Characteristic Parameters of a Granular Superconductor*

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(Received 22 September 1967)

Starting from a simple microscopic model of a granular superconductor (cubic grains of superconductor weakly coupled by Josephson junctions separating adjacent cubes), a calculation is made of the characteristic parameters $\mathfrak{I}$ and C of the continuum model of a granular superconductor, $\mathfrak{I}$ being the proportionality constant of the tunneling energy, and C the proportionality constant of the electrostatic energy density. From these are calculated the characteristic frequency ω for the onset of electromagnetic oscillations, the characteristic length ξ, the Ginzburg-Landau coherence distance ξ_{GL}, and the magnetic penetration depth λ. The last two are identical with the corresponding quantities for a dirty superconductor in the vicinity of the superconducting transition temperature. In the limit of weak magnetic fields, the identity of λ holds for all temperatures.

I. INTRODUCTION

CONSIDER a system composed of many microscopic grains of homogeneous superconductor, with each grain boundary being an insulating layer that is thin enough to allow appreciable tunneling by the Cooper pairs of the superconductor. In other words, adjacent grains are separated by Josephson junctions.[1] Such a system, a so-called granular superconductor, has been the subject of a number of experimental investigations.[2-6] The writer[7] has recently developed a theory of some of the properties of a granular superconductor.

In the present paper we wish to calculate two phonomenological parameters $\mathfrak{I}$ and C that enter the theory. These two parameters are, respectively, the coefficients of the tunneling energy density and the electrostatic energy density of the granular superconductor. To see how $\mathfrak{I}$ and C enter the theory, we consider the Hamiltonian density $\mathcal{H}(\mathbf{R})$. It is convenient to express $\mathcal{H}$ in terms of isospin operators[8] $\mathbf{s}_k(\mathbf{R})$, spin-up representing absence, spin-down presence, of a Cooper pair of internal momentum $\hbar\mathbf{k}$ in a grain located in space at $\mathbf{R}$. The Cooper-pair annihilation and creation operators are equal to the isospin angular momentum step-up and step-down operators, respectively. $\mathcal{H}(\mathbf{R})$ can be written

$$\mathcal{H}(\mathbf{R})=\mathcal{H}_K(\mathbf{R})+\mathcal{H}_V(\mathbf{R})+\mathcal{H}_T(\mathbf{R})+\mathcal{H}_E(\mathbf{R}). \quad (1)$$

Here

$$\mathcal{H}_K=-2\sum_k \epsilon_k s_{3k} \quad (2)$$

is (aside from an ignorable additive constant) just the one-electron Hamiltonian of the BCS theory,[9] with ϵ_k being the one-electron energy, and s_{3k} the z component of $\mathbf{s}_k$. (The x and y components will be designated by subscripts 1 and 2, respectively.) In terms of

$$\mathbf{S}\equiv\sum_k \mathbf{s}_k, \quad (3)$$

we can write

$$\mathcal{H}_V=-V[S_1^2+S_2^2] \quad (4)$$

and

$$\mathcal{H}_T=+\mathfrak{I}[(\nabla_R S_1)^2+(\nabla_R S_2)^2], \quad (5)$$

$\mathcal{H}_V$ being the electron-electron interaction Hamiltonian of the BCS theory, and $\mathcal{H}_T$ resulting from tunneling of Cooper pairs at grain boundaries. The sums over $\mathbf{k}$ in Eqs. (2) and (3) are restricted to the region of k space where $|\epsilon_k|<\hbar\omega$, this being the region over which there is an attractive electron-electron interaction in the BCS theory. Defining a (complex) order parameter

$$\Delta\equiv V(S_1+iS_2), \quad (6)$$

we can rewrite Eqs. (4) and (5) as

$$\mathcal{H}_V=-(1/V)\,|\Delta|^2, \quad (7)$$

$$\mathcal{H}_T=+(\mathfrak{I}/V^2)\,|\nabla_R\Delta|^2. \quad (8)$$

Here we have made use of the fact that $\mathbf{S}$ may be treated as a macroscopic, classical quantity with components that commute. This follows from the very large number of isospins $\mathbf{s}_k$ which go to make up $\mathbf{S}$. Finally,

$$\mathcal{H}_E=+CE^2 \quad (9)$$

results from the electrostatic energy associated with the junctions when the *surfaces* of the grains are electrically charged. From the macroscopic point of view, there can be a finite density of electric *dipoles*, although the electric *charge* density must always vanish. The electric

* A portion of the work reported here was carried out at RCA Laboratories. The author wishes to express his appreciation for the hospitality shown him and for the stimulating interaction with his colleagues there.

