Quark-Diquark Model of Baryons and $SU(6)$*

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A model in which a baryon can be regarded as a bound state of two particles has been generalized to be approximately invariant under $SU(6)$. In the model, one of the constituent particles of a baryon can be regarded as a quark and the other particle can be considered as a tightly bound state of two quarks, or diquark. The quark is taken to belong to a six-dimensional representation of $SU(6)$, while the diquark is taken to belong to a twenty-one-dimensional representation. With this model, which can be considered as a specific dynamical approximation to the three-quark model, the baryon medium-strong mass splittings are calculated beyond lowest-order perturbation theory. The model provides a mechanism for breaking the Gell-Mann–Okubo baryon-octet mass formula while breaking the baryon-decuplet equal-spacing rule by a smaller amount.

1. INTRODUCTION

THE nonrelativistic quark model[1,2] has been used by many authors[3] to calculate the baryon mass spectrum. In the model, a baryon is a state of three quarks bound deeply by their mutual attractive interactions. The model is simple conceptually, but has the calculational difficulties associated with the three-body problem. One approximate way of treating this problem has been to replace the sum of two-body interactions by an effective attractive potential well in which all three quarks are bound.

In a previous paper,[4] which we shall refer to as LT, two of us have introduced a model which makes use of an entirely different approximation. In the model of LT, a baryon is assumed to be a bound state of two particles, one with spin one-half and the other with spin one. The spin-one-half particle can be regarded as a quark and the spin-one particle as a bound state of two quarks, or diquark. This model may have some validity if a bound state of two quarks is so tightly bound that it has a smaller mass than that of a single quark, although of course, not so low that it would have been observed.

It is interesting to point out that in this model the lowest-mass diquark has charge $\frac{4}{3}$ of the proton charge, and so may not have been detected by the usual methods to look for fractionally charged particles with charge smaller than the proton charge. The possibility that the lowest-mass quarklike object may have a charge greater than the proton charge was pointed out by Gell-Mann[5] in his original paper on quarks and later by de Swart.[6]

Once the assumption is made that a baryon is a bound state of two particles, one with integral spin and one with half-integral spin, the additional assumption that the particle with integral spin is a bound state of two quarks can be relaxed. Then the model can be considered as a two-field model, of a different kind from those proposed by Gürsey, Lee, and Nauenberg.[7] In LT, this two-particle model of baryons was called a "boson-fermion" model. However, since the boson is most simply regarded as a diquark, and since the quarks may obey parastatistics rather than Fermi statistics,[8] we adopt here the name "quark-diquark" model. The model is not as simple conceptually as one in which a baryon is composed of three quarks, but has the calculational advantage that a baryon is made up of two particles rather than three. This enables us to make calculations to a high degree of precision more easily than can be made with a quark model.

The electromagnetic mass splittings and the magnetic moments of baryons have also been considered in the model.[9] It was found that in order to obtain agreement with the observed baryon magnetic moments the two constituent particles must be given fractional charge. This consequence of the model makes it even more attractive to regard the spin-one particle as a diquark.

One feature of the quark model, which our model did not have in its previous version, is that in the quark

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model the baryons can be considered as belonging to a 56-dimensional multiplet of $SU(6)$. However, in the quark-diquark model as originally introduced, $SU(6)$ is in some sense maximally violated. This is because if we assume $SU(6)$ invariance and wish to place baryons in a 56 multiplet, we must form the baryons from a 21-dimensional diquark multiplet and from a six-dimensional quark multiplet. Now the $SU(3)$ content of the 21-dimensional diquark is an $SU(3)$ sextet of spin one and a triplet of spin zero belonging to the $\bar{3}$ representation of $SU(3)$. The baryon decuplet turns out to be made entirely of the spin-one sextet and the quark. However, the baryon octet is composed of an equal mixture of the spin-one sextet and the spin-zero triplet. Since the spin-zero triplet was missing in the model of LT, a baryon could not even come close to belonging to a 56-dimensional representation of $SU(6)$.

In the present paper, we remedy this defect of the model by generalizing to include approximate $SU(6)$ invariance by forming a baryon from a diquark and quark, which in the lowest-order approximation belong to $SU(6)$ multiplets. We then break the symmetry to obtain the medium-strong baryon mass splittings.

