# Quantized Phase Effect and Josephson Tunneling

MICHAEL MARTIN NIETO
*Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York*
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By using properly defined sine and cosine phase variables, a model is proposed which yields a rigorous quantum-mechanical derivation of Josephson tunneling with exact current eigenstates. The model predicts that for dc tunneling the coherent fluids on both sides of the junction must be in the same state, and the "phase is quantized." This prediction can in principle be investigated experimentally, and may afford a fundamental test of quantum mechanics.

SINCE Josephson first predicted coherent tunneling for superconductors,[1] this tunneling has been observed both in superconducting junctions[2–5] and in superfluids.[6] However, the standard derivations of the effect[7–9] are all "semiclassical" in the sense that they consider a classical phase variable that is conjugate to the number operator, $N$, such that

$$[N,\phi]=i \quad \text{(WRONG)}. \tag{1}$$

But it is known that (1) is true only in the classical limit, $N\gg 1$. (It obviously is wrong if $N\ll 1$ as this implies that for a coherent fluid, $\Delta\phi\gg 1$.) Instead of $\phi$ one should use sine ($S$) and cosine ($C$) variables.[10–12]

$$S\equiv(E_- - E_+)/(2i) \tag{2}$$

$$C\equiv(E_- + E_+)/2 \tag{3}$$

$$E_+=E_-^\dagger\equiv a^\dagger(N+1)^{-1/2}. \tag{4}$$

The $E_+$ ($E_-$) are raising (lowering) operators and $a^\dagger$ is the creation operator. In the classical limit, (2) and (3) become $\sin\phi$ and $\cos\phi$.

If one has two coupled systems with number operators $N_1$ and $N_2$, one can similarly define phase difference operators:

$$S_{12}=-S_{21}=(E_{1-}E_{2+}-E_{1+}E_{2-})/(2i)\sim\sin(\phi_1-\phi_2) \tag{5}$$

$$C_{12}=C_{21}=(E_{1-}E_{2+}+E_{1+}E_{2-})/2\sim\cos(\phi_1-\phi_2). \tag{6}$$

Defining the total number operator ($N$) by

$$N=N_1+N_2, \tag{7}$$

we obtain the commutation relations

$$[N,S_{12}]=[N,C_{12}]=0 \tag{8}$$

$$[N_1,S_{12}]=iC_{12} \tag{9}$$

$$[N_1,C_{12}]=-iS_{12} \tag{10}$$

$$[S_{12},C_{12}]=[(\mathcal{P}^{00})_1-(\mathcal{P}^{00})_2]/(2i). \tag{11}$$

In the above, $(\mathcal{P}^{pq})_k$ is a number state projection operator for the type $k$ particle.

$$(\mathcal{P}^{pq})_k=|p_k\rangle\langle q_k|. \tag{12}$$

Equation (8) is very important, for it tells us that we can have states which are eigenstates of both total number and cosine or sine of the phase difference, even though $N_1$ and $N_2$ are not conserved individually. The normalized and complete eigenstates of $N$ and $C_{12}$ are[11,13]

$$|\cos\theta_{nr}\rangle=\left(\frac{2}{n+2}\right)^{1/2}\sum_{m=0}^{n}\sin(m+1)\theta_{nr}\,|m\rangle\,|n-m\rangle \tag{13}$$

$$\theta_{nr}\equiv\phi_{nr}+\tfrac{1}{2}\pi=r\pi/(n+2),\qquad r=1,2,\cdots,n+1, \tag{14}$$

with eigenvalues $n$ and $\cos\theta_{nr}$ respectively. The eigenstates of $N$ and $S_{12}$ are

$$|\sin\phi_{nr}\rangle=\left(\frac{2}{n+2}\right)^{1/2}\sum_{m=0}^{n}(-i)^m\sin(m+1)\theta_{nr}\,|m\rangle\,|n-m\rangle, \tag{15}$$

with eigenvalues $n$ and $\sin\phi_{nr}$ respectively. We see that $\cos\theta_{nr}$ and $\sin\phi_{nr}$ are discrete phase eigenspectra with $n+1$ values, that become quasicontinuous for $n\gg 1$.