[1] B. D. Josephson, Advan. Phys. **14**, 419 (1965).
[2] W. Buckel and R. Hilsch, Z. Physik **138**, 109 (1954).
[3] I. S. Khukhareva, Zh. Eksperim. i Teor. Fiz. **43**, 1173 (1962) [English transl.: Soviet Phys.—JETP **16**, 828 (1963)].
[4] O. F. Kammerer and M. Strongin, Phys. Letters **17**, 224 (1965); M. Strongin, A. Paskin, O. F. Kammerer, and M. Garber, Phys. Rev. Letters **14**, 362 (1965).
[5] B. Abeles, R. W. Cohen, and G. W. Cullen, Phys. Rev. Letters **17**, 632 (1966); R. W. Cohen, B. Abeles, and G. S. Weisbarth, *ibid.* **18**, 336 (1967).
[6] B. Abeles, R. W. Cohen, and W. R. Stowell, Phys. Rev. Letters **18**, 902 (1967).
[7] R. H. Parmenter, Phys. Rev. **154**, 353 (1967). This paper will be referred to as A. Equations from A will be identified by the prefix A, e.g., Eq. A (4.7), etc. In so far as is possible, the notation of A will be used in the present paper.
[8] P. W. Anderson, Phys. Rev. **112**, 1900 (1958).
[9] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

field **E**, appearing in Eq. (9), is related to the phase of the order parameter. Defining ϕ such that

$$\Delta = |\Delta| \exp(i\phi), \quad (10)$$

then

$$\mathbf{E} = -\nabla_R(\hbar/2e)(d/dt)\phi. \quad (11)$$

Comparing Eqs. (4), (5), and (9), we see that $\mathfrak{J}$ is the phenomenological parameter for $\mathcal{H}_T$ and C is that for $\mathcal{H}_E$, in much the same way that the BCS matrix element V is for $\mathcal{H}_V$.

In situations where there are real magnetic fields present, we must replace ∇_R by $\nabla_R-(2e/\hbar c)\mathbf{A}$ in Eqs. (5) and (8), and we must add a term $-c^{-1}(d\mathbf{A}/dt)$ to Eq. (11). Here $\mathbf{A}(\mathbf{R})$ is the magnetic vector potential. At finite temperatures, the k sums of Eqs. (2) and (3) must be modified by multiplying the summands by

$$(1-2f_k) = \tanh\tfrac{1}{2}\beta E_k, \quad (12)$$

f_k being the Fermi factor associated with the quasiparticle excitation energy E_R.

In the following section we will calculate $\mathfrak{J}$ and C for a very simple idealized model of a granular superconductor, namely, a regular array of stacked cubes, each of dimension a, with identical tunneling junctions separating adjacent cubes. In addition to being weak, the one-electron tunneling is assumed to be a diffuse, rather than a specular, process (i.e., there is no conservation of transverse momentum during the tunneling). Having found $\mathfrak{J}$ and C, we will calculate three characteristic lengths ξ, ξ_{GL}, and λ, and one characteristic frequency $\tilde{\omega}$, associated with the granular superconductor. The characteristic length ξ is defined by the relation

$$\xi^2 = \mathfrak{J}/V. \quad (13)$$

The quantity ξ_{GL}, proportional to ξ, is what corresponds to the coherence distance of the Ginzburg-Landau phenomenological theory of superconductivity.[10] The length λ is the effective magnetic-penetration depth of the material. The frequency $\tilde{\omega}$ is the characteristic threshold frequency for electromagnetic oscillations of the type analogous to those discussed by Josephson[1] for a single Josephson junction. We will find that ξ_{GL} and λ (considered as functions of temperature T and conductivity mean free path l) are identical with the corresponding expressions for a dirty type-II superconductor, at temperatures near the superconducting transition temperature T_c. In the limit of weak magnetic fields, the identity of λ holds over all temperatures. As has already been emphasized by the writer,[7,11] this suggests a strong analogy between a granular superconductor and a conventional dirty superconductor. This behavior of ξ_{GL} is in agreement with the calculations of Cohen,[6] who used a very different model of a granular superconductor.[12] This is circumstantial evidence that the calculations of the present paper are model-independent.

Here we shall ignore one rather striking property of a granular superconductor, namely, the tendency for the material to have a higher transition temperature than that of the corresponding pure superconductors. A number of suggestions have been made as possible explanations. Examples are: (1) Ginzburg's theory[13] of surface-enhanced superconductivity; (2) attractive coupling between pair states on opposite sides of a junction[14] induced by the Josephson tunneling; (3) pairing of electrons on opposite sides of the junction via virtual phonons passing through the junction[15]; and (4) enhancement due to energy quantization of the one-electron orbitals in each grain.[16] For the purposes of the present paper, we may assume any such enhancement to be equivalent to having an effective V slightly larger than the BCS value associated with bulk material.

