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Ladder-Graph Approximation in Positron Annihilation

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The annihilation rate of the positron in a pure Fermi gas has been calculated (a) by using a ladder-graph approximation with an exponentially screened Coulomb interaction (which is nearly the static limit of the random-phase-approximation interaction), (b) by taking into account certain graphs up to third order in the same interaction, and (c) by using a better approximation for the interaction in a ladder-graph series. The contributions from graphs other than ladders was found to be 2–3% up to second order and 4–5% to third order. The results appear to establish that a ladder sum with a quite simple interaction approximates well the two-body propagator. The modification of the interaction has only a slight effect on the rate. At low electron densities, the ladder sum diverges because of the existence of bound states, which are proved to exist at any density as poles in the two-body propagator.

1. INTRODUCTION

THE problem of calculating the positron annihilation rate in an electron gas (e.g., in solid matter) has not yet been solved accurately even in the simple case of a pure electron gas, in spite of numerous efforts in this direction. The difficulty lies in the fact that the interaction between the positron and the electrons is essentially "strong," which causes troubles in calculating the two-body propagator for the positron-electron pair. Some earlier papers[1–3] suggest that the commonly used method of first integrating with respect to particle energies in Feynman diagrams and then summing over the order of the interaction, may be adequate at high electron densities (i.e., metallic), while it has been known to fail at low densities.[3,4] In this paper, we do not solve the essential difficulties; we only try to point out some characteristic features of the problem, using the above-mentioned perturbation techniques and making comparisons between more and less exact approximations (comparison with measured rates alone does not tell much about the correctness of calculations and approximations).

In Secs. 2 and 3, the ladder sum is compared with the sum of a certain class of Feynman graphs up to third order in the interaction. It is known that the ladder-type sum is exact if one uses a certain effective-interaction operator, which in turn is the sum of an infinite series of diagrams.[5] (Then, in the ladders, real- and not free-particle propagators should be used.) With this fact in mind, it is clear that the reliability of the results originating from a ladder sum depends essentially on the "elementary" interaction that is used. Therefore, to test the effect of this interaction, we have redone the ladder calculation using a slightly modified interaction, the modification being essentially the simplest one possible. In Sec. 4, we consider briefly the "divergence" in annihilation rate which results from this calculation method and prove this to be due to the existence of at least one bound-state pole in the approximate energy-dependent two-body propagator.

2. LADDER SUM

The first thing to do in order to calculate the interaction that is used in the ladder sum is to obtain the propagator of the interaction (which in this case contains, e.g., parts due to plasmons). It is known[6] that this interaction is approximated rather well by the simple bubble-diagram sum which, in its static limit, results in

$$u_s(k)=e^2\Big/\left\{k^2+\tfrac{1}{2}\kappa^2\left[1+\left(\frac{k_0}{k}-\frac{k}{4k_0}\right)\ln\left|\frac{2k_0+k}{2k_0-k}\right|\right]\right\}, \quad (1)$$

[1] S. Kahana, Phys. Rev. **129**, 1622 (1963).
[2] J. P. Carbotte, Phys. Rev. **144**, 309 (1966).
[3] J. Crowell, V. E. Anderson, and R. H. Ritchie, Phys. Rev. **150**, 243 (1966).
[4] A. Held and S. Kahana, Can. J. Phys. **42**, 1908 (1964).
[5] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Quantum Field Theoretical Methods in Statistical Physics* (Pergamon Press Inc., New York, 1965), p. 153.
[6] D. Pines and P. Nozières, *The Theory of Quantum Liquids*, (W. A. Benjamin, Inc., New York, 1966), Vol. I, pp. 292–294.

