to a good approximation, because of the slow change of the total density ρ from 1°K to the λ point. From (B6) and (B7) we have finally

$$\frac{\Delta\mu}{\mu}=\frac{\epsilon_0}{ed}\left(\frac{\mu_0}{\langle v_c\rangle}\right)\left(\frac{\rho_s}{\rho}\right). \tag{B8}$$

Taking $\epsilon_0=0.22$ eV, $d=0.4$ cm, one gets

$$\Delta\mu/\mu=5.5\times10^{-1}(\rho_s/\rho)(\mu_0/\langle v_c\rangle). \tag{B9}$$

The ratio $\Delta\mu/\mu$ calculated from (B9) for positive ions, using the experimental data for μ_0 and $\langle v_c\rangle$, is plotted in Fig. 12.

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Flow of Superfluid Helium in a Porous Medium

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Flow of superfluid liquid helium has been studied experimentally in a porous medium formed by packing fine powder into a hollow glass sphere. The experiments fall into two categories, (1) the study of torsional oscillations of such a sphere filled with helium II as a function of temperature, and (2) the study of persistent circulating flow of the superfluid in such a sphere as a function of temperature and angular velocity of preparation, using a gyroscopic technique. The results at low angular velocities are shown to be in accord with an analysis of the motion of the liquid based on the two-fluid model of helium II and indicate that the superfluid can exhibit dissipationless potential flow in a porous medium with circulation constants which are remarkably stable.

I. INTRODUCTION

THE flow of liquid helium II in fine channels is of considerable interest because while the motion of the normal component of the liquid is increasingly impeded as channel size is reduced, owing to its viscosity, the ability of the superfluid component to flow with little or no dissipation is enhanced. A convenient method of forming fine channels for the study of this flow is to pack a fine powder into a container or tube. When this is done the open space remaining in the container assumes a complicated multiply connected geometry, and a network of interconnected flow channels is formed. This paper describes two related sets of experiments which we have carried out to study the flow of helium II inside a powder-filled sphere and presents an analysis of the flow taking place in these experiments based on the two-fluid model.

The first type of experiment represented a modification of the classic Andronikashvili oscillating-pile-of-disks experiment[1] in which the usual pile of disks was replaced by the powder-filled sphere. In effect, these experiments yielded measurements of the angular momentum imparted to the liquid inside the sphere as the sphere was put into rotation from rest at a temperature below T_λ, the temperature of the lambda transition.

The second set of experiments involved the use of a gyroscopic technique to study the behavior of persistent metastable circulating flow of the superfluid in the powder-filled sphere. Ever since the discovery of superfluidity in liquid helium, it has been natural to wonder whether the superfluid can flow with identically zero dissipation. A particularly sensitive test for the presence of dissipation can be made by preparing a coasting circulating current of superfluid and measuring the rate at which the flow decays. When the present work was undertaken, a number of experiments had already provided evidence that such circulating currents can have lifetimes much larger than the expected lifetimes of similar currents in helium I.[2–6] The gyroscopic technique developed in the present work,[7] and an elegant alternative gyroscopic technique developed by Reppy and co-workers,[8,9] have made it possible to establish in a particularly clear-cut way, the existence of persistent superfluid flow by providing a means of making repeated observations of the angular momentum of the flow without destroying the flow.

In our experiments of this type, long-lived circulating

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[1] E. L. Andronikashvili, J. Phys. (U.S.S.R.) **10**, 201 (1946); Zh. Eksperim. i Teor. Fiz. **16**, 780 (1946); **18**, 424 (1948); E. M. Lifschitz and E. L. Andronikashvili, *A Supplement to "Helium"* (Consultants Bureau Enterprises, Inc., New York, 1959), pp. 70–76.

[2] H. E. Hall, Phil. Trans. Roy. Soc. (London) **A250**, 359 (1957).

[3] W. F. Vinen, Proc. Roy. Soc. (London) **A260**, 218 (1961).

[4] P. J. Bendt, Phys. Rev. **127**, 1441 (1962).

[5] D. Depatie, J. D. Reppy, and C. T. Lane, in *Proceedings of the Eighth International Conference on Low-Temperature Physics*, edited by R. O. Davies (Butterworths Scientific Publications, Ltd., London, 1963), p. 75.

[6] J. D. Reppy and D. Depatie, Phys. Rev. Letters **12**, 187 (1964).

[7] J. B. Mehl and W. Zimmermann, Jr., Bull. Am. Phys. Soc. **10**, 30 (1965), Phys. Rev. Letters **14**, 815 (1965).

[8] J. Clow, J. C. Weaver, D. Depatie, and J. D. Reppy, in *Proceedings of the Ninth International Conference on Low Temperature Physics*, edited by J. G. Daunt, D. O. Edwards, F. J. Milford, and M. Yaqub (Plenum Press, Inc., New York, 1965), p. 328.

[9] J. D. Reppy, Phys. Rev. Letters **14**, 733 (1965).

flow was observed in the powder-filled sphere, and studies of the stability and temperature dependence of this flow were made. In these measurements persistent flow was normally generated by cooling the helium-filled sphere through the λ point in rotation and subsequently stopping rotation of the sphere, leaving the superfluid in motion. The angular momentum of the coasting superfluid was then measured by measuring the torque impulse necessary to reorient the sphere. Studies of the angular momentum of the current so formed as a function of the angular velocity of rotation were also made.

The results of our work are consistent with the assumptions that under appropriate circumstances (1) the superfluid in a porous medium can undergo potential flow without dissipation, and (2) the circulation of the superfluid around any closed path in the fluid can remain remarkably stable, even as the sphere is rotated or as the temperature is varied. In addition, evidence suggests that the equilibrium state of the superfluid in a rotating porous medium is a state with angular momentum approximately equal to the solid-body value.

The following text of this paper is divided into three main sections. In Sec. II we present a hydrodynamic analysis of the motion of the liquid in the porous medium provided by the powder-filled sphere. In Sec. III the torsional-oscillation experiments are described and their results analyzed and discussed. In Sec. IV we give a similar presentation of the persistent-flow experiments. These sections are then followed by a brief concluding section.[10]

II. HYDRODYNAMIC ANALYSIS

A. Introduction

We begin the analysis with the basic assumption that the flow of helium II in a porous medium is governed by the equations proposed for the two-fluid model by Landau[11,12]:

$$\nabla\times\mathbf{v}_s=0\,,\tag{1}$$

$$\partial\rho/\partial t+\nabla\cdot(\rho_s\mathbf{v}_s+\rho_n\mathbf{v}_n)=0\,,\tag{2}$$

$$\partial(\rho s)/\partial t+\nabla\cdot(\rho s\mathbf{v}_n)=0\,,\tag{3}$$

$$\partial\mathbf{v}_s/\partial t+\nabla[\tfrac{1}{2}(\mu+\mathbf{v}_s^2)-\rho_n(\mathbf{v}_n-\mathbf{v}_s)^2/2\rho]=0\,,\tag{4}$$

$$\partial(\rho_s v_{si}+\rho_n v_{ni})/\partial t+\partial\Pi_{ik}/\partial x_k=0\,,\tag{5}$$

where $\mathbf{v}_s$ and $\mathbf{v}_n$ are the velocities of the superfluid and normal-fluid components, respectively, ρ_s and ρ_n are the densities of the superfluid and normal-fluid components, $\rho=\rho_s+\rho_n$ is the total density, s is the entropy per unit mass, μ is the chemical potential per unit mass, and Π is the momentum flux density tensor. In Eq. (5) a component notation is used in which the additional subscripts refer to the three Cartesian axes and a term containing a subscript twice is to be summed over that subscript.

A further assumption, which will enter into the analysis in a somewhat less direct way than the Landau equations, is that the superfluid circulation is quantized. That is, we assume that the superfluid circulation κ around any closed path in the liquid which cannot be shrunk down upon itself without crossing a region not in the liquid is given by the Onsager-Feynman relation

$$\kappa\equiv\oint\mathbf{v}_s\cdot d\mathbf{r}=nh/m\,,\tag{6}$$

where n is an integer, h is Planck's constant, and m is the mass of a helium atom.[13,14] The existence of such irreducible paths arises from the multiple connectivity of the region occupied by the liquid. The additional possibility that quantum superfluid vortices may be present in the liquid, a possibility which would require some modification of the Landau equations, will be discussed at several points further on in the paper.

When the Landau equations are applied to the flow that takes place in our experiments a great simplification takes place. First, we argue that the flows of both the normal and superfluid components may separately be regarded as incompressible. Flow in an ordinary liquid may be regarded as incompressible when the flow velocity v is everywhere small compared to c, the speed of sound ($v\ll c$), and when changes in flow take place with characteristic times τ long compared to the time for sound to propagate a distance l characteristic of the distances over which significant changes of v take place ($l/\tau\ll c$).[15] In the case of helium II it is plausible that both the mass density ρ and the entropy density ρs may be regarded as constants whenever the quantities v and l/τ characterizing the flows of the component fluids are small compared to the speeds of first and second sound. In the present experiments both v and l/τ were of the order of 10 cm sec^{-1} or less, in comparison with a first-sound speed of roughly 2×10^4 cm sec^{-1} and a second-sound speed of roughly 2×10^3 cm sec^{-1} (except near T_λ). Thus it seems likely that for our purposes ρ and ρs can be regarded as constants. Equations (2) and (3) then reduce to the equations

$$\nabla\cdot\mathbf{v}_s=0\,,\tag{7}$$

$$\nabla\cdot\mathbf{v}_n=0\,,\tag{8}$$

which characterize incompressible flow for both the superfluid and normal-fluid components separately. It should be noted that since under isothermal conditions

[10] A more detailed account of the present work has been given by J. B. Mehl, Ph.D. thesis, University of Minnesota, 1966 (unpublished).

[11] L. D. Landau, J. Phys. (U.S.S.R.) **5**, 71 (1941); **8**, 1 (1944).

[12] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Ltd., London, 1959), Chap. XVI.

[13] L. Onsager, Nuovo Cimento **6**, Suppl. 2, 249 (1949).

[14] R. P. Feynman, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Co., Amsterdam, 1955), Vol. I, Chap. II.