We assume that the diquark is a multiplet of 21 states formed as the symmetric combination of two quarks. The $SU(6)$ multiplicities contained in a two-quark state are given as follows:

$$6\otimes 6=21\oplus 15.$$

The 15-dimensional diquark we assume to lie much higher in energy than the 21, if indeed it is bound at all. Then a 21 diquark can combine with a third quark as follows:

$$21\otimes 6=56\oplus 70.$$

We assume that the members of the baryon octet and decuplet belong to the 56-dimensional representation as usual and that the particles forming the 70 lie higher in energy. The $SU(3)$ content of the 21-dimensional diquark is as follows:

$$21\supset {}^3 6+{}^1\bar{3},$$

where our notation is that the spin multiplicity is written as a left superscript on the $SU(3)$ multiplicity. Thus, the diquark is a mixture of a spin-one particle of $SU(3)$ multiplicity six and a spin-zero particle of $SU(3)$ multiplicity three. We shall call these two diquarks a sextet and a triplet, respectively. If the forces are invariant under $SU(6)$, the sextet and triplet are degenerate in mass.

One simple way to break the $SU(6)$ symmetry is to assume that the sextet diquark and the triplet diquark have different masses. Another way is to assume that the interaction between a quark and a sextet is different from the interaction between a quark and a triplet. We shall consider both of these possibilities.

We shall also break $SU(3)$ invariance by letting the quark with hypercharge $Y=-\frac{2}{3}$ have a somewhat heavier mass than the two quarks with $Y=\frac{1}{3}$. As a consequence of this, the sextet will break up into three distinct masses; there will be an isospin triplet with one mass, a doublet with a somewhat higher mass, and a singlet with a still higher mass. Similarly, the triplet quark will break into a singlet and doublet in isospin.

If the interaction between the two quarks to form a diquark is invariant under $SU(3)$, the splitting of the masses within the sextet and within the triplet will be closely related to the mass splitting within the quarks themselves. In fact, if we use a nonrelativistic model to describe the interaction between the two quarks, the mass splitting within the diquark will be just the same as the mass splitting within the quark. However, we shall relax this assumption and allow the mass splittings within the diquarks to be free parameters. This assumption can follow either from a relativistic model of the diquark or from an interaction between the two quarks which depends on the hypercharge and breaks $SU(3)$.

With these assumptions we list in Table I the masses and other quantum numbers of the quark and of the triplet and sextet diquarks. In Table I, the difference between m_s and m_t is a measure of the violation of $SU(6)$ symmetry in the medium-strong interaction and the parameters δ_q, δ_t, δ_s are a measure of the violation of $SU(3)$ symmetry. For completeness we have also included in Table I the parameters $\epsilon_i(i=1, 2, \cdots, 4)$. In the expressions for the masses of the quark, sextet, and triplet, these parameters ϵ_i are assumed to arise from electromagnetic effects and are assumed to be small compared to the δ's. In writing down only four different ϵ_i, we are assuming that the electromagnetic splittings are U-spin invariant. In this paper, however, we are interested in the medium-strong mass splittings rather than the electromagnetic effects and shall put the ϵ_i equal to zero.

In Sec. 2 we shall give the lowest-order perturbation theory results for the baryon mass splittings using this model. However, the virtue of the model is that it enables us to make calculations going beyond lowest-

TABLE I. Quantum numbers of quark, sextet and triplet. The quark has baryon number $B=\frac{1}{3}$ and spin $S=$one-half, the sextet $B=\frac{2}{3}$, $S=1$, and the triplet $B=\frac{2}{3}$, $S=0$.

	Symbol	Mass	Isospin	Hypercharge	Charge
Quark	q_1	m_q	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$
	q_2	$m_q+\epsilon_1$	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$
	q_3	$m_q+\delta_q+\epsilon_1$	0	$-\frac{2}{3}$	$-\frac{1}{3}$
Sextet	s_1	m_s	1	$\frac{2}{3}$	$\frac{4}{3}$
	s_2	$m_s+\epsilon_2$	1	$\frac{2}{3}$	$\frac{1}{3}$
	s_3	$m_s+\epsilon_3$	1	$\frac{2}{3}$	$-\frac{2}{3}$
	s_4	$m_s+\delta_s+\epsilon_2$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$
	s_5	$m_s+\delta_s+\epsilon_3$	$\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{2}{3}$
	s_6	$m_s+2\delta_s+\epsilon_3$	0	$-\frac{4}{3}$	$-\frac{2}{3}$
Triplet	t_1	m_t	0	$\frac{2}{3}$	$\frac{1}{3}$
	t_2	$m_t+\delta_t$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$
	t_3	$m_t+\delta_t+\epsilon_4$	$\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{2}{3}$