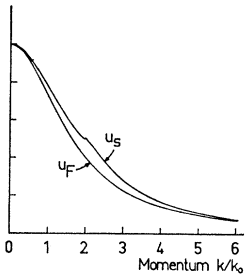

FIG. 1. The static limit of the random-phase-approximation interaction (u_s) and the Fermi-Thomas screened interaction (u_F) at $r_s=4$ in comparable units.

where k_0 is the Fermi momentum, and κ is the Fermi-Thomas screening constant given by

$$\kappa^2=me^2k_0/(\pi^2)=4k_0/(\pi a_0),$$

where a_0 is the Bohr radius. The units are chosen so that $c=\hbar=\epsilon_0=1$. Turning to the conventional parameter $r_s=r_i/a_0$, we find it to be

$$r_s=\pi\gamma\kappa^2/(4k_0^2),$$

where $\gamma=(9\pi/4)^{1/3}\approx1.92$. The function $u_s(k)$ shows a peculiar behavior near $k=2k_0$, where its derivative is infinite. This leads to a weak "long-range" oscillating potential[6] for which the length of the period is π/k_0, which is about $\frac{1}{2}$ of the shortest normally encountered electron wavelength. Thus it is expected that the electrons do not "feel" these rapid small changes very much; we therefore leave them out of the formalism.[7] This smoothing has been done by taking the simple exponentially screened Fermi-Thomas potential

$$u_F(k)=e^2/(k^2+\kappa^2), \tag{2}$$

which, according to Fig. 1, does not deviate very much from u_s. They both tend to the bare Coulombic potential when $k\to\infty$ and to each other when $k\to0$.

There is some evidence[8] that using the static interaction without considering the self-energy effects is at least in some cases almost the same as accounting for the dynamical and self-energy effects, because these effects largely cancel each other. In addition, it seems that the pure plasmonic effects contribute very little to the annihilation rate. For example, if we calculate the simplest diagrams, which correspond to the case where the final state contains one plasmon after annihilation, they are found to give rather small contributions. For these reasons, we have not taken into account any electronic or plasmonic self-energy effects, and we have used the simplified static potential u_F. It is to be expected that even in more accurate approximations the results will be qualitatively of the same kind.

In what follows we use wave functions that are determined by periodic boundary conditions and are normalized in a volume V. This leads to the well-known point set in momentum space, each point $\mathbf{k}$ corresponding to the wave $\exp(i\mathbf{k}\cdot\mathbf{x})/V^{1/2}$. Starting from quantum electrodynamics, one can show that in the nonrelativistic limit the partial annihilation rate $R(\mathbf{P})$ may be expressed equivalently by

$$R(\mathbf{P})=\sum_{\text{all }\nu}|\chi_\nu(\mathbf{P})|^2, \tag{3}$$

where

$$\chi_\nu(\mathbf{P})=r_0(2\pi/V)^{1/2}\int d\mathbf{x}\exp(-i\mathbf{P}\cdot\mathbf{x})\times\sum_{\alpha,\beta}\epsilon_{\alpha\beta}\langle f_\nu|T\{\phi_\alpha(\mathbf{x},t)\psi_\beta(\mathbf{x},t)S\}|0\rangle, \tag{3'}$$

or by

$$R(\mathbf{P})=2\pi r_0^2V^{-1}\sum_{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\iint d\mathbf{x}d\mathbf{y}\exp[i\mathbf{P}\cdot(\mathbf{y}-\mathbf{x})]\times\langle0|\tilde\psi_\alpha^\dagger(\mathbf{y})\tilde\phi_\beta^\dagger(\mathbf{y})\tilde\phi_\delta(\mathbf{x})\tilde\psi_\gamma(\mathbf{x})|0\rangle, \tag{4}$$

where ϕ_α refers to a positron of spin α, and ψ to an electron; $\tilde\phi$ and $\tilde\psi$ denote Heisenberg operators, while ϕ and ψ are expressed in the interaction picture. $|0\rangle$ is the full Fermi sphere of electrons plus one positron at rest, and $|f_\nu\rangle$ denotes any final state that lacks an electron-positron pair.

$$\epsilon_{\uparrow\uparrow}=\epsilon_{\downarrow\downarrow}=0;\ \epsilon_{\uparrow\downarrow}=-\epsilon_{\downarrow\uparrow}=1.$$