[15] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Ltd., London, 1959), Sec. 10.

there are no sources or sinks of heat within the sphere, there is at the walls no conversion of the superfluid into normal fluid or vice versa, and hence the components of $\mathbf{v}_s$ and $\mathbf{v}_n$ normal to the wall at the wall must equal the normal component of the wall velocity itself.

An important consequence of Eqs. (1) and (7) is that $\mathbf{v}_s$ can be set equal to the gradient of a scalar velocity potential ϕ which satisfies the Laplace equation

$$\nabla^2\phi=0. \tag{9}$$

Because of the multiple connectivity of the region ϕ will in general be a multiple-valued function of position. However, by the Onsager-Feynman relation, Eq. (6), the various values of ϕ at a given position can only differ by integral multiples of h/m.

A further simplification results from the fact that in the fine channels of our powder-filled spheres the normal-fluid component is effectively constrained by viscous effects to move with the powder. In order to gain a quantitative idea of this effect, it is useful to consider a simple example. Consider a long tube of length l having a circular cross section of radius $a\ll l$ closed upon itself to form a slender torus of radius $R=l/2\pi$ as shown in Fig. 1. Suppose that this torus is filled with helium II and that the torus together with its contents is initially at rest. If the torus is now suddenly brought into steady rotation about its axis of circular symmetry, Eqs. (4) and (9) together with the boundary condition on $\mathbf{v}_s$ mentioned above predict that the superfluid component will remain at rest. The behavior of the normal fluid can then be determined from Eq. (5), making use of the boundary condition that $\mathbf{v}_n$ at the wall must equal the wall velocity. In the present case of incompressible flow for each of the components separately the momentum flux density tensor assumes the form

$$\Pi_{ik}=P\delta_{ik}+\rho_s v_{si}v_{sk}+\rho_n v_{ni}v_{nk} -\eta_n(\partial v_{ni}/\partial x_k+\partial v_{nk}/\partial x_i)\,, \tag{10}$$

where P is the total pressure and η_n is the coefficient of viscosity of the normal component.[11,12] As a result Eq. (5) reduces to an equation identical to the Navier-Stokes equation for an ordinary incompressible fluid with density ρ_n and viscosity η_n flowing with velocity $\mathbf{v}_n$. The solution to the problem of an ordinary fluid in such a torus in the limit that $R\gg a$, so that in each section of the torus the curvature can be ignored, has the property that the fluid will be pulled by the walls into solid-body motion along with the walls with a fundamental time constant τ_1 given by the expression

$$\tau_1=a^2\rho_n/x_{01}{}^2\eta_n\,, \tag{11}$$

where x_{01} is the first root of the zeroth-order Bessel function $J_0(x)$ and equals 2.4.

For superfluid helium ρ_n/η_n ranges from 0.2×10^3 sec cm^{-2} at 1.2°K to 6×10^3 sec cm^{-2} at the lambda temperature T_λ. Taking a to be 10^{-4} cm, representative of the size of flow channel we have used, τ_1 then ranges from 4×10^{-7} to 1×10^{-5} sec. Although the geometry of the flow channels in our experiment is much more complicated than that of the simple torus considered above, it is plausible that these time constants are reasonable estimates for the motion of the normal component in the sphere relative to that of the sphere and powder. Since changes of the velocity of the sphere occur with characteristic times of 0.1 sec or longer, it is thus a safe assumption that the normal component moves with the sphere and powder. This behavior is, of course, consistent with the condition that $\nabla\cdot\mathbf{v}_n=0$.

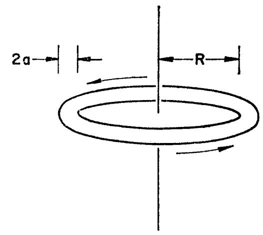

FIG. 1. A flow channel in the shape of a torus of circular cross section which can be set into rotation about its axis of circular symmetry.

Hence for our present purposes the motion of the normal component is known immediately once the motion of the sphere and powder is given. A determination of the motion of the superfluid involves finding solutions to Eq. (9) subject to the appropriate boundary conditions. This problem is the subject of Secs. II B and II C below.

It is worthwhile noting at this point that while there is good reason to believe that under isothermal conditions the normal fluid follows the motion of the powder and sphere very closely, estimates indicate that the pores still are large enough to permit thermal gradients in the sphere to relax relatively rapidly by superfluid–normal-fluid counterflow. For a characteristic pore size of 10^{-4} cm the fundamental thermal relaxation time of the sphere is thought to be less than 0.1 sec at all temperatures below T_λ reached in this work except those very close to T_λ itself. Thus, in particular, there is no reason to expect difficulty in obtaining thermal equilibrium within the sphere after changes in bath temperature take place, except possibly very close to T_λ.

B. Velocity Potential in a Multiply Connected Container

The following treatment draws heavily on ideas presented by Lamb.[16] Consider an arbitrary multiply connected region V bounded by a surface S. The connectivity of such a region may be defined as follows. Let a barrier be defined as an imaginary surface drawn in V such that it is bounded entirely by a curve lying on S. If at most n barriers can be drawn in V without dividing V into separate parts, then V is said to be $(n+1)$-ply connected. Now let the n barriers be numbered by positive integers m where m runs from 1 to n. Associated

[16] H. Lamb, *Hydrodynamics* (Dover Publications, Inc., New York, 1945), 6th ed., Nos. 47–52, 132.

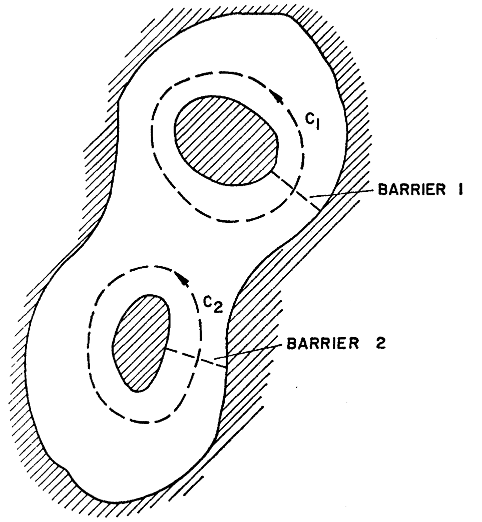

FIG. 2. A triply connected region in two dimensions. A possible choice of two barriers which render the region simply connected is shown, and for each barrier a possible choice of an associated closed path.

with each of the n barriers is at least one closed path c_m which cuts the mth barrier once and only once and cuts no other barriers. A simple two-dimensional illustration is given in Fig. 2. For a given multiple-valued velocity potential $\phi(\mathbf{r})$ in V there will be associated with each barrier m a circulation constant κ_m given by the relation

$$\kappa_m = \oint_{c_m} \nabla\phi \cdot d\mathbf{r}, \tag{12}$$

where the integral is taken around the closed path c_m.

Although for each barrier m there will, in general, be infinitely many paths which will satisfy the definition for c_m, Stokes's theorem may be used to show that the circulations defined above will not depend on the particular choice of path. It should also be noted that the choice of the n barriers is not unique for a given region. However, a different choice of barriers will always give rise to a set of n circulations which are linearly dependent on the original circulations.

Making use of the above concepts it is possible to obtain for an $(n+1)$-ply connected region the following uniqueness theorem, whose proof may be found in Lamb[16]: If ϕ satisfies the Laplace equation $\nabla^2\phi = 0$ everywhere in V, if in addition the normal derivative of ϕ at S, $\partial\phi/\partial n$, is specified everywhere on S, and finally, if a set of n independent circulation constants κ_m are given for a given set of n barriers, then ϕ is uniquely determined in V to within an arbitrary additive constant.

Suppose that ϕ represents the potential flow of an incompressible liquid in an $(n+1)$-ply connected container which is at rest. The component of velocity normal to the wall is then zero everywhere along the wall, so that $\partial\phi/\partial n$ is zero everywhere on S. Hence in this case the velocity field is determined once n independent circulations are known.

To take a more general case, suppose that the container above can move as a solid body with instantaneous translational velocity $\mathbf{u}$ and rotational angular velocity $\boldsymbol{\omega}$. In this case $\partial\phi/\partial n$ at S must satisfy the relation

$$(\partial\phi/\partial n)_S = \mathbf{n}\cdot(\mathbf{u}+\boldsymbol{\omega}\times\mathbf{r}) = \mathbf{u}\cdot\mathbf{n}+\boldsymbol{\omega}\cdot(\mathbf{r}\times\mathbf{n}), \tag{13}$$

where $\mathbf{n}$ is a unit vector normal to S. Under these circumstances a unique determination of the velocity field of the fluid is provided by specifying the three components of $\mathbf{u}$, the three components of $\boldsymbol{\omega}$, and n independent circulation constants. Since these quantities may all be specified independently and since ϕ is linearly related to all of these quantities, the resulting unique ϕ can be written in the form

$$\phi(\mathbf{r}) = \mathbf{u}\cdot\mathbf{A}(\mathbf{r}) + \boldsymbol{\omega}\cdot\mathbf{B}(\mathbf{r}) + \sum_{m=1}^{n} \kappa_m b_m(\mathbf{r}) + \phi_0, \tag{14}$$

where the components of $\mathbf{A}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ and all n of the $b_m(\mathbf{r})$ individually satisfy the Laplace equation and appropriate boundary conditions, and ϕ_0 is an arbitrary constant. In particular, the components of $\mathbf{A}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are each required to have zero circulation around all closed paths in the fluid, and for each value of m, $b_m(\mathbf{r})$ is required to have unity circulation around contour c_m and zero circulation around the remaining $n-1$ independent contours. The velocity field $\mathbf{v}(\mathbf{r}) = \nabla\phi$ is thus uniquely determined by the motion of the container together with the specification of n independent circulations.