II. MATHEMATICAL ANALYSIS

We think of the grains of superconductor as cubes of length a, all stacked together without gaps. Separating adjacent grains are barriers of thickness d, where $d \ll a$. We first calculate the parameter C, this being very easy to do. From the form of Eq. (9), we see that $8\pi C$ plays the role of an effective dielectric constant. Since our assumed model is cubic, the dielectric constant must be isotropic, so there is no loss of generality in assuming that the macroscopic electric field **E** is parallel to one of the three symmetry directions (say along the x axis). Since the *microscopic* field must vanish in each superconducting cube, there is a *microscopic* electric field $(a/d)\mathbf{E}$ across each junction lying in the yz plane. For simplicity, we assume a microscopic dielectric constant of unity associated with the insulating material composing the junction. Thus there is a total electrostatic energy of

$$(1/8\pi)(a/d)^2E^2(a^2d) = (E^2/8\pi)(a^4/d), \quad (14)$$

associated with each junction in the yz plane. Since there is one such junction per cube, the *average* electrostatic energy density is (14) divided by a^3. It follows that

$$8\pi C = (a/d). \quad (15)$$

The characteristic velocity for electromagnetic radiation

[10] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950). For an English translation of this work, see the commentary and reprint volume by D. ter Haar, *Men of Physics: L. D. Landau* (Pergamon Press, Oxford, England, 1965), Vol. I, p. 138.

[11] R. H. Parmenter, Phys. Rev. **158**, 314 (1967).

[12] Cohen's model consists of barriers that are easily tunnelable (in the sense that the barrier potentials represent weak perturbations on the situation of an ideal bulk superconductor). The barriers are specular rather than diffuse.

[13] V. L. Ginzburg, Phys. Letters **13**, 101 (1964).

[14] See footnote 14a of Ref. 7.

[15] M. H. Cohen and D. H. Douglass, Jr., Phys. Rev. Letters **19**, 118 (1967).

[16] R. H. Parmenter, Phys. Rev. **166**, 392 (1968).

is [see Eq. A (5.13)]

$$v=c/(8\pi C)^{1/2}=c(d/a)^{1/2}, \tag{16}$$

where c is the velocity of light in vacuum.

In order to express the parameter $\mathfrak{J}$ in terms of the conductivity mean free path l, we first must relate the one-electron tunneling probability to the normal-state conductivity. Let $T_{kk'}$ be the one-electron tunneling matrix element coupling one-electron state $\mathbf{k}$ in one cube with one-electron state $\mathbf{k}'$ in an adjacent cube. The one-electron tunneling probability per unit time from state $\mathbf{k}$ to state $\mathbf{k}'$ is

$$P_{kk'}=(2\pi/\hbar)a^3N(0)\mid T_{kk'}\mid^2, \tag{17}$$

where $a^3N(0)$ is the density of one-electron states per unit energy, for a given spin, at the Fermi surface in the cube of normal metal. Thus $N(0)$ is the corresponding quantity *per unit volume of metal*, in agreement with BCS notation. We are, of course, only interested in tunneling at energies close to the Fermi energy in the metal. It is convenient to define a tunneling time t_0,

$$t_0\equiv 1/\langle P_{kk'}\rangle, \tag{18}$$

where the averaging of $P_{kk'}$, which amounts to the averaging of $\mid T_{kk'}\mid^2$, is over all possible orientations of $\mathbf{k}$ and $\mathbf{k}'$.

We assume the dc normal-state conductivity σ is limited by the tunneling junctions of the granular superconductor. Just as with the dielectric constant, σ must be isotropic because of the assumed cubic symmetry, and we can put the macroscopic electric field $\mathbf{E}$ parallel to the x axis, one of the symmetry axes. Since there is a voltage drop of aE across each junction in the yz plane, the Fermi levels on the two sides of any such junction are separated by an energy aeE (e = charge of the electron). Thus there are $2a^3N(0)[aeE]$ states occupied on one side of a junction which are isoenergetic with empty states on the opposite side. Here we are talking about a junction of area a^2 separating two cubes. (The above factor of 2 results from spin.) This corresponds to a current density

$$J=(e/a^2)[2a^3N(0)][aeE]\langle P_{kk'}\rangle. \tag{19}$$

In writing Eq. (19), we are assuming *diffuse* tunneling of electrons through the barrier. It is reasonable to assume no conservation of transverse $\mathbf{k}$ vector during the tunneling process for junctions far from geometrical perfection, undoubtedly true in practice. Since the normal conductivity is

$$\sigma\equiv J/E=n_0e^2\tau/m, \tag{20}$$

where n_0 is the conduction-electron density and τ the conductivity lifetime, and since

$$\begin{aligned} N(0)&=\tfrac{1}{2}(dn_0/dE_F)\\ &=\tfrac{3}{4}(n_0/E_F)\\ &=(3n_0/2mv_F^2) \end{aligned} \tag{21}$$