Keeping in mind a concrete picture of a Feynman graph corresponding to expression (4), we see that in order to contribute to $R(\mathbf{P})$, its interaction lines must be horizontal (time axis pointing upwards), and all "particle" lines (for electron, $p>k_0$; for positron, $\mathbf{p}\neq\mathbf{0}$) must point up, and hole lines (electron, $p<k_0$; positron, $\mathbf{p}=\mathbf{0}$) down. There must be only one positron hole line. To get $R(\mathbf{P})$, all diagrams must be summed to give the two-body propagator, in which we then perform the energy integrations. Supposing that all energy integrations may be performed term by term in the perturbation series, we obtain for an Nth-order graph (after letting V go to infinity and denoting $\mathbf{P}/k_0=\mathbf{K}$), the general expression

$$R^{N,i}(\mathbf{P})=2\pi r_0^2D^{N,i}(\mathbf{K})/V, \tag{5}$$

where

$$D^{N,i}(\mathbf{K})=(-1)^H\left(\frac{\beta}{8\pi}\right)^N\int\cdots\int d\mathbf{x}_1\cdots d\mathbf{x}_N\times\left[\prod_{\nu=1}^N\frac{1}{e^\nu-i\eta}\right]\times\left[\prod_{\tau=1}^N\frac{1}{\mathbf{y}_\tau^2+\beta}\right]. \tag{5'}$$

In this expression, H is the total number of hole lines in the diagrams, and $\mathbf{x}_1,\cdots,\mathbf{x}_N$ are the free momenta (in units of k_0) connected to fermion lines (in each graph, one and only one positron line is a hole line and has $\mathbf{x}=\mathbf{0}$; in addition, we require that the total momentum of the annihilating pair be equal to $\mathbf{K}$; these considerations and the fact that the momentum is conserved restrict the number of free momentum parameters to N). Integrations over the momenta must

[7] The difficulty with the induced charge density mentioned in Ref. 6 is not even physically important here because the interaction is attractive.

[8] J. P. Carbotte and S. Kahana, Phys. Rev. **139**, A213 (1965).

TABLE I. Ladder-sum results.

r_s	Ladders $\phi(K)$ $K=0$	$K=0.3$	$K=0.6$	$K=0.9$	Total rate (10^9/sec)	Ladders with modified interaction $\phi(K=0)$	Total rate[a] (10^9/sec)
2	1.733	1.749	1.804	1.948	5.362	1.715	4.431
3	2.195	2.220	2.305	2.533	2.650		
4	2.770	2.807	2.936	3.289	1.858	2.716	1.389
6	4.519	4.605	4.916	5.830	1.660	4.329	1.045
8	8.208	8.458	9.415	12.742	3.087	7.377	1.281
10						15.115	2.753

[a] Computed as if $\phi(K)=\phi(0)$ when $K<1$.

be restricted so that an electron hole has $x<1$, etc. The quantity $\mathbf{y}_\tau$ is the momentum transfer connected to the τth interaction, and $e^r=m(E_{\rm up}-E_{\rm down})/(k_0^2)$, when $E_{\rm up}-E_{\rm down}$ is the energy difference between the upward and downward lines that lie between two succeeding interaction lines. (In this respect, we also consider the annihilation as an interaction.) We have put $\beta=(\kappa/k_0)^2=4r_s/\pi\gamma$. The total-annihilation rate is then given by

$$R=\sum_{i,N,\mathbf{P}} R^{N,i}(\mathbf{P})=2\pi r_0^2\int d\mathbf{x} \times\sum_{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\langle 0\mid\tilde{\psi}_\alpha^\dagger(\mathbf{x})\tilde{\phi}_\beta^\dagger(\mathbf{x})\tilde{\phi}_\delta(\mathbf{x})\tilde{\psi}_\gamma(\mathbf{x})\mid 0\rangle. \quad (6)$$