C. Angular Momentum of Helium II in a Powder-Filled Sphere

Now we apply the assumptions and results of the preceding sections to the case of superfluid flow in a powder-filled sphere which can undergo rotational motion. In particular, we shall concern ourselves with the angular momentum of the liquid. In accord with the arguments of the previous sections, the normal component is assumed to move in solid-body motion with the sphere. The superfluid component is assumed to be governed by a velocity potential satisfying the Laplace equation and is thus uniquely determined by the state of rotational motion of the sphere together with a specification of n independent circulation constants, where $n+1$ is the connectivity of the open space inside the sphere. These circulation constants will be assumed to take on only the quantum values given by Eq. (6). It should be pointed out that the equation of superfluid motion, Eq. (4), leads to the result that the superfluid circulations will all remain constant in time even as the sphere undergoes changes in state of rotation. Such constancy might also result from some intrinsic stability of the quantum levels of circulation against change. It must be remarked, however, that this result would be compromised by the existence and motion of quantum superfluid vortices.

First consider the case in which no circulation is present about any of the paths in the sphere. If we adopt Cartesian coordinates with origin at the center of the sphere and suppose that rotation takes place about the z axis with angular velocity ω, ϕ takes the simple form $\phi=\omega B_z(\mathbf{r})$. The angular momentum of the superfluid component is then given by the expression

$$\mathbf{L}_b=\rho_s\int_V \mathbf{r}\times\mathbf{v}_s d^3\mathbf{r}=\rho_s\omega\int_V \mathbf{r}\times\nabla B_z(\mathbf{r})d^3\mathbf{r}. \quad (15)$$

The integral on the right depends entirely upon the geometry of the open space inside the sphere. The subscript b refers to the fact that $\mathbf{L}_b$ is entirely due to boundary motion.

It is reasonable to assume that in some average sense the pore structure possesses sufficient symmetry about the z axis that the x and y components of $\mathbf{L}_b$ vanish. The z component remaining may be written

$$L_b=\rho_s\omega\int_V (x\partial B_z/\partial y-y\partial B_z/\partial x)d^3\mathbf{r}=\rho_s\omega\chi I_\lambda/\rho_\lambda, \quad (16)$$

where we have for convenience replaced the integral, whose exact value would be impossible to calculate, by the constant $\chi I_\lambda/\rho_\lambda$. Here I_λ is the solid-body moment of inertia of all the liquid in the sphere at T_λ, ρ_λ is the total density of liquid helium at T_λ, and χ is a dimensionless geometrical parameter to be determined by experiment.

To understand the meaning of the last equation, consider the following extreme cases. If the inside of the sphere were filled with a stack of closely spaced disks whose planes lay perpendicular to the axis of rotation, χ would vanish, for although the normal fluid would move with the disks the superfluid would not be brought into motion by rotation of the sphere. If, on the other hand, the flow channels were nearly blocked or just very tortuous, the superfluid would be forced to move with the sphere in nearly solid-body motion, and χ would approach unity. It seems likely that χ must always lie between 0 and 1.

If we now assume that macroscopically the powder-filled sphere is sufficiently symmetric so that the same χ and I_λ apply to all axes of rotation, then Eq. (16) can be written in the vector form

$$\mathbf{L}_b=(\rho_s/\rho_\lambda)\chi I_\lambda\boldsymbol{\omega}. \quad (17)$$

In the general case the circulation constants will be nonzero, and in this case the total angular momentum of the superfluid $\mathbf{L}_s$ may be written

$$\mathbf{L}_s=\mathbf{L}_p+\mathbf{L}_b, \quad (18)$$

where $\mathbf{L}_p$ is the angular momentum due to nonzero circulation constants in the absence of rotation and is given by the expression

$$\mathbf{L}_p=\rho_s\sum_{m=1}^{n}\kappa_m\int_V \mathbf{r}\times\nabla b_m(\mathbf{r})d^3\mathbf{r}. \quad (19)$$

To complete the picture we must add to $\mathbf{L}_s$ the angular momentum of the normal component $\mathbf{L}_n$ to get the total angular momentum of the liquid $\mathbf{L}_l$. Since $\mathbf{L}_n$ is simply a solid-body angular momentum, it can conveniently be written in the form

$$\mathbf{L}_n=(\rho_n/\rho_\lambda)I_\lambda\boldsymbol{\omega}. \quad (20)$$

When torsional oscillations of the sphere are excited from rest at some temperature below T_λ, as occurs in our first set of experiments, the liquid contribution to the total angular momentum of the rotating system will come entirely from $\mathbf{L}_b+\mathbf{L}_n$, assuming that the circulation constants all remain zero. The fact that $\mathbf{L}_b$ and $\mathbf{L}_n$ are both proportional to $\boldsymbol{\omega}$ means that the liquid contributes an effective moment of inertia to the sphere I_{eff} given by the expression

$$I_{\text{eff}}=(\rho_s/\rho_\lambda)\chi I_\lambda+(\rho_n/\rho_\lambda)I_\lambda, \quad (21)$$

which can be determined from measurements of the period of torsional oscillation. In fact, for this result to be valid it is not really necessary that the circulation constants be initially zero, but only that they remain constant during the torsional oscillations.

When the sphere is cooled through T_λ in steady rotation, as in the persistent-current experiments, the superfluid is likely to be formed in a state with nonzero circulations. Thus, in steady rotation at some temperature below T_λ, the total angular momentum of the liquid $\mathbf{L}_l$ will have contributions from all three parts, $\mathbf{L}_b$, $\mathbf{L}_p$, and $\mathbf{L}_n$. When rotation of the sphere is subsequently brought to rest, $\mathbf{L}_b$ and $\mathbf{L}_n$ are reduced to zero, and $\mathbf{L}_p$ alone will remain as the angular momentum of persistent flow, assuming no change in the circulation constants take place. A gyroscopic technique can then be used to measure $\mathbf{L}_p$. Thus, the torsional-oscillation experiments and the persistent-flow experiments complement each other in measuring different contributions to the total angular momentum of the liquid in the sphere.

There are two aspects of these considerations which are of particular interest. The first concerns the state of superfluid circulation formed while cooling through T_λ in steady rotation. It is plausible that a state of circulation forms which most nearly permits $\mathbf{v}_s$ to approximate solid-body rotation when averaged over a volume which is macroscopically small but which nevertheless contains many pores, and that this state is the true equilibrium state of the liquid in rotation below T_λ.[17] In this case the

[17] For recent discussions of the equilibrium state of the superfluid in rotating multiply connected containers see R. J. Donnelly and A. L. Fetter, Phys. Rev. Letters **17**, 747 (1966); A. L. Fetter, Phys. Rev. **152**, 183 (1966); **153**, 285 (1967).

equilibrium value of total angular momentum would be given simply by the expression

$$\mathbf{L}_l = (\rho/\rho_\lambda) I_\lambda \omega. \tag{22}$$

As a result the angular momentum of the persistent current left after the sphere is brought to rest should be given by the expression

$$\mathbf{L}_p = \mathbf{L}_l - \mathbf{L}_n - \mathbf{L}_b = (\rho_s/\rho_\lambda)(1-\chi) I_\lambda \omega. \tag{23}$$

Since χ in this expression can be obtained from torsional-oscillation measurements, it thus is possible to test the hypothesis leading to Eq. (22) by measurements of $\mathbf{L}_p(\omega)$.

The second aspect of interest concerns the temperature dependence of the $\mathbf{L}_p$ of a persistent current in a stationary sphere. According to Eq. (19), as long as the circulations remain constant $\mathbf{L}_p$ will be directly proportional to ρ_s and hence will vary with temperature without any manipulation of the sphere. This remarkable behavior, which has been verified in the work of Reppy and co-workers,[6,9] as well as in the present work, depends critically on the constancy of the circulations during changes of temperature. It should be pointed out without delay that no violation of conservation of angular momentum is involved; the changes in liquid angular momentum with temperature must be accompanied by torques exerted on the liquid by the sphere and powder.

In closing this section it should be remarked that the analysis just presented applies only to flow at sufficiently low velocities. In general, there exists in every superfluid-flow situation some geometry-dependent value of velocity called the critical velocity above which strong resistance to flow is present, presumably due to the creation and motion of quantum superfluid vortices. In the present experiments this resistance would act to dissipate the motion of the superfluid relative to the powder and in so doing to alter the circulation constants. The inclusion of these effects would require an extension of the Landau equations on which the analysis was based.

It is conceivable that quantum superfluid vortices are also present in the liquid under subcritical flow conditions but are effectively immobilized by being pinned to the walls at their points of contact. In this case each vortex may simply be regarded as adding to the degree of connectivity of the open region inside the sphere, and the analysis of this section should continue to apply.

III. TORSIONAL-OSCILLATION EXPERIMENTS

A. Introduction

In the classic experiments carried out by Andronikashvili to measure $\rho_n(T)$, use was made of a stack of closely spaced disks which was free to undergo torsional oscillation about an axis perpendicular to the plane of the disks. The disks were immersed in helium II. Because in these experiments the viscous normal fluid was entrained by the disks while the superfluid remained at rest, only the normal fluid contributed to the moment of inertia of the rotating system, and thus $\rho_n(T)$ could be determined by making period measurements.

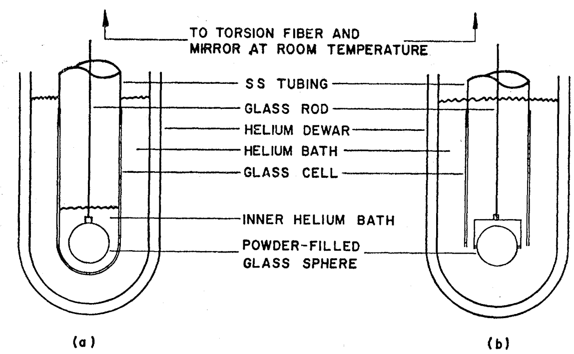

FIG. 3. Sketches of the low-temperature part of the apparatus showing the two configurations used.

In the present experiments we have made measurements similar to those of Andronikashvili with a torsion suspension in which a powder-filled sphere containing helium II replaced the stack of disks. The analysis of Sec. II indicates that because of the irregular boundaries provided by the powder not only the normal fluid but some fraction of the superfluid may be expected to contribute to the effective moment of inertia of the sphere as it undergoes torsional oscillation. The present experiments serve as a test of the assumptions and analysis of Sec. II and provide measurements of the parameter χ which is characteristic of the geometry of the open space inside the sphere.