Usually the starting point for deriving Josephson tunneling is a Hamiltonian of the form

$$H=\omega_1N_1+\omega_2N_2+VN_1+Ta^\dagger b+T^*b^\dagger a, \tag{16}$$

where $a^\dagger$ ($b^\dagger$) creates a particle on side one (two) of

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the junction. $\omega_1$ ($\omega_2$) gives the zeroth-order energy ($\hbar=1$) on side one (two). The middle term is the potential ($V$) applied to one side of the junction and the last two terms are the tunneling Hamiltonian with tunneling matrix element $T$.[14]

What we propose is the Hamiltonian

$$H=\omega_1 N_1+\omega_2 N_2+VN_1+ZC_{12}, \quad (17)$$

where $Z$ is the new tunneling matrix element, taking the place of $T$ and $T^*$. Eq. (17) is similar in structure to Eq. (16) and implies that the tunneling is a quantum-phase-correlation phenomena.

In the Heisenberg representation the current $\dot{N}_1$ is given by

$$i\dot{N}_1=[N_1, H]=-iZS_{12}, \quad (18)$$

which is valid at $t=0$. To find $S_{12}$ as a function of $t$ we commute $S_{12}$ with $H$ twice to get

$$\ddot{S}_{12}=-(\omega_1-\omega_2+V)^2 S_{12}-\tfrac{1}{2}Z^2\left[\frac{E_{1-}(\mathcal{P}^{10})_2-E_{1+}(\mathcal{P}^{01})_2}{2i}\right] +\tfrac{1}{2}Z^2\left[\frac{(\mathcal{P}^{10})_1E_{2-}-(\mathcal{P}^{01})_1E_{2+}}{2i}\right]. \quad (19)$$

The last two terms can be considered to be the sine phase correlations with the vacuum, since they contribute only for those parts of the states which relate to the vacuum on one of the sides of the junction. For a relatively small value of $n$ these terms become insignificant, and we will drop them. (This should not change our later conclusions.) From (18) and (19) we then have

$$S_{12}(t)=\{E_{1-}E_{2+}\exp[-i(\omega_1+V-\omega_2)t] -E_{1+}E_{2-}\exp[i(\omega_1+V-\omega_2)t]\}/2i, \quad (20)$$

$$\dot{N}_1=-ZS_{12}(t). \quad (21)$$

We can now draw our first conclusion. *For the dc Josephson effect* ($V=0$), $\omega_1=\omega_2$. *If this were not the case we would have ac current immediately. Thus, if the two regions are in phase correlation, the coherent fluids must be in the same energy state.* With $\omega_1=\omega_2$ and $V\neq 0$ the ac Josephson condition is obtained.

Although there exist views that the phase concept is nonessential,[15] we feel otherwise. Our belief was first based on the necessity of a quantum operator to serve as the conjugate variable to the well-understood number operator. If phase is physical, a coherent fluid tunneling through a barrier will have different transmission characteristics than the ordinary quantum-mechanical wave.

A way of looking at this is from the concept that time can be thought of as a phase variable.[13] In ordinary waves one looks for the $t\rightarrow\infty$ solution. But if phase is physical, in a coherent fluid it will be locked in, and this solution does not arrive. So to speak, each region of the superconductor does not "see" the other sections (phase change, i.e., time for the region being stopped) and the $t\rightarrow\infty$ solution is not set up. Similarly, one might discuss a coupling energy locking the phases[16] or think of reflected waves not being able to be observed.[17] (We add that Kao[17] is currently conducting an experiment that has an important bearing on this point.)

In any event, given phase operators, we have from (20) and (21) our second conclusion that *the dc current operator is quantized with eigenvalues* $-Z\sin\phi_{nr}$.[18] *Thus, according to this model the dc current is not continuous, but has a structure, and is excited from level to level.* The excitation comes from a temporary small potential which produces an ac current until the next level is excited, removing the potential.

An objection that might be raised over the existence of phase quantization is the possibility that strong outside forces not considered in the model (electrostatic in the superconducting case) would wipe out the proposed quantization. However, although the electrostatic energies are intrinsically larger than the phase-quantization energies, that does not necessarily mean that the quantization is unobservable. Coherence seems to average out external influences (the aforementioned phase locking). An example of this is the existence of Cooper pairs. Naively one would expect correlation or electrostatic energies to overcome the small energy gap binding the pairs.

Nevertheless, this point is certainly pertinent, and deserves further investigation. However, if indeed the above objection can be disregarded, then even though the particular phase eigenspectra may be model-dependent, the existence of phase quantization should not be.

*We then have a prediction that may afford a fundamental test of quantum mechanics.* As with the number operator, the properly defined phase difference operator is meaningful, and it should exist and be subject to quantization. Given a sufficiently small number of particles this quantized phase effect is in principle accessible to experiment.