The usual ladder sum is easily found to be the square of the sum of all "half-diagrams" shown in Fig. 2. Thus we may write

$$D(\mathbf{K})=\phi(\mathbf{K})^2, \quad (7)$$

where

$$\phi(\mathbf{K})=1+\int_{x>1} d\mathbf{x}\, T(\mathbf{x},\mathbf{K})/(\mathbf{x}^2-\mathbf{x}\cdot\mathbf{K}) \quad (7')$$

and

$$T(\mathbf{x},\mathbf{K})=\frac{\beta}{8\pi[(\mathbf{x}-\mathbf{K})^2+\beta]}+\frac{\beta}{8\pi}\times\int_{y>1} d\mathbf{y}\,\frac{T(\mathbf{y},\mathbf{K})}{(\mathbf{y}^2-\mathbf{y}\cdot\mathbf{K})[(\mathbf{x}-\mathbf{y})^2+\beta]}. \quad (8)$$

Denoting $\xi=\cos(\mathbf{x},\mathbf{K})$ and $T(\mathbf{x},\mathbf{K})=T(x,K,\xi)$, we integrate Eq. (8) to get

$$T(x,K,\xi)=\frac{\beta}{8\pi(x^2+K^2+\beta-2xK\xi)}+\tfrac{1}{4}\beta\int_1^\infty dy\, y\times\int_{-1}^{+1} d\eta\,\frac{T(y,K,\eta)}{y-K\eta}\times[(x^2+y^2+\beta-2xy\xi\eta)^2-4x^2y^2(1-\xi^2)(1-\eta^2)]^{-1/2}. \quad (8')$$

For practical purposes, we change the limits of y integration from $(1,\infty)$ to $(0,1)$, defining

$$H(t,K,\xi)=2\pi T(1/t,K,\xi)/[t^2(1-tK\xi)],$$

and thus obtain

$$\phi(K)=1+\int_0^1 dt\int_{-1}^{+1} d\xi\, H(t,K,\xi), \quad (9)$$

with H obeying Eq. (10);

$$H(t,K,\xi)=\frac{\beta}{4(1-tK\xi)[1-2tK\xi+(K^2+\beta)t^2]}+\frac{\beta}{4(1-tK\xi)}\int_0^1 d\tau\int_{-1}^{+1} d\eta\,\tau^2 H(\tau,K,\eta)\times[(t^2+\tau^2+\beta t^2\tau^2-2t\tau\xi\eta)^2-4t^2\tau^2(1-\xi^2)(1-\eta^2)]^{-1/2}. \quad (10)$$

This equation has been solved numerically for various electron densities and values of total momentum K. The results are presented in Table I, where the values of

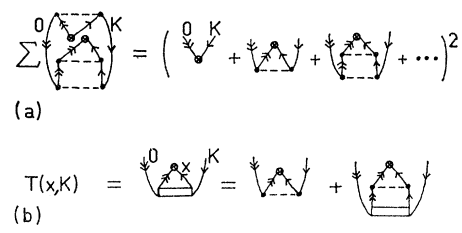

FIG. 2. (a) Ladder series, (b) Eq. (8).

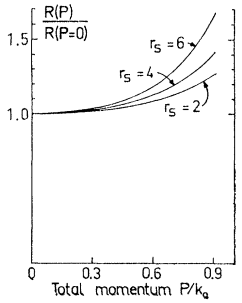

FIG. 3. Partial-annihilation rate versus total momentum at various densities.

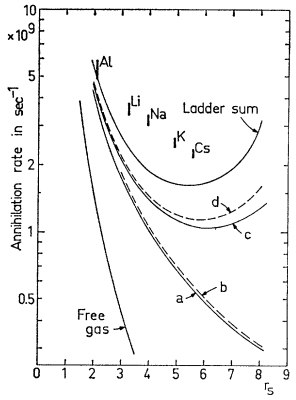

FIG. 4. Total-annihilation rate in various approximations. (a) The diagrams of Fig. 5; (b) ladder graphs up to third order; (c) modified ladder sum; (d) original ladder sum. Graphs (a)–(d) have been calculated from the values $R(0)$, assuming $R(K)=R(0)$. The experimental points are due to Bell and Jørgensen [Can. J. Phys. **38**, 652 (1960)].