(m, v_F, and E_F having their usual meanings), we get

$$\tau=3(a/v_F)^2(1/t_0). \tag{22}$$

If we prefer to write the conductivity in terms of mean free path

$$l=v_F\tau, \tag{23}$$

then (22) becomes

$$(l/a)=3(a/v_Ft_0). \tag{24}$$

Following Wallace and Stavn,[17] who first treated Josephson tunneling by means of the isospin formalism,[8] the Josephson tunneling Hamiltonian for one of our elementary junctions is

$$\mathfrak{H}_T'=-\sum_{kk'}\mathfrak{J}_{kk'}'(s_{1kL}s_{1k'R}+s_{2kL}s_{2k'R}), \tag{25}$$

where

$$\mathfrak{J}_{kk'}'=4a^6\mid T_{kk'}\mid^2(E_k+E_{k'})^{-1}. \tag{26}$$

The subscripts L and R on the isospin operators designate the two superconducting cubes separated by the junction in question. E_k and $E_{k'}$ are the quasiparticle excitation energies, as in Eq. (12). The factor of a^6 in (26) arises for the same reason that the factor a^3 occurs in (17). We wish to follow the BCS convention that $\sum_k$ is a summation over states in k space (of a given electron spin) having a density appropriate to a crystal of unit volume. Thus we must multiply each k sum by the factor a^3 to account for the fact that we are talking about one-electron states in a cube of volume a^3. Comparing (26) with the corresponding expression in Wallace and Stavn [Eq. (A9) of Ref. 17], we see an extra factor of 4 in the former. This is a result of the fact that in Ref. 17, all sums are over both $\mathbf{k}$ and electron spin, unlike the sums in the present paper.

At finite temperatures, the junction-tunneling Hamiltonian becomes

$$\mathfrak{H}_T'=-\sum_{kk'}\mathfrak{J}_{kk'}'(s_{1kL}s_{1k'R}+s_{2kL}s_{2k'R})(1-2f_k)(1-2f_{k'}), \tag{27}$$

where now

$$\mathfrak{J}_{kk'}'=\frac{4a^6\mid T_{kk'}\mid^2}{(1-2f_k)(1-2f_{k'})}\left[\left(\frac{(1-f_k)(1-f_{k'})-f_kf_{k'}}{E_k+E_{k'}}\right)+\left(\frac{f_k(1-f_{k'})-f_{k'}(1-f_k)}{E_k-E_{k'}}\right)\right]. \tag{28}$$

The dependence of $\mathfrak{J}_{kk'}'(1-2f_k)(1-2f_{k'})$ on the f's and the E's can be checked by inspection of Table III of the BCS paper.[18] Equation (28) can be rewritten

$$\mathfrak{J}_{kk'}'=\frac{4a^6\mid T_{kk'}\mid^2}{(1-2f_k)(1-2f_{k'})}\left(\frac{E_k(1-2f_{k'})-E_{k'}(1-2f_k)}{E_k^2-E_{k'}^2}\right). \tag{29}$$

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[17] P. R. Wallace and M. J. Stavn, Can. J. Phys. **43**, 411 (1965).

[18] The terms of Table III (using the upper sign) that are proportional to (ϵ_0^2/EE') are those that contribute to the junction pairing energy.

In A, it was assumed that $\mathfrak{J}_{kk'}'$ (and thus $\mathfrak{J}$) was the same for all junctions in the granular superconductor, and the same for all time. However, whenever the quasiparticle excitation energy E_k is a function of time t and/or position $\mathbf{R}$ in the superconductor (as it in general will be for the solutions to the isospin equations of motion discussed in A), actually $\mathfrak{J}_{kk'}'$ (and thus $\mathfrak{J}$) will be a function of t and/or $\mathbf{R}$. Thus, in order to maintain consistency with the original derivation of the isospin equations of motion, we will evaluate $\mathfrak{J}_{kk'}'$ only for the two time- and position-independent solutions to the equations of motion, namely the time- and position-independent superconducting solution (BCS ground state) and the time- and position-independent normal state (referred to as cases I and II, respectively, in A). For these two cases, we have

$$E_k=(\epsilon_k^2+\Delta_0^2)^{1/2}, \tag{30}$$

where $\Delta_0=\epsilon_0(T)$ (the BCS half-energy gap at temperature T) for case I, and $\Delta_0=0$ for case II. Which of the two resultant values of $\mathfrak{J}_{kk'}'$ (and thus of $\mathfrak{J}$) we insert into the isospin equations of motion will depend on which of the two time- and position-independent solutions more closely approximates the solution of interest.[19]