B. Apparatus

The apparatus consisted essentially of a torsion pendulum formed by attaching a powder-filled sphere to the lower end of a long glass rod, the upper end of which was suspended from a sensitive torsion fiber at room temperature. This suspension was hung in a long, tubular enclosure which was mounted inside a liquid-helium Dewar. In an experimental run, the Dewar and the lower part of the enclosure were filled with liquid helium so that the sphere was well immersed. Helium was able to fill the powder-filled sphere through a small hole at the top of the sphere.

Five powder-filled spheres were used in the work. All five spheres were thin-walled glass spheres of the type used for Christmas-tree ornaments (with the silvering removed), and had average diameters of 3.0 cm and masses when empty of from 1.2 to 1.9 g. Two of the spheres were filled with a fine phenolic resin powder[18] with a particle size of about 20 μ as observed under a microscope. The density within a powder particle of this type was 1.36 g cm^{-3}. Each of these two spheres was

[18] Durez No. 791 fines, Durez Plastics Division, Hooker Chemical Co., North Tonawanda, N. Y.

attached to the glass rod by cementing a fitting to the neck of the sphere as shown in Fig. 3(a). The neck of one of these spheres was sealed shut in a flame to prevent helium from entering the sphere. This sphere was used to evaluate experimentally the effects due to the liquid outside the sphere. The remaining three spheres were filled with an alumina powder[19] with a particle size of about 1 μ, according to the manufacturer. In this case the density within a powder particle was 3.97 g cm^{-3}, according to the manufacturer. Each of these three spheres was attached to the glass rod by a yoke-and-pivot arrangement as shown in Fig. 3(b). This arrangement was designed for use in the persistent-current experiment. Two of the spheres filled with alumina powder had the necks removed, and the remaining sphere had the neck sealed shut in a flame to prevent helium entry.

Filling of the spheres with powder was carried out by adding powder in small amounts at a time, each time agitating the sphere to settle the powder and tamping the powder to compact it. The alumina powder packed very firmly at a low filling factor, while the phenolic resin powder was not as firm at a much higher filling factor. Here the filling factor is defined as the ratio of volume occupied by the powder to the total volume inside the sphere.

Two types of enclosure were used, as shown in Fig. 3. In the apparatus shown in Fig. 3(a) use was made of an enclosure which isolated the suspension from the outer bath and permitted period measurements to be made in low-density helium gas at low temperatures. The enclosure could be filled with liquid helium by condensation of helium gas. In the apparatus shown in Fig. 3(b), use was made of an open enclosure which filled with liquid helium from the bath.

Band-shaped torsion fibers of 90% Pt–10% Ni were used.[20] The fiber used with the apparatus of Fig. 3(a) had a torsion constant of 2.47 dyn cm rad^{-1}, which corresponded to a 12-sec period of the suspension, and the fiber used with the apparatus of Fig. 3(b) had a torsion constant of 0.81 dyn cm rad^{-1}, corresponding to a 28-sec period.

Near the upper end of the glass rod was mounted a galvanometer mirror which was used with a lamp-and-scale arrangement for observing angular deflections. Angular deflections could be observed with an accuracy of 2×10^{-3} rad. A solid-state photocell was mounted behind a small hole in the scale. The passing light beam activated the photocell which was wired to trigger a bistable multivibrator. The multivibrator output was connected to the input of an interval timer. This arrangement permitted period measurements to be made with an accuracy of about 1 msec for periods typically in the 10–30-sec range. With the enclosure evacuated the period was observed to be independent of amplitude to within 2 msec for amplitudes up to 3 rad.

[19] Alumina Abrasive 1.0 C, Linde Division, Union Carbide Co., East Chicago, Ill.

[20] Sigmund Cohn Corporation, Mount Vernon, N. Y.

Just below the mirror an aluminum cylinder was mounted concentric with the glass rod. This cylinder formed the rotor of a simple induction motor which was used to excite and damp torsional oscillations of the suspension.

The helium-bath temperature was stabilized below T_λ to better than 0.5 m°K, using a carbon or germanium resistance thermometer and an electronic regulator similar in principle to the one described by Blake and Chase.[21] Temperature measurements were made using an Octoil-S manometer and the 1958 He^4 temperature scale.[22]

C. Experimental Method

The torsional periods of both the open-sphere and sealed-sphere suspensions were measured with the spheres immersed in liquid helium at several temperatures between 1.25°K and T_λ. In a typical experimental run with the apparatus of Fig. 3(a), the period was first measured with the enclosure filled with helium gas at 4.2°K at several pressures in the range 0.02 to 0.20 atm. These period measurements, extrapolated to zero pressure, gave a value of P_0, the period of the system in a vacuum at 4.2°K. Then enough helium gas was condensed to fill the lower enclosure to the point where the liquid level was a few cm above the top of the sphere. The temperature was then lowered below T_λ. After stabilizing the temperature, period measurements were made as a function of amplitude of oscillation. It was observed that consecutive single-period measurements with both the open-sphere and sealed-sphere systems were reproducible to within ±2 msec as long as the amplitude of oscillation was kept below about 0.8 rad. Above 0.8 rad the period increased with amplitude, presumably due to entrainment of the superfluid, an effect which would be expected to occur when the critical velocity is exceeded in some part of the system. Unfortunately, the increased period seemed to be due to a combination of effects within the sphere and outside the sphere, which made it difficult to infer much quantitative information about the critical phenomena.[23] Accurate measurements could not be made at amplitudes below 0.15 rad.

Measurements of the type described above were repeated at several temperatures below T_λ. Mechanical instabilities, presumably due to lack of thermal equilibrium, made measurements above T_λ impossible. It was, however, possible to make measurements just below T_λ. In general, measurements were repeated after intervals of several hours to check reproducibility.

The measurements made with the apparatus of Fig. 3(b) were made during runs which were devoted pri-

[21] C. Blake and C. E. Chase, Rev. Sci. Instr. **34**, 984 (1963).

[22] F. G. Brickwedde, H. van Dijk, M. Durieux, J. R. Clement, and J. K. Logan, *The 1958 He⁴ Scale of Temperatures* (National Bureau of Standards, 1960), Vol. 10.

[23] A detailed account of these amplitude-dependent effects has been given by J. B. Mehl, M. S. thesis, University of Minnesota, 1964 (unpublished).

marily to persistent-current studies. Because of lack of sufficient time, these period measurements were neither as thorough nor as accurate as the measurements made with the apparatus of Fig. 3(a). The measurements were made with oscillation amplitudes of from 0.2 to 0.5 rad. Although detailed studies of amplitude dependence were not made, it was verified that the measured periods did not vary with amplitude by more than about 5 msec.

D. Analysis

It has been shown in Sec. II that in regard to torsional oscillation the liquid inside the sphere contributes an effective moment of inertia to the torsion suspension given by Eq. (21), provided that the circulation constants remain unchanged during oscillation. The total moment of inertia of the suspension also includes a term I_0 due to the solid parts and a term I_s which represents the inertial effects of the liquid outside the sphere. The latter effects may be functions of geometry, temperature, and period of oscillation.

The period of torsional oscillation P is then given by the expression

$$P^2=(4\pi^2/k)(I_0+I_s+\rho_s\chi I_\lambda/\rho_\lambda+\rho_n I_\lambda/\rho_\lambda)\,, \qquad (24)$$

where k is the torsion constant of the suspension. It was verified experimentally that the correction to this expression due to damping was negligible. The period P_s of a suspension with a sealed sphere attached is given by the expression

$$P_s^2=(4\pi^2/k)(I_{0,s}+I_{s,s})\,. \qquad (25)$$

Since I_s is difficult to estimate theoretically it is convenient that it can be eliminated by a subtraction method. The open and sealed spheres were chosen to be nearly identical in shape, so that it may be assumed that differences between I_s and $I_{s,s}$ due to geometrical effects are unimportant. It may also be assumed that the dependences of I_s and $I_{s,s}$ on period are weak enough so that differences between I_s and $I_{s,s}$ due to differences in period may also be neglected. Then for measurements at the same temperature I_s and $I_{s,s}$ are equal, and thus for P and P_s at the same temperature it is possible to subtract Eq. (25) from (24), eliminating I_s and $I_{s,s}$ and yielding the relation

$$P^2(T)-P_s^2(T)=(4\pi^2/k)(I_0-I_{0,s}+\rho_s\chi I_\lambda/\rho_\lambda+\rho_n I_\lambda/\rho_\lambda)\,. \qquad (26)$$

In helium II the temperature dependence of $P^2(T)-P_s^2(T)$ arises entirely from the last two terms on the right-hand side of Eq. (26),

$$\rho_s\chi I_\lambda/\rho_\lambda+\rho_n I_\lambda/\rho_\lambda=(\rho/\rho_\lambda)I_\lambda-(\rho_s/\rho_\lambda)(1-\chi)I_\lambda\,. \qquad (27)$$

If we ignore the small variation of ρ with temperature, the entire temperature variation of this expression is contained in ρ_s, and a plot of $P^2(T)-P_s^2(T)$ versus ρ_s should be linear.

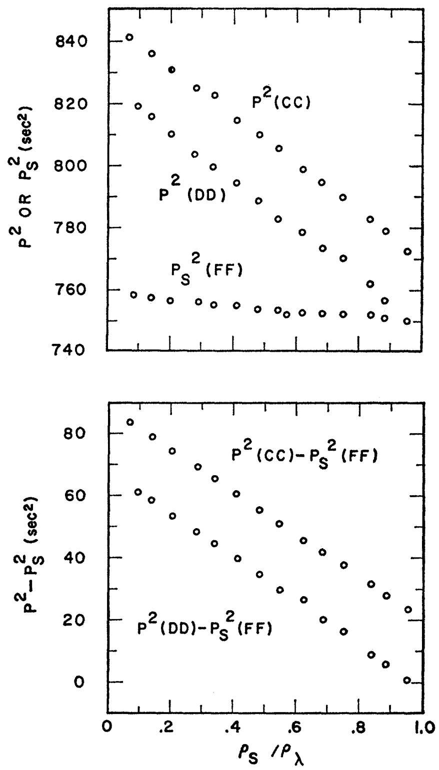

FIG. 4. Representative results from the torsional-oscillation experiments. P^2 for open spheres *CC* and *DD*, P_s^2 for sealed sphere *FF*, and $P^2-P_s^2$ for the combinations *CC-FF* and *DD-FF*, all as functions of ρ_s/ρ_λ.