$\phi(K)$ are given. In Fig. 3, we have plotted $D(K)$ versus K. Figure 4 shows the total rate compared with experimental points and the free-electron gas result, which has $D_0(K)=\theta(1-K)$ and

$$R_0=\sum_{\mathbf{P}} 2\pi r_0^2/V$$
$$=\tfrac{1}{2}(2\pi r_0^2/V)N_e$$
$$=\pi r_0^2 n_e \approx 12.05\times 10^9 r_s^{-3}\ \text{sec}^{-1}.$$

Comparison with measured rates reveals that the calculated rates are too small at metallic densities. Our calculated rates are also a bit smaller than some other numerical results.[1,3] At large values of r_s, our ladder sum is invalid, for it can be shown to diverge (Sec. 4).

3. OTHER GRAPHS AND MODIFIED LADDERS

To compare the ladder sum with the sum of at least a little larger set of diagrams, we have computed, at $\mathbf{P}=\mathbf{0}$ up to third order, all "generalized ladders" shown in Fig. 5. Of these, all but 4 and 5 must be taken twice, because of the existence of symmetric graphs with respect to the horizontal line through the point of annihilation. The sign of each contribution is shown in the figure. For each ladder graph of order N, there are $N!-1$ other graphs of this generalized type. Because they contain more hole lines than the ladder graph, the integration volume in the momentum space is more restricted. Thus one is led to guess that their contributions are smaller than those of the ladders, but on the other hand, they are much more numerous than the proper ladders. If we use the unscreened Coulomb force, we obtain, e.g., $D_2^u=\frac{1}{64}\beta^2(4+\pi^2)$ and $D_3^u=-\frac{1}{64}\beta^2\pi^2$ for graphs 2 and 3. They are even of the same order of magnitude.

The calculations have been performed only for zero total momentum (many earlier calculations as well as the present one indicate that the rate does not depend very strongly on $\mathbf{P}$ in the case of a pure electron gas), and to save computing time, the contributions from diagrams 8–11, 14, 16, and 17 have been calculated from approximation formulas. Because of the smallness of the contributions, these approximations are not serious, although the tiny D_{14} turns out to have the wrong sign. The results are shown in Table II. It is seen that, because of the use of a "self-consistent" effective screened interaction, the contributions from outside the proper ladders (which are the zeroth-order graph+1+2+4+6+12) are very strongly reduced. Up to second order, these outside contributions vary between 2 and 3%, and up to third order, they are about 4–5%. There are, of course, numerous other diagrams besides the ones considered, but those all have more than one electron loop (not counting the bubbles implicitly contained in the interactions) and are thus geometrically of an essentially different kind, and also very laborious to calculate. They can be considered as vertex corrections and corrections to the interaction in the generalized ladder sum.

As previously mentioned, the exact two-particle Green's function can be calculated as a "proper" ladder sum, where each line denotes the exact fermion propagator, and the interaction $U(p_1p_2p_3p_4)$ is the sum of all diagrams that cannot be divided into two parts connected by one electron and one positron line pointing in the same direction. Thus the series for U begins as in Fig. 6. In order to find out how much these corrections to the interaction affect the ladder sum, we have taken as a new interaction the sum of the first two terms in the series and computed the annihilation rate $R_m(\mathbf{P})$, with $\mathbf{P}=\mathbf{0}$. The details of this calculation are explained in the Appendix. We see in Fig. 4 that the modification only slightly decreases the annihilation rate at lower densities.

4. DIVERGENCE OF THE LADDER SUM

Figure 4 shows that the calculated rate begins to increase when the electron density is lowered below

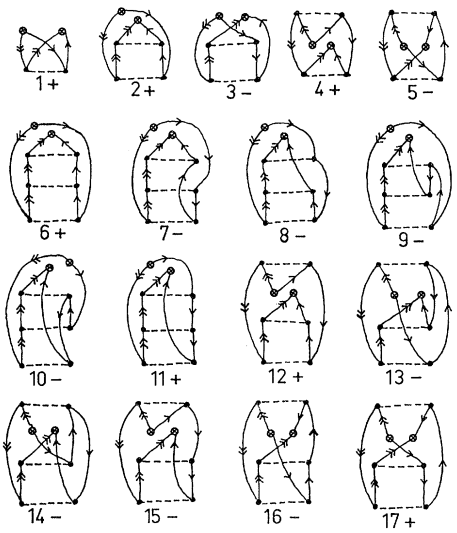

FIG. 5. The computed generalized ladder graphs.

