area ρ, is given by

$$F(\xi)=\frac{1}{\pi}\sum_{n>0\text{ odd}}^{\infty}\frac{(-1)^{(n-1)/2}}{n}\left[\frac{e^{3ni\xi}}{3n+1}-\frac{e^{-3ni\xi}}{3n-1}\right].\qquad(3.15\text{b})$$

The function $F(\xi)$, which is periodic with period $\frac{1}{3}2\pi$, can be evaluated most easily by noting its resemblance to $F^0(\xi)$, the Fourier-series representation of the saw-tooth function of the same period

$$F_0(\xi)\equiv\tfrac{1}{2}\xi;\quad -\tfrac{1}{6}\pi<\xi<\tfrac{1}{6}\pi\qquad(3.16\text{a})$$

$$\equiv\tfrac{1}{6}\pi-\tfrac{1}{2}\xi;\quad \tfrac{1}{6}\pi<\xi<\tfrac{1}{2}\pi\qquad(3.16\text{b})$$

$$\equiv\frac{1}{i\pi}\sum_{n>0}\frac{(-1)^{(n-1)/2}}{n}\left[\frac{e^{3ni\xi}}{3n}-\frac{e^{-3ni\xi}}{3n}\right].\qquad(3.16\text{c})$$

The difference between $F(\xi)$ and $iF_0(\xi)$ is a series whose convergence is much more rapid than either. Two terms for the real part and one for the imaginary part

$$F(\xi)\approx-\frac{1}{4\pi}\left(\cos3\xi-\frac{\cos9\xi}{30}\right)+i\left(F_0(\xi)+\frac{\sin3\xi}{12\pi}\right)\qquad(3.17)$$

provide accuracy of better than 1%.

We use (3.15a) to evaluate the denominator of (3.14). The terms k_0 and $(N-1)\beta$ in (3.14) are discarded as being $1/\rho$ smaller than those retained (at metallic densities and frequencies below the infrared). The roots k_n in (3.14) have been evaluated by (3.6a), and so the denominator of (3.14) is

$$\beta\rho\Big[\sum_{\substack{N\text{ values}\\\text{of }l}} e^{i(\varphi_l-\xi)}-N+\tfrac{3}{2}-F(\xi)\Big]\equiv\beta\rho G(\xi)\,.\qquad(3.18\text{a})$$

There is undoubtedly some Fourier expansion we have failed to take advantage of here, for after a few moments of numerical evaluation, it becomes clear that

$$G(\xi)=(2/\sqrt{3})\exp i(\pi/3-\xi)\,.\qquad(3.18\text{b})$$

Inserting (3.18) into (3.14) gives

$$e(L)=\frac{3k_0}{2\beta\rho^2}e^{2i(\xi-\varphi/3)}\int_0^\infty W(u)g(u)e^{i\beta uL}du\,.\qquad(3.19)$$

In Appendix B, we study the function $W(u)$ and show that for u less than $\rho^{1/2}$, it is real and independent of all the physical parameters. The important feature of its u dependence in this range is that it is of order unity and decreases smoothly to zero at $u=1$, approaching zero with a slope S also of order unity. This behavior is enough to fix the form of the emergent field in thick slabs, i.e., for $L/l=L\,\text{Im}\beta>1$. If L is large enough, multiple reflection [higher iterates of (3.8b)] can be ignored and we can take $g(u)=1$. Then in evaluating the integral in (3.19). we can use the form of $W(u)$ appropriate to smallest u, namely

$$\int_1^\infty W(u)e^{i\beta uL}du=S\int_1^\infty (u-1)e^{i\beta uL}du=-Se^{i\beta L}/(\beta^2L^2)\,,$$

where S is the slope of W at $u=1$. Using this evaluation, and the definitions (3.2) and (3.4) give

$$e(L)=[3k_0S\rho/2(\rho/r)^3L^2]\exp i(\beta L-2\pi/3-\xi)\,.\quad(3.20)$$

[11] It does depend on the form of the Fermi surface, which we have taken to be a sphere.