An interesting possibility is the $CP$-violating interference from coherent regeneration of $K_S$ by $K_L$. From Eqs. (8) to (11) we see, for $\langle N\rangle=1$, that $N$, $C_{12}$, $S_{12}$, and $(N_1-N_2)$ correspond to the $T=\frac{1}{2}$ system of $SU(2)$,

[14] In this discussion we are considering true bosons instead of the pairs in superconductivity. Also, all of the coherent particles on one side of the junction are taken to be in a single-energy state.

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[18] One should realize that the Hamiltonians (16) and (17) ignore the particles leaving and entering the normal states at opposite ends of the superconducting region. This introduces a character that can prevent the $|\cos\theta_{nr}\rangle$ states from being the eigenfunctions in time of $H$. (See Ref. 13.) However, if we truly have a dc current, then we can use Eq. (15) for current eigenfunctions.

transforming as $I$, $T_x$, $T_y$, and $T_z$, respectively. This correspondence holds only for $\langle N\rangle=1$ ($T=\frac{1}{2}$) because the projection operators on the right-hand side of Eq. (11) complete the group structure only for this case. Unfortunately, it is just the $CP$ violation that destroys the $SU(2)$ symmetry of the $K_S$, $K_L$ system. Even worse, the mass difference of $K_S$, $K_L$, which would correspond to the potential term of (16), is not small with respect to the $K_S$ lifetime. These facts, and the unknown details of the strong interactions make it difficult to find new results here.

It seems then, that at present it would be more fruitful to think about low-energy systems for possible applications. Because of the number factor, the system should be small, perhaps a dilute helium gas. But in any case, it would certainly be of great interest if attempts could be made to observe quantized phase-difference phenomenon in a coherent boson-type medium.

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# Coherence and Quantization in Nearly Superconducting Rings

J. E. Zimmerman* and A. H. Silver
*Scientific Laboratory, Ford Motor Company, Dearborn, Michigan*
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New experiments demonstrating macroscopic coherence and quantization in partially resistive superconducting rings with point contacts are described. These include: (1) the observation of quantized magnetic transitions for frequencies $\omega>\omega_1=R/L$, where $R$ is the resistance and $L$ the inductance of the ring, and (2) the measurement of coherent oscillations at frequencies limited by thermal noise fluctuations, $8k_BTR/\Phi_0^2$, where $k_B$ is Boltzmann's constant, $T$ the absolute temperature, and $\Phi_0$ the flux quantum. These results are interpreted by introducing into the phenomenological model of weakly connected superconducting rings a continuous time dependence of the quantum states.

We describe new experiments which demonstrate macroscopic coherence and quantization in partially *resistive* superconducting rings. Two principal effects are displayed with the use of a superconducting point contact in series with a small inductance $L$ and a small normal resistance $R$. First, for frequencies $\omega>\omega_1=R/L$ there are quantized magnetic transitions as previously reported for superconducting rings.[1–3] The classical decay $\exp(-\omega_1 t)$ in the flux-screening current can be represented as a continuous exponential shift in the quantum number of the state. Second, any constant voltage across the resistance generates an oscillating current[4] similar to the Josephson ac effect[5] and this oscillation has been detected[6] at frequencies much below $\omega_1$; in fact, the lower frequency limit is approximately[6,7] $8k_BTR/\Phi_0^2$, where $k_B$ is Boltzmann's constant, $T$ the absolute temperature, and $\Phi_0$ the flux quantum.

These properties can be described by an extension of the phenomenological model of weakly connected superconducting rings.[1–3] The primary modification for resistive rings is to permit a continuous time dependence of the macroscopic quantum states; hence, the integral quantum numbers, $k$, for the London fluxoid become continuous variables. However, transitions between quantum states, which occur when the current in the weak contact equals the critical current $i_c$, satisfy the selection rule $|\Delta k|=1$. The picture which emerges is that of a discrete set of macroscopic states all moving in unison.

The general equivalent circuit of the system is shown in Fig. 1, where $\Phi_x$ represents an applied magnetic field in terms of the intercepted magnetic flux, $i$ is the loop current flowing in $R$, $L$, and the point contact, $i_1$ and $i_2$ are branch currents, and $V_J$ is the voltage across the point contact. Construction of such circuits has been previously detailed.[4,8] For $R\equiv 0$ we have demonstrated[3] the complete equivalence of an input current $I$ and applied magnetic field, $\Phi_x$. Such an equivalence no longer applies for resistive circuits and the two external fields, $\Phi_x$ and $I$, must be considered separately.

* Present address: Applied Research Lab., Philco-Ford Corp., Newport Beach, Calif.
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