the value corresponding to $r_s \approx 6$. It has been shown previously[4] that a positron can form a bound state with an electron in a sufficiently diffuse gas. In fact, it can be shown that within the ladder-sum approximation and using the static random-phase-approximation interaction, a bound-state pole appears in the electron-positron propagator at all densities.

Here we consider the function $G(p_1p_2q_1q_2)$, where 1 refers to the positron and 2 to the electron, and p and q are the outgoing and incoming energy momenta, respectively. It is easily found that in the resulting integral equation for the ladder sum we can perform the integrations with respect to all the variables of G except the total momentum $\mathbf{P}$, total energy E, and one of the momenta, say, $\mathbf{p}_2=\mathbf{p}$. Thus we are left with

$$\sum_{\mathbf{q}} \iint dp^0 dq^0\, G(\mathbf{P}-\mathbf{p}, \mathbf{p}, \mathbf{P}-\mathbf{q}, \mathbf{q}; E-p^0, p^0, E-q^0, q^0) = G(\mathbf{p}PE),$$

$$G(\mathbf{p}PE) = 2\pi i m \times\left\{\frac{\theta(k_0-p)\,\delta(\mathbf{p},\mathbf{P})}{mE-\frac{1}{2}\mathbf{P}^2-i\epsilon} - \frac{\theta(p-k_0)\,(1-\delta(\mathbf{p},\mathbf{P}))}{mE-\frac{1}{2}\mathbf{P}^2-(\mathbf{p}^2-\mathbf{p}\cdot\mathbf{P})+i\epsilon}\right\} \times[1+(2\pi i V)^{-1}\sum_{\mathbf{k}} u(\mathbf{p}-\mathbf{k})G(\mathbf{k}PE)], \quad (11)$$

where $\delta(\mathbf{p}, \mathbf{P})$ denotes the Kronecker δ. Again putting $\mathbf{P}=k_0\mathbf{K}$, $\mathbf{p}=k_0\mathbf{x}$, $\mathbf{k}=k_0\mathbf{y}$, $k_0^2\omega = mE-\frac{1}{2}\mathbf{P}^2$, $u(\mathbf{p}) = e^2U(\mathbf{x})/k_0^2$, and $G(\mathbf{p}PE) = 2\pi i m Q(\mathbf{x}\mathbf{K}\omega)/k_0^2$, we obtain

TABLE II. Contributions from the diagrams in Fig. 5. The numbers in the table are the ratios of the contributions to that of the zeroth-order graph. In the total sum, all graphs except 4 and 5 have been taken twice. R_t and R_l are the corresponding total-annihilation rates in units 10^9 sec^{-1}. They have been calculated as if the partial rates were independent of the total momentum.

Graph \ r_s	2	3	4	6	8
1	0.493	0.673	0.831	1.103	1.337
2	0.170	0.314	0.478	0.845	1.245
3	−0.019	−0.025	−0.030	−0.038	−0.043
4	0.243	0.453	0.690	1.217	1.788
5	−0.014	−0.017	−0.019	−0.021	−0.022
6	0.051	0.129	0.242	0.570	1.024
7	−0.008	−0.016	−0.023	−0.038	−0.053
8	−0.004	−0.007	−0.010	−0.017	−0.023
9	−0.001	−0.002	−0.004	−0.006	−0.009
10	−0.001	−0.003	−0.004	−0.008	−0.012
11	0.001	0.001	0.002	0.002	0.003
12	0.084	0.211	0.397	0.932	1.665
13	−0.006	−0.011	−0.015	−0.023	−0.029
14	0.001	0.001	0.002	0.002	0.003
15	−0.009	−0.017	−0.025	−0.041	−0.058
16	−0.001	−0.002	−0.003	−0.005	−0.007
17	0.001	0.002	0.002	0.003	0.003
Total+zeroth order	2.729	3.934	5.350	8.757	12.858
R_t	4.110	1.756	1.007	0.489	0.303
Ladder-types total	2.837	4.107	5.587	9.117	13.330
R_l	4.275	1.835	1.052	0.508	0.314