E. Results and Discussion

Measurements of P as a function of T were made for each of the unsealed spheres, and measurements of P_s as a function of T for each of the sealed spheres. Some typical results, those for the spheres filled with the alumina powder, are shown plotted against $\rho_s(T)/\rho_\lambda$ in the upper part of Fig. 4. Use was made here of values of $\rho_s(T)$ recently calculated by Hussey *et al.* from second-sound velocities and thermal data.[24] Notice that some temperature dependence of P_s^2 is present, although it is much weaker than that of P^2. Hence, it was important to take surface effects into account, although the effects were not large.

Plots of $P^2(T)-P_s^2(T)$ versus $\rho_s(T)/\rho_\lambda$ were then made for each open sphere paired with a similar sealed sphere. Such plots for the spheres filled with alumina powder are shown in the lower part of Fig. 4. The linearity expected on a basis of the analysis developed in Sec. II was indeed observed, giving support to the correctness of the ideas presented there.

The difference between the $\rho_s/\rho_\lambda=0$ intercept of a plot of $P^2-P_s^2$ versus ρ_s/ρ_λ and the $\rho_s/\rho_\lambda=1$ intercept is equal to $(4\pi^2/k)(1-\chi)I_\lambda$. Since k had been measured and I_λ had been determined from measurements of the

[24] R. G. Hussey, B. J. Good, and J. M. Reynolds, Phys. Fluids **10**, 89 (1967).

TABLE I. Representative powder-filled spheres.

Sphere	Powder type	Filling factor[a]	χ[b]
A	Resin	0.41 ±0.01	0.32±0.01
CC	Alumina	0.176±0.002	0.15±0.01
DD	Alumina	0.173±0.002	0.13±0.01

[a] The filling factor represents the ratio of volume occupied by the powder particles to the total volume.
[b] χ is a geometrical parameter defined in Sec. II.

sphere volume and the mass of powder in the sphere, knowing the density of the material in the powder particles, it was therefore possible to determine the geometrical parameter χ. In the measurements using the enclosure of Fig. 3(a) it was also possible to determine I_λ from measurements of P and P_s in vacuum and at T_λ. Some results of these determinations are given in Table I along with values of the filling factor. It may be noticed that for both the resin and alumina powders χ is approximately 80% of the filling factor. Although it is hard to attach much fundamental significance to this relation, it is in accord with the rough idea that as the interior of the sphere is opened out a decreasing fraction of the superfluid will be pushed into motion by motion of the powder.

IV. PERSISTENT-FLOW EXPERIMENTS

A. Introduction

The experiments described in this section involved the use of a gyroscopic technique to study various properties of long-lived metastable states of circulating flow of superfluid helium. These experiments were carried out using powder-filled spheres similar to and in some cases the same as those used in the torsional-oscillation experiments. In so doing, the present experiments complement the torsional-oscillation experiments in testing the theoretical analysis presented in Sec. II.

Persistent currents were prepared by either of two methods: (1) cooling the powder-filled sphere containing helium through T_λ in steady rotation, and (2) bringing the sphere into rapid rotation from rest at a temperature below T_λ after cooling through T_λ without rotation. For both methods the final step in the preparation of a current was to bring the sphere carefully to rest. Were the analysis of Sec. II valid under all circumstances, no persistent current would be generated by method (2); the method depends on a breakdown of the analysis at sufficiently high velocities of rotation which allows the superfluid circulation constants to change.

The angular momentum of the persistent current was measured by measuring the torque impulse needed to reorient the sphere. Assuming that the superfluid circulation constants remain unchanged during this reorientation the angular-momentum vector of the persistent current rotates with the sphere, and the torque impulse gives a direct measure of the persistent angular momentum. Furthermore, the measurement may be repeated at will.

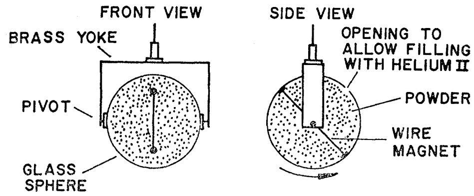

FIG. 5. Details of the sphere and mounting for the configuration of Fig. 3(b). The sphere is shown in its equilibrium position. The arrow shows the direction of tilt as the vertical magnetic field is applied.

B. Apparatus

The apparatus used in these measurements was basically the same as was used for the torsional-oscillation experiments described in Sec. III in the configuration shown in Fig. 3(b). The torsion suspension and enclosure were mounted on a rotating turntable which rested on a 12-cm-diam ball bearing at the top of the Dewar assembly. The rotating vacuum seal between the turntable and the stationary parts of the Dewar assembly was made with a commercial oil seal.[25] The turntable could be driven at steady angular velocities up to 6 rad sec^{-1} using a continuously variable transmission unit driven by a synchronous motor.[26]

The mounting of the sphere is shown in more detail in Fig. 5. The sphere was mounted in a yoke-and-pivot arrangement so that it could be tilted about a horizontal axis. A small bar magnet was cemented in place inside the sphere, and the interaction between this magnet and a vertical magnetic field supplied from outside the Dewars provided the torque needed to tilt the sphere. This vertical field was homogeneous enough and could be well enough aligned so that it exerted a negligible amount of torque on the sphere about a vertical axis.

The principle of our measurement technique is as follows. If a persistent current has been prepared with the sphere in its equilibrium position with the magnetic field turned off, the angular momentum in that position will lie in the vertical direction. As the sphere is tilted by application of the magnetic field, and the angular momentum vector of the persistent flow rotates with the sphere, the change in vertical component of superfluid angular momentum must be accompanied by a torque about the vertical axis exerted on the fluid by the powder and walls of the sphere. The reaction to this torque is an opposite and equal torque exerted on the powder and sphere by the fluid, which tends to deflect the torsion suspension. Since no torque acts about the vertical axis due to the magnetic field, measurement of this deflection gives a direct and repeatable measure of the persistent angular momentum.

The sphere was balanced so that it normally rested in the position shown in Fig. 5. The magnet had a dipole

[25] Model 55354S, National Seal Division, Federal-Mogul-Bower Bearings, Inc., Detroit, Mich.
[26] Model N30VM, Graham Transmissions, Inc., Menomonee Falls, Wisc.

moment of 2.5 dyn cm G^{-1}, and fields of 50 to 100 G were used to tilt the sphere through angles of approximately 90°. In the tilted position the center of mass of the sphere was raised, so that upon removal of the field the sphere returned to its equilibrium position. When the magnetic field was applied as a linear function of time with a total rise time of 3 to 6 sec, the sphere tilted smoothly in accord with the magnetic-field strength. The coils supplying this field were powered by a dc power supply with a motor-driven autotransformer input stage. By this means the field could be increased and decreased in a linear and reproducible manner, and the resulting tilts and returns were quite reproducible.

A uniform and accurately vertical magnetic field was supplied by a Helmholtz pair of coils external to the Dewars in conjunction with additional coils to cancel the horizontal components of the earth's magnetic field. Adjustments of the coils so as to eliminate vertical components of torque acting on the sphere due to magnetic effects were made in the following way. First, the currents in the earth's field canceling coils were set so that the zero of the torsion suspension as measured on a scale which rotated with the entire suspension and enclosure was independent of the orientation of the suspension and enclosure. With a torsion constant k of 0.8 dyn cm rad^{-1}, it was possible to adjust these currents so that a zero shift of no more than 0.004 rad occurred between any two orientations of the suspension and enclosure. The next step was to align the axis of the Helmholtz coils until little or no deflection of the suspension occurred when the sphere was tilted through 90° in the absence of a persistent current. It was usually possible to reduce this spurious deflection to less than 0.002 rad. In addition, to reduce transient torques due to dynamical effects during the tilt it was necessary to adjust the yoke so that the tilt axis was as nearly horizontal as possible. In this way the solid parts of the sphere had a minimum vertical component of angular momentum during the tilt. With all these adjustments properly made it was possible to tilt the sphere when immersed in liquid helium II with no persistent current present through an angle of about 90° without introducing oscillation of the torsion pendulum with an amplitude greater than 0.004 rad. It may be noted that a static deflection of this size corresponded to a torque of about 0.003 dyn cm, roughly a factor of 10^5 smaller than the torque applied about a horizontal axis to tilt the sphere.

C. Analysis of the Measurement Process

When no torques other than that due to the torsion fiber act on the torsion suspension about a vertical axis, its motion can be described accurately by the equation of motion of a damped oscillator,

$$I\ddot{\theta}+2\delta I\dot{\theta}+k\theta=0\,, \tag{28}$$

where θ is the angular deflection measured from the rest position of the suspension, δ is a damping coefficient, and I is the total moment of inertia of the solid parts and liquid that appears in Eq. (24). It is convenient to rewrite Eq. (28) in the form

$$\ddot{\theta}+2\delta\dot{\theta}+(\omega_0^2+\delta^2)\theta=0\,, \tag{29}$$

where $\omega_0^2+\delta^2$ equals k/I. The general solution of Eq. (29) is

$$\theta(t)=\theta_0 e^{-t\delta}\sin(\omega_0 t+\beta)\,, \tag{30}$$

where θ_0 and β are undetermined constants.

Assume that a persistent current is present in the sphere with an angular momentum $\mathbf{L}_p$ which lies initially in the vertical direction. Assume further that the torsion suspension is initially at rest. An approximate estimate of the amplitude of oscillation following a tilt can now be made by assuming that the sphere is tilted instantaneously. Suppose that the sphere and along with it $\mathbf{L}_p$ are tilted through an angle α. Then the decrease in the vertical component of $\mathbf{L}_p$ is $L_p(1-\cos\alpha)$. In accord with conservation of angular momentum, this angular momentum will be transformed into that of rotation of the torsion suspension according to the relation

$$I\omega_0\theta_0=L_p(1-\cos\alpha)\,. \tag{31}$$

Under the assumption that $\delta\ll\omega_0$, the maximum deflection following tilt will be given to a good approximation by the expression

$$\theta_{\max}=L_p(1-\cos\alpha)(\omega_0/k)e^{-P\delta/4}\,, \tag{32}$$

where $P=2\pi/\omega_0$ is the period of the suspension. In these measurements δ was typically of the order of 0.001 sec^{-1} in relation to an ω_0 of about 0.22 sec^{-1}.