We now replace $\mathfrak{J}_{kk'}'$ by the average

$$\mathfrak{J}'\equiv\langle\mathfrak{J}_{kk'}'\rangle, \tag{31}$$

where $|T_{kk'}|^2$ is averaged over orientations of $\mathbf{k}$ and $\mathbf{k}'$, whereas the rest of $\mathfrak{J}_{kk'}'$ is averaged over energies ϵ_k and $\epsilon_{k'}$. In performing this latter averaging, we use the weighting factors $(1-2f_k)/E_k$ and $(1-2f_{k'})/E_{k'}$, these being the two energy-dependent factors multiplying $\mathfrak{J}_{kk'}'$ in $\mathfrak{K}_T'$, Eq. (27). Furthermore, we restrict the energy averaging to be over the energy range where the BCS electron–electron interaction is attractive. Thus

$$\mathfrak{J}'=8a^6\langle|T_{kk'}|^2\rangle I_1/I_2^2, \tag{32}$$

where

$$I_1\equiv\int_0^{\hbar\omega}\int_0^{\hbar\omega}d\epsilon d\epsilon'\,\frac{\tanh[\frac{1}{2}\beta(\epsilon'^2+\Delta_0^2)^{1/2}]}{(\epsilon^2-\epsilon'^2)(\epsilon'^2+\Delta_0^2)^{1/2}}, \tag{33}$$

$$I_2\equiv\int_0^{\hbar\omega}d\epsilon\,\frac{\tanh[\frac{1}{2}\beta(\epsilon^2+\Delta_0^2)^{1/2}]}{(\epsilon^2+\Delta_0^2)^{1/2}}. \tag{34}$$

In writing (32), we have made use of Eqs. (12) and (30). I_2 is immediately evaluated:

$$\begin{aligned} I_2&=1/N(0)V, && \text{if } \Delta_0=\epsilon_0(T)\\ &=\ln(T_c/T)+1/N(0)V, && \text{if } \Delta_0=0. \end{aligned} \tag{35}$$

In order to evaluate I_1, we rewrite it as

$$I_1=\lim_{b\to0}\mathcal{P}\int_b^{\hbar\omega}d\epsilon'\,\frac{\tanh[\frac{1}{2}\beta(\epsilon'^2+\Delta_0^2)^{1/2}]}{(\epsilon'^2+\Delta_0^2)^{1/2}}\int_b^{\hbar\omega}\frac{d\epsilon}{\epsilon^2-\epsilon'^2}, \tag{36}$$

where $\mathcal{P}$ indicates that the principal part of the integral of ϵ past ϵ' is to be taken.[20] In (36), we may safely replace both upper limits $\hbar\omega$ by infinity, since the integrand drops off rapidly enough as ϵ and ϵ' increase. Since

$$\mathcal{P}_{\epsilon'\geq b}\int_b^\infty\frac{d\epsilon}{\epsilon^2-\epsilon'^2}=(2\epsilon')^{-1}\ln\left(\frac{\epsilon'+b}{\epsilon'-b}\right), \tag{37}$$

we get

$$\begin{aligned} I_1&=\lim_{b\to0}\int_b^\infty d\epsilon'\,\frac{\tanh[\frac{1}{2}\beta(\epsilon'^2+\Delta_0^2)^{1/2}]}{2\epsilon'(\epsilon'^2+\Delta_0^2)^{1/2}}\ln\left(\frac{\epsilon'+b}{\epsilon'-b}\right)\\ &=\frac{\tanh(\frac{1}{2}\beta\Delta_0)}{2\Delta_0}\lim_{b\to0}\int_b^\infty\frac{d\epsilon'}{\epsilon'}\ln\left(\frac{\epsilon'+b}{\epsilon'-b}\right), \end{aligned} \tag{38}$$

where, in the last step, we have used the fact that the entire contribution to the ϵ' integral comes from values of ϵ' comparable with b (which is approaching zero). Defining $y=b/\epsilon'$, we can rewrite (38) as

$$\begin{aligned} I_1&=\frac{\tanh(\frac{1}{2}\beta\Delta_0)}{2\Delta_0}\int_0^1\frac{dy}{y}\ln\left(\frac{1+y}{1-y}\right)\\ &=(\pi^2/8\Delta_0)\tanh(\tfrac{1}{2}\beta\Delta_0). \end{aligned} \tag{39}$$