order perturbation theory. In Sec. 3, we shall apply a formalism which enables us in principle to obtain the baryon masses as eigenvalues of simultaneous differential equations. We shall get approximate solutions to these equations assuming that the potential is very deep and that the wave functions vanish at the boundary. This is the approximation previously made in LT, and enables us to solve the equations in closed form as algebraic equations. Unfortunately in several cases the structure of these closed-form solutions is so complicated that examining them does not lead to any useful insight. However, by treating the $SU(6)$ and $SU(3)$ symmetry breaking to second order, we obtain results which go beyond the usual lowest-order perturbation theory and which illustrate the calculational advantage of using a two-particle model for the baryons.

2. PERTURBATION-THEORY RESULTS

We assume that the interaction between the quark and a diquark is a square-well potential with the following characteristics: It has a major term which is $SU(6)$ invariant (when acting on states with zero orbital angular momentum). As only the 56- and 70-dimensional representations occur in our model, this major term can be written as

$$V_{56}P_{56}+V_{70}P_{70},$$

where P_{56} and P_{70} are projection operators for the baryon states belonging to the 56- and 70-dimensional representations, respectively. The interaction contains small symmetry-breaking terms of two kinds, those which are $SU(3)$-invariant terms and those which break $SU(3)$ invariance. For the $SU(3)$-invariant terms we include a potential v_s which acts only between a sextet and a quark combined to form a member of the baryon octet, and a potential v_t which acts between a triplet and a quark. These two potentials split the symmetry between the decuplet and the octet of the 56. For states of the 56, we also introduce two $SU(3)$-breaking interactions proportional to the hypercharge of the system: Yv_{10} for the decuplet and Yv_8 for the octet. The hypercharge Y is introduced to indicate explicitly that these potentials are proportional to the hypercharge. The subscripts 10 and 8 explicitly show that their magnitudes can be different in the decuplet and octet baryon states. We are not directly interested in the states of the 70-dimensional representation but only in their perturbing effect on the states of the 56-dimensional representation, and so we do not need such details as the $SU(3)$-breaking interaction for states of the 70. With these assumptions, the potential V between the quark and the diquark can be written as follows:

$$V=(V_{56}+Yv_{10})P_{10}+(V_{56}+Yv_8)P_8 +V_{70}P_{70}+v_sP_s+v_tP_t\,, \quad (1)$$

where P_{10} and P_8 are projectors for the decuplet and octet parts of the 56-dimensional representation of $SU(6)$, P_s is a projector for the baryon octet state composed of a sextet and a quark, and P_t is a projector for an octet from a triplet and quark. The operators P_{10}, P_8, P_s and P_t are also spin projection operators; P_{10} being a projector for spin three-half, and P_8, P_s, and P_t being projectors for spin one-half. The interaction of Eq. (1) is quite different in its spin and unitary spin dependence from the interaction postulated in LT. We shall assume that V_{56}, V_{70}, and $V_{70}-V_{56}$ are much larger in magnitude than the symmetry-breaking terms v_{10}, v_8, v_s, and v_t.

The remaining part of the Hamiltonian includes the rest energy and kinetic energy of the quark and diquark. In LT, we used a Klein-Gordon equation to describe this part of the Hamiltonian. While we can do so here, we believe that little is gained by this, since our approximations in evaluating the model have their greatest validity if the kinetic energy is small. Therefore we shall use the nonrelativistic expression for the kinetic energy, although we can carry through the argument with the relativistic expression, using somewhat different approximations. We can write the kinetic and rest energies in terms of projection operators on states with particles of given mass. Let $P_s(I_sI_q)$ be a projection operator for a state consisting of a member of the sextet with isospin I_s and a quark with isospin I_q, and let $P_t(I_tI_q)$ be a projection operator for a triplet diquark of isospin I_t and quark with isospin I_q. Also let the expression $T(m_1,m_2)$ for the kinetic and rest energy of particles with masses m_1 and m_2 and relative momentum p be defined by

$$T(m_1,m_2)=m_1+m_2+\tfrac{1}{2}p^2/(\mu_{12})\,, \qquad \mu_{12}=m_1m_2/(m_1+m_2)\,. \quad (2)$$