IV. DISCUSSION OF THE TRANSMISSION AMPLITUDE

The field $e(L)$ at the emergent face of the slab, as given by (3.20) can be rewritten, using definitions (3.2), (3.3), and (3.4) as

$$e(L)=-\frac{2}{\pi}\frac{v_F c}{\omega_p{}^2L^2}S\rho e^{i(\beta L-2\pi/3-\xi)}, \qquad (4.1a)$$

where S is a constant of order unity. The amplitude of $e(L)$ depends on magnetic field only through the factor

$$\rho=\left[\frac{3\pi}{4}\left(\frac{v_F}{c}\right)^2\omega\omega_p{}^2\right]^{1/3}\bigg/\left[(\pm\omega_c-\omega)^2+1/\tau^2\right]^{1/2}, \qquad (4.1b)$$

which gives the square root of a Lorentzian for the line shape near cyclotron resonance. The attenuation of the field is $L^{-2}\exp(-L/l)$, characteristic of single-particle excitations near a limiting point of the Fermi surface.[3,1] The phase of the emergent field exhibits the Gantmakher-Kaner oscillations [the term $L\,\mathrm{Re}\beta=(\pm\omega_c-\omega)L/v_F$ in the exponent] which are linear in magnetic field. The periodicity of this series of oscillations is spoiled at cyclotron resonance where the phase slips by an additional amount π (the term $\xi=\tan^{-1}1/(\pm\omega_c+\omega)\tau$ in the exponent) across the resonance peak.

There is a reasonably simple interpretation of the oscillations, the resonance peak, and the phase slip across the peak, which is based on the idea that the field of the incident wave penetrates a short distance into the slab, leaving the interior relatively field-free.[12] In the situation we have been calculating the electric field of the incident wave is parallel to the slab surface, and circularly polarized so that it rotates in the plane of the surface with a frequency ω. An electron moving across the slab will receive an impulse from the electric field only while it is near the incident surface. Thereafter, its motion carries it through the slab with velocity v_z, during which time, it experiences only the steady magnetic field which bends its path into a helix. If this electron is observed at a time t after receiving the original impulse, its position is v_zt, and the direction of the current it carries at that time is at an angle ω_ct with the original direction. Another electron of velocity v_z observed closer to the incident face at this same time will have been at the face more recently than the first, and hence will have suffered less rotation. It will, however, have received an impulse from the electric field at a later time, when the direction of that field is somewhat rotated. It is clear then that at any instant, the direction of the impulse carried by electrons varies with distance across the slab and that the angle describing this direction is $(\pm\omega_c+\omega)z/v_z$ where the $\pm$ refers to the two possible senses of field rotation relative to cyclotron rotation. At any given point the direction itself rotates with frequency ω, as does the initial-field direction. Thus the impulse I carried by the electrons of velocity v_z is

$$I_x\approx\cos[(\pm\omega_c+\omega)z/v_z-\omega t],$$
$$I_y\approx\sin[(\pm\omega_c+\omega)z/v_z-\omega t].$$

Although this looks like the description of a wave, there is no wave in the conventional sense of energy flow between oscillators. There is just a rotating helix. As the cyclotron frequency ω_c approaches the field frequency ω, the pitch of the helix (for the resonant direction of rotation) lengthens and, at cyclotron resonance, becomes infinite. When the cyclotron frequency passes the field frequency, the pitch of the helix shortens up, but this time the handedness of the helix is opposite to what it previously was.

In the metal, electrons have a continuous distribution of v_z values. The contribution of each of these electrons, each with its own helical pitch, must be added. Electrons whose motion is exactly across the slab arrive soonest and (all other things being equal) in greatest number because their direct path across the slab makes them least subject to the collisions which inevitably interrupt their flight. These electrons would therefore determine the dominant helical pitch, or apparent wavelength of the signal.