Thus

$$\begin{aligned} I_1&=(\pi^2/8\epsilon_0)\tanh(\epsilon_0/2k_BT), && \text{if } \Delta_0=\epsilon_0(T)\\ &=(\pi/4)^2(1/k_BT), && \text{if } \Delta_0=0. \end{aligned} \tag{40}$$

Once we replace $\mathfrak{J}_{kk'}'$ by $\mathfrak{J}'$ in (27), we can rewrite $\mathfrak{K}_T'$ as

$$\mathfrak{K}_T'=-\mathfrak{J}'(S_{1L}S_{1R}+S_{2L}S_{2R}), \tag{41}$$

where

$$\mathbf{S}=\sum_k\mathbf{s}_k(1-2f_k), \tag{42}$$

the sum running over the range $|\epsilon_k|\leq\hbar\omega$, just as in Eq. (3). $\mathbf{S}$ is the total isospin for a given grain. Invoking the continuum approximation,[21] we consider $\mathbf{S}$ to be a continuous function of $\mathbf{R}$, $\mathbf{S}(\mathbf{R})$ being the total isospin of the grain situated at $\mathbf{R}$. Thus we can rewrite the isospin factor in Eq. (41) as

$$(S_{1L}S_{1R}+S_{2L}S_{2R})=S_1(\mathbf{R})S_1(\mathbf{R}+\mathbf{a})+S_2(\mathbf{R})S_2(\mathbf{R}+\mathbf{a}). \tag{43}$$

[19] Actually, E_k, and thus $\mathfrak{J}'_{kk'}$ and $\mathfrak{J}$, are t- and $\mathbf{R}$-independent for that $\mathbf{R}$-dependent solution to the equations of motion which was considered in Ref. 11 (the case of a steady uniform current flow in a granular superconductor having dimensions small enough for magnetic effects to be ignored). The zero-current BCS and normal-metal solutions are the two limiting cases for this solution. Because of the functional form of E_k [see Eq. A (7.3)] for this solution, the integral determining $\mathfrak{J}'$ cannot be evaluated analytically. We will not consider this case further here.

[20] Essentially the same method was used by BCS in evaluating an integral in Appendix C of Ref. 9.

[21] For the analogous discussion of the continuum approximation in ferromagnetism, see, e.g., C. Kittel, in *Low Temperature Physics*, edited by C. De Witt, B. Dreyfus, and P. G. de Gennes (Gordon and Breach Science Publishers, Inc., New York, 1962), p. 459.

The vector $\mathbf{a}$ (of magnitude a) connects the centers of the two adjacent cubes whose common junction we are describing. As previously, consider an elementary junction in the yz plane, so that $\mathbf{a}$ is along the x axis. Expanding in a power series, we have

$$S_{1,2}(\mathbf{R}+\mathbf{a})=S_{1,2}(\mathbf{R})+a(\partial/\partial x)S_{1,2}(\mathbf{R})+\tfrac{1}{2}a^2(\partial^2/\partial x^2)S_{1,2}(\mathbf{R})+\cdots. \quad (44)$$

The first term on the right-hand side of (44) contributes to (41) a term $-\mathfrak{J}'(S_1^2+S_2^2)$ having the same form as $\mathcal{H}_V$, so that we may assume this term subsumed into the electron-electron interaction,[14] as has already been discussed in the previous section. The second term of (44) contributes to (41) a term which vanishes when we add contributions from adjacent junctions. The third term of (44) contributes to (41) a term of the form

$$-\tfrac{1}{2}a^2\mathfrak{J}'[S_1(\partial^2/\partial x^2)S_1+S_2(\partial^2/\partial x^2)S_2].$$

By doing an integration by parts with respect to $\mathbf{R}$, this term transforms into

$$\mathcal{H}_{Tx}\equiv+\tfrac{1}{2}a^2\mathfrak{J}'[(\partial S_1/\partial x)^2+(\partial S_2/\partial x)^2]. \quad (45)$$

(The summation over different junctions corresponds to an integration over $\mathbf{R}$ in the continuum limit, so that the above partial integration is justified.)