Equation (32) might be expected to be a reasonable approximation as long as the tilt time is short compared with the period of the pendulum. In practice, the tilt time was as long as a quarter of a period, so that a more accurate calculation of $\theta_{\max}$ is desirable. In order to take the effects of a finite tilt time into account, a driving term $N(t)$ must be added to Eq. (28). This driving term will be equal to the negative time rate of change of the vertical component of persistent angular momentum. If the tilting occurs in a time interval $0\leq t\leq\tau$, $N(t)$ will differ from zero only in this interval. Equation (29) will take the form

$$\begin{aligned}\ddot{\theta}+2\delta\dot{\theta}+(\omega_0^2+\delta^2)&=N(t)/I\,, \quad 0\leq t\leq\tau\\ &=0\,, \qquad\qquad \tau<t.\end{aligned} \tag{33}$$

In order to calculate the amplitude of the first maximum deflection following a tilt, it is only necessary to solve Eq. (33). The solution requires a knowledge of the driving term $N(t)$, which in the present experiment was not known accurately. Fortunately, however, detailed calculations based on several physically reasonable forms of $N(t)$ indicate that in the present case where $\delta\ll\omega_0$, $\theta_{\max}$ is not sensitive to the exact form of $N(t)$ as long as the tilt time τ is less than a quarter period.[10]

These calculations show that to a good approximation the right-hand side of Eq. (32) need only be multiplied by a correction factor $F(\tau/P)$ which is unity for $\tau=0$ and which decreases as τ increases. For example, the choice $N(t)\propto\sin(\pi t/2\tau)$ yields the expression

$$F(\tau/P)=\frac{[1-(8\tau/P)\sin(2\pi\tau/P)+(4\tau/P)^2]^{1/2}}{1-(4\tau/P)^2}, \quad (34)$$

which for $\tau/P=\frac{1}{4}$ equals 0.93. Other reasonable choices for $N(t)$ lead to expressions for $F(\tau/P)$ which differ from the one above by less than 1% for $\tau/P<\frac{1}{4}$. With the correction factor included, Eq. (32) may be rearranged to give

$$L_p=\frac{kPe^{P\delta/4}}{2\pi(1-\cos\alpha)F(\tau/P)}\theta_{\max}. \quad (35)$$

All of the quantities appearing on the right-hand side of Eq. (35) are observable quantities. Measurement of these quantities thus enables a determination of the persistent angular momentum L_p. Note that the magnitude of L_p is proportional to the observed maximum deflection. For a series of identical tilts at a fixed temperature, Eq. (35) states that the maximum deflections following tilt depend on L_p alone. If only relative changes in L_p are important, it is thus sufficient to measure $\theta_{\max}$ alone. For a series of identical tilts at different temperatures, Eq. (35) predicts to a good approximation the proportionality

$$L_p\propto(P/P_\lambda)e^{P\delta/4}\theta_{\max}, \quad (36)$$

where P_λ is the period at T_λ. Equation (36) will be sufficient for studying the temperature dependence of L_p. The complete form of Eq. (35) need only be used when the absolute magnitude of L_p is required. It should also be added that the detailed calculations predict that the maximum deflection following a return to equilibrium will be almost exactly equal in magnitude and opposite in direction to that following a tilt.

D. Stability and Temperature Dependence of Persistent Currents

Experimental studies of the stability and temperature dependence of persistent currents were carried out using three different powders. The first of these was the phenolic resin powder with 20-μ particles used in the torsional-oscillation experiments, the second the alumina powder with 1-μ particles also used in the torsional-oscillation experiments, and the third a silica powder.[27] According to the manufacturer the fundamental particles of this last powder are ≈ 10 mμ in size and are aggregated into clusters $\approx 1\ \mu$ in size. Because it is generally found to be true that critical flow velocities for the superfluid tend to increase as channel size decreases,[28,29] it was expected that as the particle size decreased the stability of persistent currents would be enhanced and that persistent currents of higher angular momentum could be prepared. This expectation was indeed fulfilled in the work described below.

A typical experimental run began as follows. After the helium Dewar and suspension enclosure were filled with liquid helium, the system was cooled to a temperature slightly below T_λ to ensure complete filling of the sphere. During the cooldown the suspension was lifted at a point below the torsion fiber in order to prevent torsional oscillation. Once the temperature was stabilized below T_λ the suspension was lowered so that it was free to oscillate. Several tilts and returns were made to ensure that the apparatus was operating properly and to record the tilt angle α. Adjustments of the magnetic field coils were then made as described in Sec. IV B above. After complete adjustment of the coils had been made, several tilts and returns were made to observe any residual deflections which might be present. Then the suspension was lifted, the system warmed up through T_λ, and the entire suspension and enclosure rotated for about 4 min at 2.20°K. Still in steady rotation, the system was cooled back down through T_λ, and at some temperature below T_λ the rotation was slowly and smoothly brought to a halt.

A typical angular momentum measurement sequence began with the suspension at rest. The sphere was carefully tilted by increasing the current in the Helmholtz coils. The maximum deflection of the suspension following tilt was recorded and the suspension then brought to rest with the sphere in its tilted position. Next, the current in the Helmholtz coils was carefully decreased back to zero, allowing the sphere to return to its original equilibrium position. The maximum deflection following this return was recorded, and finally the suspension was again brought to rest. Such a measurement sequence was repeated many times in a typical run. At the end of the run a check was made of the tilt angle α and the adjustments of the magnetic field coils. No significant drifts were ever observed.

The principal results of these experiments are (1) that long-lived persistent currents can indeed be prepared and that repeated measurements of the persistent angular momentum can be made without dissipating that angular momentum, and (2) that the angular momentum of a persistent current is a reversible function of temperature in direct proportion to $\rho_s(T)$, as suggested by the analysis of Sec. II. Samples of the data which led to these results are shown in Figs. 6–10.

Figure 6 shows a sequence of measurements which were made at 1.24°K after a circulating current had been prepared in a sphere filled with the resin powder by rotation at $\omega=0.27$ rad sec^{-1}. The maximum deflections

[27] Quso F22, Philadelphia Quartz Co., Philadelphia, Pa.

[28] K. R. Atkins, *Liquid Helium* (Cambridge University Press, London, 1959), pp. 198–201.

[29] W. M. van Alphen, G. J. van Haasteren, R. de Bruyn Ouboter, and K. W. Taconis, Phys. Letters **20**, 474 (1966).

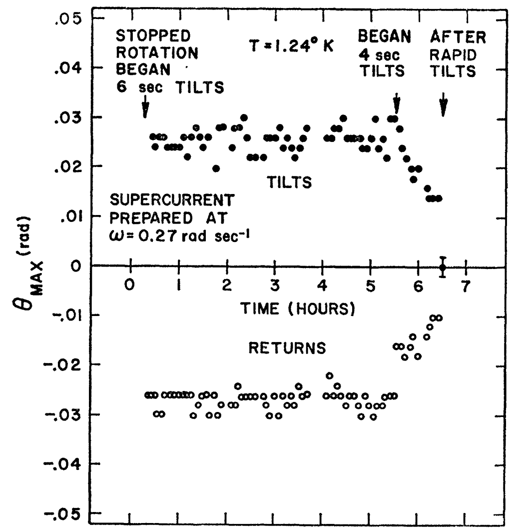

FIG. 6. Maximum deflections $\theta_{\max}$ following tilts and returns as functions of time for a persistent current in the resin powder at constant temperature. For identical tilts at constant temperature, $\theta_{\max}$ is directly proportional to the persistent angular momentum.

following tilts and returns are seen to be equal and opposite as expected. During the first 5 h the tilt time τ was 6 sec, and no loss of angular momentum was observed. At the end of this period the tilt time was decreased to 4 sec, and partial destruction of the persistent current was observed. Finally a series of rapid tilts ($\tau<1$ sec) brought about total destruction of the current. In another run with the same sphere, 5-sec tilts were made every 5 min for a period of several hours. A slow decay of angular momentum was observed over the entire period, with the total loss after 5 h equal to about 15% of the original angular momentum. Thus it appears that for slow enough tilts dissipation of the circulating flow does not occur. Above some critical tilt rate, however, dissipation is present and increases with increasing tilt rate.

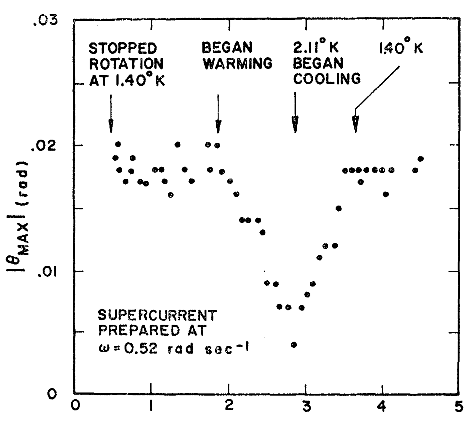

FIG. 7. $|\theta_{\max}|$ as a function of time for a persistent current in the resin powder, showing the reversible variation of persistent angular momentum with temperature. Each point represents the average of the two readings taken during a measurement cycle.

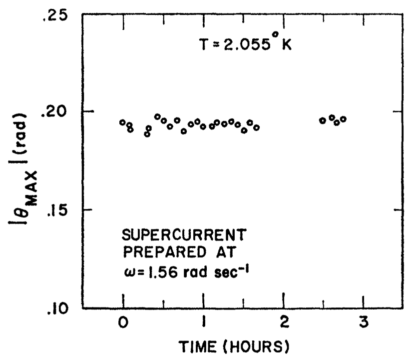

FIG. 8. $|\theta_{\max}|$ as a function of time for a persistent current in the alumina powder at constant temperature, showing the stability of the persistent angular momentum.