All other things are, however, *not* equal. In the first place, the electrons which travel directly across the slab are those which originate at circumpolar latitudes on the Fermi sphere. Their motion is most nearly normal to the electric field and hence, by reason of their direction, they receive the least energy from the field. (In a degenerate Fermi gas, the current carried per particle depends on the energy it acquires from the field, not the velocity, because in effect the Pauli principle demands transitions across the Fermi energy.) They contribute least to the current for this reason. Secondly, their motion carries them out of the field region most rapidly and their effectiveness is further decreased. Nonetheless when all the individual contributions are added, the effective pitch of the helix is as *though* the circumpolar electrons were most effective, although the phase of the helix leads or lags that of the circumpolar electrons somewhat, being held back or advanced by electrons originating closer to the Fermi equator. Whether the phase shift is a lead or a lag depends on the cyclotron frequency relative to the field frequency: electrons originating closer to the equator and observed at the emergent face must have left the incident face earlier than those which originated closer to the pole. The question is whether the added cyclotron rotation in their longer flight was greater, or less than, the added rotation of the field at the incident face during the same time.

The above ideas describe the phase of the field at

[12] G. Weisbuch and A. Libchaber, Phys. Rev. Letters 19, 498 (1967).

the emergent face as a function of magnetic field: the linear increase in phase (the Gantmakher-Kaner term) arises from the steady change in the pitch of the helix, and the phase shift across the resonance peak represents the conversion of the phase lead of the equatorial electrons to a lag, relative to the phase of the polar electrons.

The ideas also describe the transmission peak at cyclotron resonance. Although the analogy is not exact, it is as though one had Fresnel zones on the Fermi surface, each zone being defined such that all electrons from that zone arrive at the emergent face with a phase along the helix which is within $\frac{1}{2}\pi$ of the average from that zone. Contributions from adjacent zones tend to cancel although (unlike in the optical case) the cancellation is not complete. The zones, constructed in this way, would be bounded by parallels of latitude whose spacing is uniform in $1/\cos\theta$, θ being the polar angle, and would have a width given by $\Delta(1/\cos\theta)=\lambda/2L$ where λ is the pitch of the circumpolar helix. The thicker the slab, the narrower are the zones and the more complete the phase cancellation will be. [Note the $1/L^2$ decrease in (4.1a), even when the exponential attenuation due to collisions can be ignored.] At resonance, however, the first Frensel zone expands to fill the entire Fermi hemisphere, and there is no cancellation of adjacent zones, so the signal is more intense.

These Fresnel-zone expansion effects by themselves would lead, in this limit, to a line shape with a Lorentzian dependence on magnetic field and a phase which slips by 2π across the resonance. [The mathematics which leads to this conclusion is formally identical with the unnumbered equation preceding Eq. (3.20).] However, this discussion has ignored what changes the driving field itself will experience. Clearly, it the portion of the field which penetrates the surface and launches these helical electrons changes its phase or intensity relative to the incident field, or, changes its depth of penetration, the intensity of the impulse given to the electrons crossing it or the phase of this impulse relative to that of the incident field will also change, and thus there will be a corresponding effect on the signal transmitted across the slab. The driving field certainly has structure of its own (i.e., wavelength, attenuation length, phase relative to the incident field). This structure can be expected to change near cyclotron resonance, because it is determined in large measure by the currents near the surface.These currents, which shield the interior of the metal, are the response of the electrons to the fields to which they are subject. Think of these electrons as harmonic oscillators with frequency ω_c. The response of the electrons will lag or lead the field, as any harmonic oscillator lags or leads the oscillatory driving force, when the frequency of that force goes through resonance with the frequency of the oscillator. How sharp the resonance is depends on the damping of the oscillator. Similar effects must occur at the emergent fact of the slab. The point is that in observing the emergent fields near cyclotron resonance, one is experiencing the effects of *three* resonances which are occurring simultaneously, one due to Fresnel-zone expansion, the other two due to the shielding currents at the two faces of the slab.

The assumption of diffusion scattering of electrons at slab boundary faces corresponds to heavy damping of the electrons whose response is the shielding current. In such a case the emergent field will exhibit more of the behavior we have described as Fresnel-zone expansion. The assumption of specular reflection of electrons from the boundary faces corresponds to no damping of these same electrons, other than that provided by bulk collisional damping. In such a case, the resonant behavior of the shielding current should augment (or diminish, since this is a resonance in the shielding which means that the field penetrates less) the effect of Fresnel-zone expansion. Specular reflection, as calculated by Gantmakher and Kaner, shows no resonance at $\omega_c=\omega$ (even when one restores terms of order ω/ω_c which were dropped because of the low frequency of interest in their paper). We can conclude that the resonant response of the electrons adjusts the field penetration in such a way as to cancel the effects of phase shift and Fresnel-zone expansion on the electrons travelling across the specular slab.[13]