Equation (45) is the contribution to the Hamiltonian from an elementary junction in the yz plane. There is one such junction for each elementary cube in the crystal. Similarly, the contribution from each elementary junction in the xz plane is

$$\mathcal{H}_{Ty}\equiv+\tfrac{1}{2}a^2\mathfrak{J}'[(\partial S_1/\partial y)^2+(\partial S_2/\partial y)^2], \quad (46)$$

and the contribution from each elementary junction in the xy plane is

$$\mathcal{H}_{Tz}\equiv+\tfrac{1}{2}a^2\mathfrak{J}'[(\partial S_1/\partial z)^2+(\partial S_2/\partial z)^2]. \quad (47)$$

Thus the total tunneling Hamiltonian per unit volume (i.e., Hamiltonian density) is given by Eq. (5), provided we take

$$\mathfrak{J}=(\tfrac{1}{2}a^2\mathfrak{J}')/a^3. \quad (48)$$

With the aid of Eqs. (17), (18), (24), (32), (35), (40), and (48), we can now evaluate the parameter $\mathfrak{J}$. For our purposes, it is more convenient to evaluate the characteristic length ξ, defined in terms of $\mathfrak{J}$ by Eq. (13). We get

$$\xi^2=[2\hbar v_F l I_1/3\pi N(0)VI_2^2]. \quad (49)$$

Notice that (49) depends on our model of a granular superconductor only through the conductivity mean free path l. The other quantities in (49) depend on the superconducting metal being used in forming the granular superconductor, but not on the model itself. In terms of

$$\xi_0=[\hbar v_F/\pi\epsilon_0(0)], \quad (50)$$

the BCS form of the Pippard coherence distance (at $T=0$) for the pure superconducting metal composing our granular superconductor, we have

$$\xi=\tfrac{1}{2}\pi[\tfrac{1}{3}N(0)V]^{1/2}(\xi_0 l)^{1/2}F_1(T), \quad (51)$$

where

$$F_1(T)\equiv\{[\epsilon_0(0)/\epsilon_0(T)]\tanh[\epsilon_0(T)/2k_BT]\}^{1/2}, \quad (52)$$

if $\Delta_0=\epsilon_0(T)$ (case I), and

$$F_1(T)\equiv[1+N(0)V\ln(T_c/T)]^{-1}[\epsilon_0(0)/2k_BT]^{1/2}, \quad (53)$$

if $\Delta_0=0$ (case II).

At $T=T_c$, both forms of $F_1(T)$ take on the value

$$F_1(T_c)=(\tfrac{1}{2}\pi e^{-\gamma})^{1/2}=0.939. \quad (54)$$

($\gamma=0.5772=$Euler's constant). Thus the characteristic length ξ remains finite[22] as $T\to T_c$. For case I, ξ changes only slightly as T is reduced, since $F_1(T)$ approaches unity as $T\to 0$. For case II, however, ξ and F_1 diverge as $T^{-1/2}[\ln(1/T)]^{-1}$ as $T\to 0$.

In Sec. V of A, it was shown that the magnetic penetration depth for a granular superconductor is

$$\lambda=(V/2\pi)^{1/2}(\hbar c/4e\xi\Delta_0). \quad (55)$$

In terms of

$$\lambda_0=(mc^2/4\pi n_0 e^2)^{1/2}, \quad (56)$$

the $T=0$ London penetration depth for the pure-bulk superconductor, we can rewrite λ as[23]

$$\lambda=\tfrac{1}{2}\pi[\tfrac{1}{3}N(0)V]^{1/2}(\lambda_0\xi_0/\xi)[\epsilon_0(0)/\Delta_0]. \quad (57)$$

For case I, where $\Delta_0=\epsilon_0(T)$, this becomes

$$\lambda=\lambda_0(\xi_0/l)^{1/2}F_2(T), \quad (58)$$

where

$$F_2(T)\equiv\{[\epsilon_0(T)/\epsilon_0(0)]\tanh[\epsilon_0(T)/2k_BT]\}^{-1/2}. \quad (59)$$

Equations (58) and (59) are equivalent to the results of Abrikosov *et al.*[24] for the penetration depth of a dirty superconductor in the weak-magnetic-field limit (this corresponding to case I). This equivalence holds for all temperatures. Near $T=T_c$, we can write[25]

$$\epsilon_0(T)=2\pi[2/7\zeta(3)]^{1/2}k_BT_c[1-(T/T_c)]^{1/2} \quad (60)$$

[$\zeta(z)$ being the Riemann zeta function], so that

$$\lim_{T\to T_c}F_2(T)=\tfrac{1}{2}[7\zeta(3)e^{-\gamma}/\pi]^{1/2}[1-(T/T_c)]^{-1/2}=0.615[1-(T/T_c)]^{-1/2}. \quad (61)$$

At $T=0$, we have $F_2(0)=1$. For case II, where $\Delta_0=0$, λ is infinite at all temperatures.

[22] In Sec. VII of A, it was stated incorrectly that ξ diverges as $T\to T_c$.

[23] Equation (57) corrects Eq. A (5.17), the latter having inadvertently dropped the factor $(2\pi)^{-1/2}$ appearing in Eq. (55) and Eq. A (5.14).

[24] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963), p. 341.