The reversible temperature variation of L_p may be seen in Fig. 7. Each of the maximum deflections $|\theta_{\max}|$ plotted in this figure and in Figs. 8–10 represents the average of the absolute values of the two readings constituting a measurement sequence, the one after tilt and the other after return. Here a persistent current was prepared at $\omega=0.52$ rad sec^{-1} in another sphere filled with the 20-μ resin powder. After a number of measurements at 1.40°K the helium bath was warmed to 2.11°K and then cooled back to 1.40°K. The angular momentum is seen to have decreased with increasing temperature but to have regained its original value after the sphere was recooled to its original temperature.

Persistent currents with enhanced stability and larger angular momenta were prepared using spheres packed with the 1-μ alumina powder, in accord with expectation. When using this powder no loss of angular momentum was ever observed with normal tilt times of 3 to 6 sec. It was possible, however, to destroy currents partially by a series of very rapid tilts ($\tau<\frac{1}{4}$ sec).

Measurements at constant temperature which followed the preparation of a current at $\omega=1.56$ rad sec^{-1} in a sphere filled with the alumina powder are shown in Fig. 8. This figure shows that no visible loss of angular

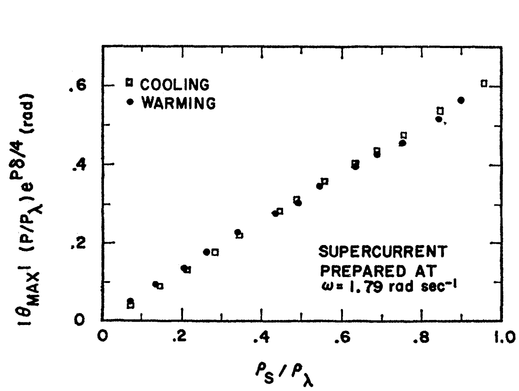

FIG. 9. The relative persistent angular momentum $|\theta_{\max}|\,(P/P_\lambda)\times e^{P\delta/4}$ as a function of ρ_s/ρ_λ for a persistent current in the alumina powder, showing the linear relationship between persistent angular momentum and superfluid density.

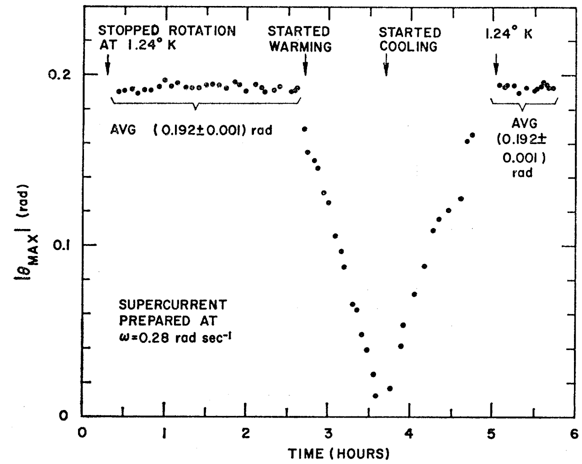

FIG. 10. $|\theta_{\max}|$ as a function of time for a persistent current in the silica powder, showing the reversible variation of persistent angular momentum with temperature.

momentum occurred over a 3-h period. The main source of scatter in the data here is thought to be slight variations in the tilt angle α from measurement to measurement. A variation in α of 1°, which was smaller than could be observed visually, could account for the observed scatter. This amount of scatter is quite typical of the amount present in the most accurate of our measurements.

Using the data of Fig. 8 it is possible to put a lower limit of about 10^6 sec on the time constant of any decay of angular momentum which may have been present. This lower limit is roughly 10^{11} times greater than the time constant of about 10^{-5} sec estimated earlier for the decay of normal-fluid flow in the same channels. Thus persistent flow experiments provide an extremely sensitive test of the absence of dissipation of a viscous type in superfluid flow.

More accurate data concerning the reversible temperature dependence of L_p were also obtained using the alumina powder. In Fig. 9 $|\theta_{\max}|(P/P_\lambda)e^{P\delta/4}$, which is directly proportional to L_p, is plotted against ρ_s/ρ_λ.[24] The measurements of $|\theta_{\max}|$ in this case were made at a number of fixed temperatures during a single cooling and warming cycle following the preparation of the persistent current at $\omega = 1.79$ rad sec^{-1}. The linear relation between L_p and ρ_s suggested in Sec. II is well borne out.

It may be remarked that in some recent work Clow and Reppy have assumed that L_p is directly proportional to ρ_s and have used this assumption to infer the behavior of $\rho_s(T)$ very near T_λ from measurements of $L_p(T)$ of a persistent current.[30] Their assumption has in effect been justified by the close agreement between the form of $\rho_s(T)$ that they obtain and that obtained very near T_λ by Tyson and Douglass using the Andronikashvili pile-of-disks method.[31]

Persistent currents were carried through temperature cycles in a number of runs using spheres packed both with the resin and the alumina powders. Complete loss of angular momentum was always observed whenever the sphere was warmed above T_λ. In many cases some loss was observed when the sphere was warmed to a maximum temperature below but near T_λ. However, it proved possible with the alumina powder to warm to within 2 m°K of T_λ without observing any measurable loss if the warming and cooling rates were carefully limited. It seems likely that the losses which did occur at temperatures below T_λ were due to supercritical flow effects associated with high superfluid–normal-fluid counterflow velocities accompanying thermal gradients during temperature change. Numerical estimates support this supposition, as does an experiment in which partial loss of angular momentum was observed after a flashlight beam had been shone on a sphere containing the resin powder.

In all cases of persistent currents prepared using the resin and alumina powders, the direction of deflection upon tilting agreed with that expected from the sense of rotation used in preparing the current, and the deflections upon return were always equal and opposite to those upon tilt, to within experimental uncertainty.

The work with spheres filled with the fine silica powder while in many ways confirming expectation gave some surprising results. Persistent currents often appeared spontaneously upon cooling through T_λ without rotation. Observations of the response of the torsion suspension during tilts indicated that the initial direction of the angular momentum of the spontaneous currents was not in general vertical. Altogether, 14 persistent currents were observed in four runs using this powder. Eleven of these currents appeared spontaneously, while 3 were prepared intentionally by cooling in rotation.

The persistent currents so formed, whether spontaneous or intentionally prepared, showed a stability somewhat greater than those using the alumina powder but in other regards behaved just like the other persistent currents described above. In particular, L_p varied reversibly with temperature below T_λ in direct proportion to ρ_s. The stability and reversible temperature variation of such a current prepared by cooling in rotation at $\omega = 0.28$ rad sec^{-1} is shown in Fig. 10. The maximum temperature reached during the warming and cooling cycle shown here was within 1 mdeg K of T_λ. It is, of course, quite possible that a spontaneous component was present in this current as well as the intended part.

The origin of these spontaneous currents is not understood. It is possible that they are generated by high-velocity counter currents of superfluid and normal fluid that occur as the sphere is cooled through T_λ.

[30] J. R. Clow and J. D. Reppy, Phys. Rev. Letters **16**, 887 (1966).

[31] J. A. Tyson and D. H. Douglass, Jr., Phys. Rev. Letters **17**, 472 (1966).

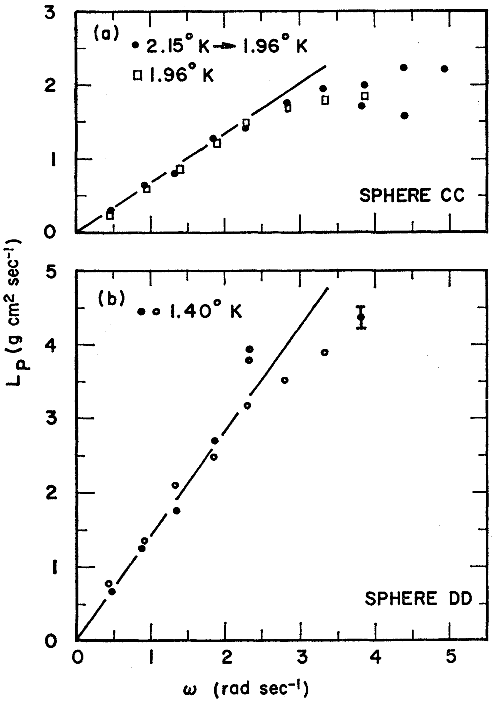

FIG. 11. The magnitude of the persistent angular momentum L_p as a function of angular velocity of preparation ω for currents in the alumina powder prepared by cooling through T_λ in rotation. The broken lines are merely straight lines fitted to the data. (a) Measurements taken at 1.96°K. Solid circles: rotation stopped at 2.15°K; system cooled to 1.96°K without rotation before measurements were taken. Open squares: rotation stopped at 1.96°K. (b) Measurements taken at 1.40°K after rotation was stopped at 1.40°K. Solid circles and open circles indicate different runs.

E. Magnitude of Persistent Angular Momentum as a Function of Angular Velocity of Preparation

Measurements of the persistent angular momentum L_p as a function of the angular velocity ω used in preparation of the persistent current were made using spheres filled with the alumina powder. The principal part of these measurements involved preparing currents by cooling through T_λ in steady rotation. The results of this part are shown as points in Fig. 11, where each point represents the result of an individual cooldown followed by several measurements of the resulting persistent angular momentum. The angular momentum L_p was determined from measured values of $|\theta_{\max}|$, P, τ, k, α, and δ using Eq. (35). It is seen in both parts of Fig. 11 that for less than a critical value of about 2.5 rad sec^{-1} the dependence of L_p on ω is linear. Above this critical value L_p tends to become saturated. It was noticed that L_p in the saturated region seemed to be less reproducible than in the linear region.

In the analysis of Sec. II it was suggested that the helium II formed by cooling through T_λ in steady rotation would appear in an equilibrium state with total angular momentum equal to the solid-body angular momentum that the liquid possessed above T_λ. In this case the resulting persistent angular momentum would be given by Eq. (23),

$$L_p = (\rho_s/\rho_\lambda)(1-\chi)I_\lambda\omega. \qquad (37)$$

TABLE II. Comparison of L_p/ω from persistent-current measurements (data of Fig. 11) with $(\rho_s/\rho_\lambda)(1-\chi)I_\lambda$ determined from torsional-oscillation measurements (data of Fig. 4).