This discussion should not be interpreted as meaning that diffuse reflection at the surface eliminates all resonant response of the shielding electrons. Clearly, those electrons which are directed into the metal at small enough angles so that they stay in the skin depth, or driving field, until they are scattered will respond resonantly whether their original encounter with the surface was specular or diffuse. Indeed, the resonant response of the shielding electrons is the most probable cause of the difference between the bare Fresnel-zone expansion effect (Lorentzian line shape and 2π phase slip) and the results (4.1) for the diffuse slab (square root of a Lorentzian and π phase slip). The bare Fresnel-zone expansion effect could be observed only if one postulated some mechanism, i.e., macroscopic surface roughness, external to this model, which could act to intercept all electrons which would otherwise give a resonant shielding.

One implication of the Fresnel-zone idea here is that in metals where the Kaner-Gantmakher signals arise from a flat portion of the Fermi surface, so that many electrons carry the signal, the opening of the Fresnel zone will not lead to as large an enhancement of the number of electrons which arrive in phase. Hence, for

[13] The absence of any resonance in the specular transmission is not strictly correct except in the large (L/l) limit. As L/l is decreased, corrections analogous to those discussed in the following paragraphs reduce the effect of the Fresnel-zone expansion so that it no longer exactly compensates the increased shielding caused by the equatorial electrons. The result is then a transmission dip at cyclotron resonance.

such metals, there is much less reason for a transmission peak at cyclotron resonance.

To return now to the calculation of the transmission amplitude, as the thickness of the slab is decreased, corrections to (4.1) should arise from two sources: first, shorter-range excitations are transmitted through the slab and second, the longer-range excitations themselves start to suffer multiple reflection. The first of these corrections enters the mathematics as the necessity for using a representation for $W(u)$ which is valid at larger values of u than is the simple $W\approx S(u-1)$ just used. [The function $W(u)$ really describes the number of electrons at polar angle θ, where $u=1/\cos\theta$, and how efficiently they obtain energy from the field just below the incident surface of the slab. The statment "all other things being equal" used in the discussion just given would translate into mathematics as "$W(u)$ being a constant".] A simple approximate form which is useful in the range $1\leq u<6$ (see Appendix B) is

$$W(u)\approx 0.218[1-0.88e^{-0.65(u-1)}-0.12e^{-7.7(u-1)}]. \quad (4.2)$$

The second correction, becomes important at smaller slab thicknesses than does the first. It enters the mathematics through the need for using higher iterates of (3.8b) to determine $g(u)$. For slabs greater than one mean free path thick, the next iterate provides a correction of about 1%, and so the major correction to (4.1) in this range arises by using (4.2) but retaining $g(u)=1$ in (3.19). Doing so gives a result similar to (4.1) but with S replaced by

$$0.218\left[\frac{1.50\text{–}5.0/i\beta L}{(1\text{–}0.65/i\beta L)(1\text{–}7.7/i\beta L)}\right]. \quad (4.3)$$

The corrections embodied in (4.3) tend to depress the peak of the line at cyclotron resonance relative to its wings, making the line shape somewhat less pronounced.

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APPENDIX A: THE REFLECTION FACTOR *J*

In this Appendix, we study the evaluation of an integral which plays a major role in determining the normalization, namely

$$K\equiv\frac{1}{2\pi i}\int_1^\infty \beta du\,\ln G(\beta u)\,, \quad (\text{A1a})$$

where

$$G(\beta u)=\psi^-(\beta u)/\psi^+(\beta u) \quad (\text{A1b})$$

and

$$\psi^\pm(\beta u)=\lim_{\epsilon\to 0+}\psi[k=\beta(u\pm i\epsilon)]. \quad (\text{A1c})$$