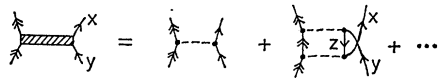

FIG. 6. Better approximation for the interaction.

for the zeroth-order part with respect to $1/V$ [we expand $G=G_0+G_1/V+\cdots$, and note that $\delta(\mathbf{p}, \mathbf{P})\to(2\pi)^3\delta(\mathbf{p}-\mathbf{P})/V$], the equation

$$Q_0(\mathbf{x}\mathbf{K}\omega) = \frac{\theta(x-1)}{\mathbf{x}^2-\mathbf{x}\cdot\mathbf{K}-\omega-i\epsilon} \times\left\{1+\frac{\beta}{8\pi}\int d\mathbf{y}\, U(\mathbf{x}-\mathbf{y})Q_0(\mathbf{y}\mathbf{K}\omega)\right\}. \quad (11')$$

In fact, this is the approximative propagator when there is no positron in the ground state. Letting $U(x, y) = U(y, x) = \int d\Omega_y U(\mathbf{x}-\mathbf{y})$, we try a spherically symmetric solution at $\mathbf{K}=\mathbf{0}$;

$$Q_0(x\omega) = \frac{\theta(x-1)}{x^2-\omega-i\epsilon}\left\{1+\frac{\beta}{8\pi}\int_1^\infty dy\, y^2U(x, y)Q_0(y\omega)\right\}. \quad (12)$$

Because $U_s(\mathbf{x}) > U_F(\mathbf{x})$ at every density, we have

$$U_s(x, y) > U_F(x, y) = \pi(xy)^{-1}\ln\frac{(x+y)^2+\beta}{(x-y)^2+\beta} > \frac{4\pi}{x^2+y^2+\beta}.$$

Let U_0 be the smallest value of $U(x, y)$ in the region $1\leqq x, y\leqq 1+\Delta$. If we write Eq. (12) as $Q=q+HQ$ and denote with subscript r functions restricted to the region $1\leqq x, y\leqq 1+\Delta$, we get

$$Q_r = q_r + (HQ)_r \geqq q_r + H_rQ_r \geqq q_r + K_rQ_r,$$

where now

$$K_r(x, y) = (\beta U_0/8\pi)y^2/(x^2-\omega).$$

The inequalities follow from the fact that if the Neumann series is required to converge, then the considered functions are all positive for $\omega<1$. Because

$$K_rq_r(x) = \frac{\beta U_0}{8\pi}(x^2-\omega)^{-1}\int_1^{1+\Delta} dy\,\frac{y^2}{y^2-\omega} = \lambda q_r(x),$$

we obtain

$$Q_r \geqq q_r + K_rq_r + K_r^2q_r + \cdots = (1+\lambda+\lambda^2+\cdots)q_r.$$

Now, if $\lambda\geqq 1$, we certainly get a divergent result. We obtain

$$\lambda(\beta, \omega) = \frac{\beta U_0}{8\pi}\left[\Delta+\tfrac{1}{2}\omega^{1/2}\ln\frac{(1+\Delta-\omega^{1/2})(1+\omega^{1/2})}{(1-\omega^{1/2})(1+\Delta+\omega^{1/2})}\right].$$

First, let $\omega=0$, as in the computed sums. Then in the u_s or u_F case when we have $U_0\geqq 4\pi/[\beta+2(1+\Delta)^2]$, we

put $\Delta=\alpha\beta^{1/2}$ and get, as $\beta\to\infty$,

$$\lambda\to\tfrac{1}{2}\alpha\beta^{1/2}(1+2\alpha^2)^{-1}\to\infty.$$

Then, given arbitrary $\beta\neq0$, we see that if $\Delta\neq0$, the lower limit of λ goes to infinity when $\omega\to1$, which is the limit of the scattering-state energies. This result can be seen to hold for an almost arbitrary attractive static interaction as long as $U_0>0$.