[25] See, e.g., Ref. 24, p. 304.

In order to make the connection between ξ and the Ginzburg-Landau coherence distance ξ_{GL}, we consider the Ginzburg-Landau-like equation derived in Sec. IV of A:

$$[\xi^2\nabla_R^2+1-|\Delta|^{-1}g(|\Delta|)]\Delta=0, \qquad (62)$$

where $g(|\Delta|)$ is defined implicitly by the relation

$$|\Delta|=N(0)Vg\int_0^{\hbar\omega}(\epsilon_k^2+g^2)^{-1/2}\tanh(\tfrac{1}{2}\beta E_k)d\epsilon_k, \qquad (63)$$

$$E_k=(\epsilon_k^2+g^2)^{-1/2}(\epsilon_k^2+|\Delta|g), \qquad (64)$$

E_k being the excitation energy. As was pointed out in A, because of the form of E_k, $g(|\Delta|)$ has no power-series expansion in $|\Delta|$ at any temperature. Despite this, we can linearize Eq. (62) for small $|\Delta|$, as a consequence of the fact that $g(|\Delta|)$ vanishes whenever $|\Delta|$ does. Thus, in the limit of small $|\Delta|$, we can set g equal to zero inside the integrand appearing in (63), and get

$$|\Delta|=N(0)Vg\int_0^{\hbar\omega}(d\epsilon/\epsilon)\tanh(\tfrac{1}{2}\beta\epsilon)$$
$$=g[1+N(0)V\ln(T_c/T)]. \qquad (65)$$

The linearized form of Eq. (62) can therefore be written

$$[\xi_{GL}^2\nabla_R^2+1]\Delta=0, \qquad (66)$$

provided we define

$$\xi_{GL}\equiv\xi\{1+[N(0)V\ln(T_c/T)]^{-1}\}^{1/2}. \qquad (67)$$

Knowing ξ as a function of l and T, we now get

$$\xi_{GL}=(\xi_0 l)^{1/2}F_3(T), \qquad (68)$$

where, for case I,

$$F_3(T)=\tfrac{1}{2}\pi\left[\frac{1}{3}\left(\frac{1+N(0)V\ln(T_c/T)}{\ln(T_c/T)}\right)\times\left(\frac{\epsilon_0(0)}{\epsilon_0(T)}\right)\tanh\left(\frac{\epsilon_0(T)}{2k_BT}\right)\right]^{1/2}, \qquad (69)$$

and, for case II,

$$F_3(T)=\tfrac{1}{2}\pi\left(\frac{[\epsilon_0(0)/2k_BT]}{3[1+N(0)V\ln(T_c/T)][\ln(T_c/T)]}\right)^{1/2}. \qquad (70)$$

At $T=0$, ξ_{GL} is the same as ξ. As $T\to T_c$, both forms of $F_3(T)$ become

$$\lim_{T\to T_c}F_3(T)=\tfrac{1}{2}\pi(\tfrac{1}{6}\pi e^{-\gamma})^{1/2}[1-(T/T_c)]^{-1/2}$$
$$=0.850[1-(T/T_c)]^{-1/2}. \qquad (71)$$

Equations (68) and (71), giving the form of ξ_{GL} as $T\to T_c$, are identical with the corresponding equation for a dirty superconductor,[26] and also identical with the result derived in Ref. 6 for a granular superconductor with easily tunnelable barriers.

We see that, for case I where $\Delta_0=\epsilon_0(T)$, ξ_{GL} varies with temperature in a manner similar to that of λ. Defining

$$\kappa\equiv(\lambda/\xi_{GL})=(\lambda_0/l)F_4(T), \qquad (72)$$

$$F_4(T)\equiv(2/\pi)\sqrt{3}\{\tanh[\epsilon_0(T)/2k_BT]\}^{-1}\times\left(\frac{\ln(T_c/T)}{1+N(0)V\ln(T_c/T)}\right)^{1/2}, \qquad (73)$$

we see that $F_4(T)$ is a slowly varying function of T. Thus

$$F_4(0)=(2/\pi)[3/N(0)V]^{1/2}=1.10[N(0)V]^{-1/2}, \qquad (74)$$

$$F_4(T_c)=\pi^{-2}[42\zeta(3)]^{1/2}=0.723. \qquad (75)$$

Finally, we mention that the characteristic frequency

$$\tilde{\omega}=v/\lambda, \qquad (76)$$

the threshold frequency for electromagnetic oscillations, can be determined immediately, since we know both λ [Eq. (58)] and v [Eq. (16)].

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[26] See, e.g., P. G. de Gennes, *Superconductivity of Metals and Alloys* (W. A. Benjamin, Inc., New York, 1966), p. 225.