Using (3.4) and (3.5) for the dispersion function $\psi(k)$, we have

$$\frac{1}{2\pi i}\ln G(\beta u)=\frac{1}{2\pi i}\ln\frac{1-ic(u)}{1+ic(u)}\equiv I(u)\,, \quad (\text{A2a})$$

where

$$c(u)=\frac{b(u)}{e^{3i\xi}-a(u)} \quad (\text{A2b})$$

$$a(u)=\frac{2}{\pi}\left(\frac{\rho}{u}\right)^3\left[\frac{1}{2}\left(1-\frac{1}{u^2}\right)\ln\frac{u-1}{u+1}-\frac{1}{u}\right] \quad (\text{A2c})$$

$$b(u)=\left(\frac{\rho}{u}\right)^3\left(1-\frac{1}{u^2}\right). \quad (\text{A2d})$$

The branch of the logarithm to be used is the one which vanishes at $u=\infty$. The same phase arguments which allow one to count the member of upper half-plane roots of ψ can be used to show that this branch yields[2]

$$I(u=1)=N-1. \quad (\text{A3})$$

We now examine $I(u)$ in various ranges of u with the aim of developing approximations compatible with the $1/\rho$ accuracy we wish to maintain.

Range 1: $1\leq u\leq \rho^{1/2}$

Throughout the range, $e^{3i\xi}$ is negligible in comparison to $a(u)$, and, on dropping this term, we have $c(u)=-b(u)/a(u)$ which is real, and independent of ξ and η. The reality of $c(u)$ means that

$$\begin{aligned}I(u)&=-\frac{1}{\pi}\tan^{-1}c(u)+m\\&=-\frac{1}{\pi}\tan^{-1}\left\{\frac{\pi(1-1/u^2)}{(1-1/u^2)\ln\dfrac{u+1}{u-1}+\dfrac{2}{u}}\right\}+m\,, \end{aligned}\quad (\text{A4})$$

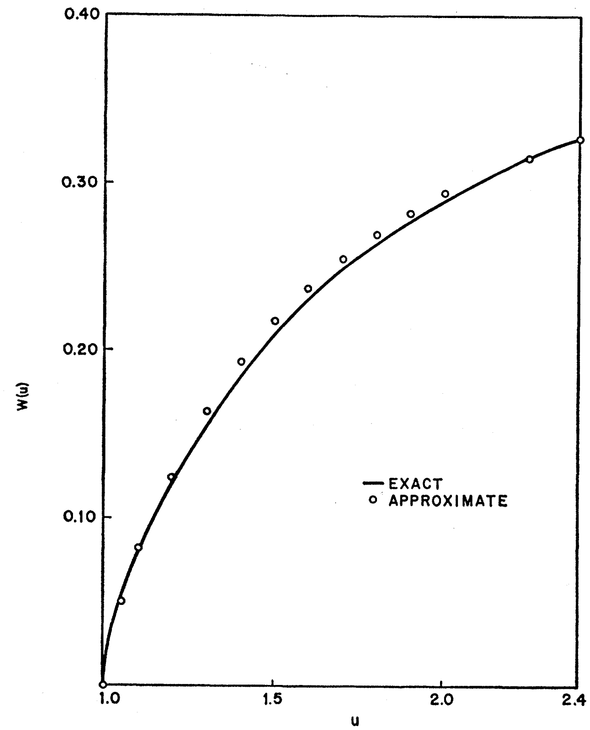

FIG. 2. The function $W(u)$ defined in (3.8c) and the approximation (3.21).

where the integer m arises because $\tan^{-1}$ is determined only to within integer multiples of π. The particular choice of branches compatible with (A3) is $m=N-1$.

At the upper end of this range, we can expand (A4) and obtain

$$I(u)=N-1-\left[\frac{1}{2}-\frac{1}{\pi^2 u}+O(1/u^3)\right],$$

where $O(1/u^3)$ stands for a series of the form

$$\frac{1}{u^3}(a_0+a_2/u^2+a_4/u^4+\cdots)$$

and so, to lowest order, at large u,

$$I(u)=N-\tfrac{3}{2}+4/\pi^2 u. \qquad \text{(A5)}$$

Comparison of the exact (A4) with the approximate (A5) indicates that because of the fortuitous size of the coefficients of the series, the approximation is valid to within about 2% even down to $u=2$. The main defect with the approximation is that at $u=1$, it goes to the wrong limit. We can force the approximation to go to the right limit by subtracting from (A5), the quantity $(4/\pi^2-\frac{1}{2})$ times a function of u which decreases smoothly from unity at $u=1$ to near zero by $u=2$. It turns out that one choice,

$$I(u)=N-\tfrac{3}{2}+4/\pi^2 u+(\tfrac{1}{2}-4/\pi^2)/u^5 \qquad \text{(A6)}$$

is not only simple enough to handle but also is remarkable for its accuracy $\sim$3% over the entire range here (see Fig. 2).