Thus the propagator diverges below those energies at which it would diverge if the interaction were zero. The smallest value of ω at which this kind of divergence occurs corresponds either to a separate pole (=bound state) or the end of a cut (=limit of energies of the interacting scattering states). The latter case does not seem plausible in this approximation, because if $\omega<1$, the kernel of our integral equation is bounded, and both $\mathrm{Tr}H$ and $\mathrm{Tr}H^2$ are limited. Thus the symmetrized kernel has only discrete eigenvalues (with respect to integration over y, ω kept fixed), leading to simple ω poles in the function $Q_0(x\omega)$. Similar poles appear also in $Q_1(x\omega)$, which is used rather than Q_0 in calculating the annihilation rate.

Because of the result that the pole appears almost for any potential, we are led to think that a kind of quasipositronium can be formed in an electron gas of arbitrary density. Because at high densities the energy of this pole (there may be of course more than one pole) is apparently only slightly below those of the lowest scattering states, we might expect transitions to occur between these states, especially in real metals. Furthermore, the energy of the pole varies as a function of the total momentum of the pair, and in more exact approximations, the interactions between this positronium and excited electrons and holes cause some smoothing of the poles. To our knowledge, these analytic properties have not been studied in detail.

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APPENDIX

Ladder sum. Equation (10) has been solved numerically by dividing both intervals, (0, 1) and (−1, +1), into 10 equivalent parts, whose end points were used to compute the values of the functions. In the integration, the simple Simpson rule was used to approximate the integrand, and the resulting 121×121 matrix was then inverted. Although the computation uses quite a few points, the error is not large because the function H is rather smooth. The computation was made with values $K=0$, 0.3, 0.6, 0.9 and $r_s=2$, 3, 4, 6, 8.

Modified ladder sum. The modified interaction is, according to Fig. 6 (using units k_0 and considering only $\mathbf{K}=\mathbf{0}$; the notation is the same as in Sec. 4),

$$U_m(\mathbf{x},\mathbf{y})=[(\mathbf{x}-\mathbf{y})^2+\beta]^{-1}-(4\pi)^{-1}\beta\times\int_{z<1}d\mathbf{z}\{[(\mathbf{x}-\mathbf{z})^2+\beta]\times[(\mathbf{y}-\mathbf{z})^2+\beta][\mathbf{x}^2+\mathbf{y}^2+(\mathbf{x}+\mathbf{y}-\mathbf{z})^2-\mathbf{z}^2]\}^{-1}.$$

Because x, $y>1$ and $z<1$, we set $\mathbf{z}=\mathbf{0}$ in the integrand as an approximation for the averaging effect of the angular integration. We then get

$$U_m(\mathbf{x},\mathbf{y})\cong[(\mathbf{x}-\mathbf{y})^2+\beta]^{-1}-\beta\{3(\mathbf{x}^2+\beta)(\mathbf{y}^2+\beta)\times[\mathbf{x}^2+\mathbf{y}^2+(\mathbf{x}+\mathbf{y})^2]\}^{-1}.$$

Putting this expression into Eq. (8) in place of $U_F(\mathbf{x}-\mathbf{y})$, one gets, after angular integration, the modified ladder sum [cf. Eq. (10)]

$$H_m(t)=\beta(6-t^2)[12(1+\beta t^2)]^{-1}+\beta(8t)^{-1}\int_0^1 d\tau\,\tau H_m(\tau)S(t,\tau),$$

where

$$S(t,\tau)=\ln\frac{(t+\tau)^2+\beta t^2\tau^2}{(t-\tau)^2+\beta t^2\tau^2}-\frac{\beta t^2\tau^2}{3(1+\beta t^2)(1+\beta\tau^2)}\times\ln\frac{t^2+\tau^2+t\tau}{t^2+\tau^2-t\tau}.$$

This equation has been solved by dividing the interval (0, 1) into 40 parts.