Sphere	Measurement temperature (°K)	L_p/ω (g cm²)	$(\rho_s/\rho_\lambda)(1-\chi)I_\lambda$ (g cm²)
CC	1.96	0.67±0.03	0.69±0.01
DD	1.40	1.40±0.07	1.35±0.02

Since as a result of torsional-oscillation experiments performed on the same spheres used in the present measurements all the quantities occurring in the formula above were known, it was possible to make a direct check of the above suggestion. In Table II we compare the values of the ratio L_p/ω pertaining to the linear, unsaturated portions of $L_p(\omega)$ shown in the two parts of Fig. 11 with the values of $(\rho_s/\rho_\lambda)(1-\chi)I_\lambda$ determined by torsional-oscillation measurements for the same spheres and temperatures of measurement, respectively. The agreement between the corresponding values is seen to be quite good.

Presumably, the saturation effects that appear in Fig. 11 above an angular velocity of about 2.5 rad sec^{-1} represent critical velocity effects. It seems likely that the total angular momentum of the liquid just before rotation is stopped is the solid-body value and that the part associated with the circulation constants is just the L_p given by Eq. (37) above. This value of L_p would result from a velocity field representing minimal relative velocities between the liquid components and the sphere and powder. However, as the sphere is stopped the relative velocity between superfluid and powder presumably becomes large enough in at least the outer regions of the sphere to allow the circulation constants to change until the superfluid velocity is sufficiently reduced, thus allowing a reduction of L_p to a more or less stable although not highly reproducible value.

The secondary part of these measurements consisted of complementary experiments to study critical velocity effects. In these experiments attempts were made to prepare persistent currents by rotating the sphere at a fixed temperature below T_λ, having cooled to that temperature without rotation. These experiments were carried out with sphere *DD*, one of the same spheres filled with alumina powder used in the $L_p(\omega)$ measurements above. The results of these attempts are shown in Fig. 12. Each point represents the result of cooling through T_λ without rotation, rotating for a total time of 30 min at a particular ω, and measuring the amount of any persistent angular momentum so formed after the rotation was finally brought to a stop. In six cases the total rotation time of 30 min was broken into three intervals of 10 min each, between which the rotation

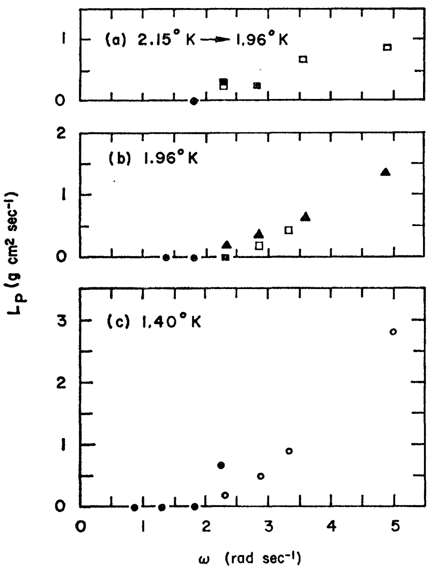

FIG. 12. L_p as a function of ω for persistent currents in the alumina powder (sphere *DD*) prepared by rotation below T_λ. (a) Rotation at 2.15°K; system cooled to 1.96°K for measurements. (b) Rotation and measurements at 1.96°K. (c) Rotation and measurements at 1.40°K. Different symbols indicate different runs.

was stopped and interim angular momentum measurements made before rotation was resumed. In these cases it was observed that no further increase in angular momentum took place after a total of 20 min of rotation. Between points the system was warmed up above T_λ to destroy any persistent current which may have resulted. The currents formed at 2.15°K were cooled to 1.96°K for measurement in order to enlarge the value of L_p, under the assumption that once formed the circulation constants of the persistent current would remain stable under temperature change.

It can be seen in Fig. 12 that for angular velocities below a critical value of about 2.0 rad sec^{-1} no persistent currents were formed, in agreement with the analysis of Sec. II. For ω's larger than 2.0 rad sec^{-1} persistent currents were generated, the persistent angular momentum so produced increasing with ω. Presumably, for these ω's the velocity of the powder relative to the fluid upon rotation becomes large enough to allow the circulation constants to take on nonzero values, which they retain at least to some degree when the rotation is stopped again.

It is interesting that the critical velocities determined by the two different methods described in this section are very similar in magnitude. There does not seem, however, to be any *a priori* reason why they should be identical. Even if it is assumed that the critical condition is governed by the attainment of a certain relative velocity between superfluid and powder, a condition which will be reached first near the equator of the sphere, it should be observed that the velocity field of the superfluid relative to the powder will not be identical on a microscopic scale in the two cases for a given subcritical value of ω. It is worthwhile noting that a critical ω of 2.0 or 2.5 rad sec^{-1} corresponds to a linear velocity of roughly 3 cm sec^{-1} at the sphere's equator. This magnitude of relative velocity between superfluid and wall agrees well with critical velocities observed in other superfluid-flow experiments with flow channels of approximately the same size.[28,29]

V. CONCLUDING REMARKS

In this paper we have presented the results of several related experiments concerning the flow of helium II in a porous medium. For flow at low enough velocities the results have been shown to be in very good agreement with an analysis based on conventional two-fluid equations for helium II in which the superfluid exhibits potential flow without dissipation.

An important assumption of the analysis whose consequences have been well verified by the experiments at low angular velocities is that the circulation constants are stable not only when the sphere itself is at rest at a constant temperature but also as the sphere itself is rotated or as the temperature is changed, even though in the latter case constant circulation implies varying superfluid angular momentum. It seems likely that this stability of circulation is closely related to the quantization of circulation of the superfluid. The first consequence of quantization of circulation is that continuous changes of circulation are ruled out. Thus, changes in circulation in which the superfluid flow remains curl-free, a transition which involves all of the fluid, must take place by finite steps and are therefore probably quite unlikely. An alternative method for quantum changes of circulation exists, however, in the possibility of the creation at one boundary of a quantum vortex which migrates through the liquid and is annihilated at some other boundary, transferring a quantum of circulation from one circuit to another. However, such a process involves an energy barrier provided by the energy needed to create a vortex. It may well be that when the local values of v_s relative to the powder are small enough, this energy barrier effectively inhibits changes of circulation, but that as v_s increases, vortex creation becomes energetically favorable, allowing circulations to change and giving rise to the critical effects seen in the experiments.

One of the interesting aspects of the results concerns the reversible changes of persistent angular momentum which take place as the sphere is warmed and cooled. Conservation of angular momentum leads to the conclusion that a torque about the vertical axis must act on the liquid as it is cooled or warmed if a persistent current is present with a vertical component of angular momentum. Thus, more or less static deflections of the torsion suspension should be present during steady changes of temperature. Such deflections have been observed, and although the smallness of the effect made it possible only to observe these deflections with an uncertainty of about 30%, the deflections were of the correct direction and appeared to be of the correct magnitude expected from the magnitude of the persistent angular momentum present and the rate of warming or cooling.

It is worthwhile considering the reversible temperature dependence of the persistent angular momentum from the viewpoint of the elementary-excitation model of Landau.[11] Consider a persistent current in a sphere at rest at absolute zero where the entire fluid is superfluid and has angular momentum **L**. Following an argument presented by Feynman,[14] as the temperature is increased from absolute zero thermal excitations appear which, in order to have zero average group velocity and remain in equilibrium with the stationary walls, have an average angular momentum which opposes and cancels part of the **L** possessed by the fluid background. Thus, a net torque must be exerted on the liquid to create these excitations which lowers the total angular momentum of the liquid. When the temperature is lowered again, these excitations disappear, giving up their angular momentum to the walls and thus removing the angular momentum which tended to cancel **L**. Hence, as the temperature returns to absolute zero, the total fluid angular momentum returns to the original **L**. The temperature variation of the persistent angular momentum can thus be interpreted as being due to the creation and destruction of excitations superimposed on a fluid background which remains in motion.

ACKNOWLEDGMENTS

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Brownian Motion of a Heavy Impurity in Harmonic Lattices

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Starting from the well-known expression for the frequency spectrum of a mass impurity in a harmonic lattice, the velocity autocorrelation function for a heavy impurity has been calculated without referring to any particular model. Using then general arguments, the postulates of Langevin's equation have been briefly discussed.

1. INTRODUCTION

THE motion of a heavy impurity in harmonic lattices has been studied by Rubin[1,2] and others[3] with a view to justifying the postulation of a phenomenological equation of the Langevin type. In particular, Mazur[3] has discussed general conditions under which a heavy mass in a harmonic lattice will perform Brownian motion. The purpose of this paper is to show how, starting from the well-known result for the spectral function of an impurity in a harmonic lattice, one can derive in a simple and straightforward manner the velocity autocorrelation function for the impurity. We have derived exact expressions for that function in the heavy-mass limit for one-, two-, and three-dimensional harmonic lattices. In the one-dimensional case, using the fluctuation-dissipation theorem of statistical physics, the postulates of the Langevin equation have been rigorously justified.

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[1] R. J. Rubin, J. Math. Phys. **1**, 309 (1960).

[2] R. J. Rubin, J. Math. Phys. **2**, 373 (1961).

2. MATHEMATICAL FORMULATION

The real part of the velocity autocorrelation is expressed in terms of the frequency distribution function $f(\omega)$ as follows[4]:

$$\mathrm{Re}\langle \mathbf{v}(t)\cdot\mathbf{v}(0)\rangle = \frac{\nu h}{2M'}\int_0^\infty \omega \coth(h\omega/2k_BT)f(\omega)\cos(\omega t)d\omega\,, \quad (1)$$

where ν stands for the dimensionality of the lattice, and the other symbols have their usual meaning.

For a harmonic solid with cubic symmetry, $f(\omega)$ for an isolated localized impurity has been given by Waller[5]

[3] P. Mazur, in *Statistical Mechanics of Equilibrium and Non-Equilibrium*, edited by J. Meixner (North-Holland Publishing Co., Amsterdam, 1956), p. 63; P. Mazur and E. Braun, Physica **2**, 1973 (1964); H. Nakazawa, Progr. Theoret. Phys. (Kyoto) Suppl. **36**, 176 (1966).

[4] See, for instance, A. Rahman, K. S. Singwi, and A. Sjölander, Phys. Rev. **126**, 986 (1962).

[5] I. Waller, Arkiv Fysik **24**, 495 (1963).