Range 2: $\varrho^{1/2}<u<\varrho^{5/6}$

It is no longer within $1/\rho$ accuracy to ignore $e^{3i\xi}$ in comparison to $a(u)$. We can however use a large u expansion of $a(u)$ and, to within $1/\rho$ accuracy, take

$$1/c(u)=e^{3i\xi}/b(u)-a(u)/b(u)$$
$$\approx(ue^{i\xi}/\rho)^3+4/\pi u.$$

Expanding the logarithm appearing in (A2a) in a power series in $1/c(u)$, and choosing the branch which is continuous with (A6), then gives

$$I(u)=N-\tfrac{3}{2}+4/\pi^2 u+(ue^{i\xi}/\rho)^3/\pi. \qquad \text{(A7)}$$

Range 3: $\varrho^{5/6}<u<\varrho$

In this range, $1/c(u)$, although less than unity, can approach it, and so the full power-series expansion of the log is needed. Again choosing the branch continuous with (A7),

$$I(u)=N-\tfrac{3}{2}-\frac{1}{i\pi}\left[\frac{1}{ic}+\frac{1}{3(ic)^3}+\frac{1}{5(ic)^5}+\cdots\right].$$

However, we can put $1/c=(ue^{i\xi}/\rho)^3$ in all terms of this series save the first, where the piece $4/\pi^2 u$ must also be included, and still stay within the $1/\rho$ accuracy. This gives

$$I(u)=N-\tfrac{3}{2}+\frac{4}{\pi^2 u}+\frac{1}{\pi}\sum_{n=\text{odd}\ n>0}\frac{(-1)^{(n-1)/2}}{n}\left(\frac{ue^{i\xi}}{\rho}\right)^{3n}. \qquad \text{(A8)}$$

Range 4: $\varrho<u$

In this range, $c(u)\leq 1$ and, to accuracy $1/\rho$,

$$c(u)=b(u)/e^{3i\xi}\approx(\rho e^{-i\xi}/u)^3.$$

We use the full power series of the logarithm and choose the branch which vanishes for large u. This gives

$$I(u)=-\frac{1}{\pi}\sum_{n\ \text{odd}\ n>0}\frac{(-1)^{(n-1)/2}}{n}\left(\frac{\rho e^{-i\xi}}{u}\right)^{3n}. \qquad \text{(A9)}$$

This form is not obviously continuous with (A8) at $u=\rho$. However, if one remembers that $N-\frac{3}{2}=\pm\frac{1}{2}$ depending on the value of ξ, and then makes use of the Fourier series for the square wave of periodicity $\frac{1}{3}2\pi$ and amplitude $\frac{1}{2}$, the continuity of these expressions is easy to demonstrate.

Using (A6)–(A9), we now can evaluate (A1a) as

$$K=\int_1^\infty I(u)du$$

$$=[N-\tfrac{3}{2}+F(\xi)]\rho+4/\pi^2\ln\rho+O(1/\rho^4)\,, \qquad \text{(A10)}$$

where $F(\xi)$ is the series given in (3.15b). At large ρ, we retain only the first term of (A10), which gives (3.15a).

APPENDIX B: THE FUNCTION $W(t)$

Utilizing (1.8), we have

$$\frac{X^{\pm}(\beta t)}{X(-\beta t)}=\frac{(1-t)^{N-1}}{(1+t)^{N-1}}\exp\frac{1}{2\pi i}\int_1^\infty du\,\ln G(\beta u)$$

$$\times\left[\frac{1}{u-t\mp i\epsilon}-\frac{1}{u+t}\right]. \qquad \text{(B1)}$$

In the range where t is of the order of unity, which is the only range where $W(t)$ will be needed, the square bracket in (B1) goes as u^{-2} for large u. We have seen that $\ln G(\beta u)$ is rather constant for large u unless ξ is such that there is a pole along the cut. The $\ln G(\beta u)$ peaks at $u=\rho$ and its integral is of order ρ. However, in (B1), this peak would be weighted by $1/u^2=1/\rho^2$, and thus the contribution of the peak to the integral will be of order $1/\rho$. Hence, for the purposes of calculating the integral in (B1), we can neglect the peak and, to an accuracy of $1/\rho$, use the form of $\ln G(\beta u)$ appropriate to the range $u<\rho^{1/2}$, namely (A4) with $m=N-1$. Then using

$$1/(u-t\mp i\epsilon)=P1/(u-t)\pm i\pi\delta(u-t)$$

the integral in the exponent of (B1) is

$$P\int_1^\infty du\left(\frac{1}{u-t}-\frac{1}{u+t}\right)\left\{(N-1)-\frac{1}{\pi}\tan^{-1}\right.$$

$$\left.\times\left[\frac{\pi(1-1/u^2)}{(1-1/u^2)\ln[(u+1)/(u-1)+2/u]}\right]\right\}\pm i\pi(N-1)\pm i$$

$$\times\tan^{-1}\left[\frac{\pi(1-1/t^2)}{(1-1/t^2)\ln[(t+1)/t(-1)+2/t]}\right]. \qquad \text{(B2)}$$

The term proportional to $(N-1)$ in (B2) is

$$(N-1)[\ln(t+1)/(t-1)\pm i\pi]=(N-1)\ln(t+1)/(1-t)\,,$$

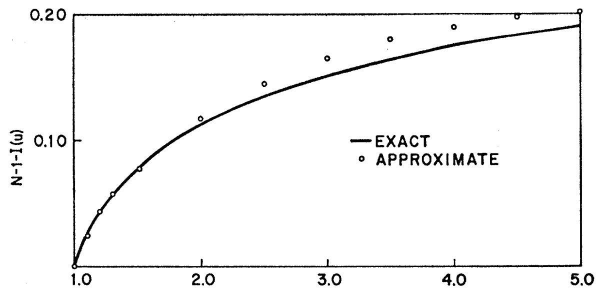

FIG. 3. The function $N-1-I(u)$ defined by (A4) and the approximation (A6).

which, when exponentiated in (B1), exactly cancels the factor in front of the exponential. This leaves

$$W(t)=\frac{1}{\pi}\exp\left\{-\frac{P}{\pi}\int_1^\infty du\left(\frac{1}{u-t}-\frac{1}{u+t}\right)\right.$$

$$\left.\times\tan^{-1}\left[\frac{(1-1/u^2)}{(1-1/u^2)\ln[(u+1)/(u-1)+2/u]}\right]\right\}$$

$$\times\sin\tan^{-1}\left[\frac{\pi(1-1/t^2)}{(1-1/t^2)\ln[(t+1)/(t-1)+2/t]}\right]. \qquad \text{(B3)}$$

This is, as we had claimed, real and independent of the physical parameters. Since this is the case and since the function is always used as a factor in an integrand, a numerical approximation to this function of fierce appearance but gentle behavior is indicated. As a first step, one can replace the $\tan^{-1}$ by the same approximation which led from (A4) to (A6), namely,

$$\frac{1}{\pi}\tan^{-1}\left[\frac{\pi(1-1/u^2)}{(1-1/u^2)\ln[(u+1)/(u-1)+2/u]}\right]$$

$$\approx\frac{1}{2}-\frac{4}{\pi^2 u}-\left(\frac{1}{2}-\frac{4}{\pi^2}\right)\frac{1}{u^5}. \qquad \text{(B4)}$$

This allows one to carry out the integration in (B3) so as to take a look at the beast which emerges. How one handles it next is a matter of choice. We chose to approximate the result by as simple a function as would reproduce the important behavior and, after some numerical evaluation, decided on the form given in (4.2). A comparison between (4.2) and the more exact expression obtained by using (B4) in (B3) is given in Fig. 3.