Then the Hamiltonian H_0 without interaction but including rest and kinetic energy is given by

$$\begin{aligned} H_0=\;&T_1P_s(1\,\tfrac{1}{2})+T_2P_s(\tfrac{1}{2}\,\tfrac{1}{2})+T_3P_s(0\,\tfrac{1}{2})+T_4P_s(1\,0)\\ &+T_5P_s(\tfrac{1}{2}\,0)+T_6P_s(0\,0)+T_7P_t(0\,\tfrac{1}{2})\\ &+T_8P_t(\tfrac{1}{2}\,\tfrac{1}{2})+T_9P_t(0\,0)+T_{10}P_t(\tfrac{1}{2}\,0)\,, \end{aligned} \quad (3)$$

where

$$\begin{aligned} &T_1=T(m_s,m_q)\,, && T_6=T(m_s+2\delta_s,\, m_q+\delta_q)\,,\\ &T_2=T(m_s+\delta_s,\, m_q)\,, && T_7=T(m_t,m_q)\,,\\ &T_3=T(m_s+2\delta_s,\, m_q)\,, && T_8=T(m_t+\delta_t,\, m_q)\,,\\ &T_4=T(m_s,\, m_q+\delta_q)\,, && T_9=T(m_t,\, m_q+\delta_q)\,,\\ &T_5=T(m_s+\delta_s,\, m_q+\delta_q)\,, && T_{10}=T(m_t+\delta_t,\, m_q+\delta_q)\,. \end{aligned} \quad (4)$$

The approximation, that the potential is so deep that the wave function vanishes at the boundary of the potential, means that the momentum p is just a constant inversely proportional to the radius of the potential. We can then obtain the perturbation-theory results for the baryon masses by taking the expectation value of this Hamiltonian between $SU(6)$-invariant states. These $SU(6)$-invariant states are given in the Appendix. The decuplet states listed in the Appendix are the same as those in LT, except that here we also include the spin wave functions, while the octet states include equal amplitudes for a sextet-quark state and a triplet-quark state. Note that the phases of the Clebsch-Gordan coefficients in the expressions for the wave functions may be different from the phases used in some other papers. Using the wave functions from the Appendix, we obtain the following expressions for the baryon masses:

$$
\begin{aligned}
N^* &= T_1 + V_{56} + v_{10}\,,\\
Y^* &= \tfrac{1}{3}T_4 + \tfrac{2}{3}T_2 + V_{56}\,,\\
\Xi^* &= \tfrac{2}{3}T_5 + \tfrac{1}{3}T_3 + V_{56} - v_{10}\,,\\
\Omega &= T_6 + V_{56} - 2v_{10}\,,\\
N &= \tfrac{1}{2}T_1 + \tfrac{1}{2}T_7 + V_{56} + v_8 + \bar{v}\,,\\
\Lambda &= \tfrac{1}{2}T_2 + \tfrac{1}{6}T_8 + \tfrac{1}{3}T_9 + V_{56} + \bar{v}\,,\\
\Sigma &= \tfrac{1}{3}T_4 + \tfrac{1}{6}T_2 + \tfrac{1}{2}T_8 + V_{56} + \bar{v}\,,\\
\Xi &= \tfrac{1}{6}T_5 + \tfrac{1}{3}T_3 + \tfrac{1}{2}T_{10} + V_{56} - v_8 + \bar{v}\,,
\end{aligned}
\tag{5}
$$

where

$$\bar{v} = \tfrac{1}{2}(v_s + v_t)\,. \tag{6}$$

If we use Eqs. (2) and (4) in Eqs. (5), and neglect δ_s, δ_t, and δ_q in the terms containing p^2, we obtain for the baryon mass differences the expressions

$$Y^* - N^* = \tfrac{2}{3}\delta_s + \tfrac{1}{3}\delta_q - v_{10}\,, \tag{7}$$

$$\Xi^* - N^* = \tfrac{4}{3}\delta_s + \tfrac{2}{3}\delta_q - 2v_{10}\,, \tag{8}$$

$$\Omega - N^* = 2\delta_s + \delta_q - 3v_{10}\,, \tag{9}$$

$$\Lambda - N = \tfrac{1}{2}\delta_s + \tfrac{1}{6}\delta_t + \tfrac{1}{3}\delta_q - v_8\,, \tag{10}$$

$$\Sigma - \Lambda = \tfrac{1}{3}(\delta_t - \delta_s)\,, \tag{11}$$

$$\Xi - \Sigma = \tfrac{2}{3}\delta_s + \tfrac{1}{3}\delta_q - v_8\,, \tag{12}$$

$$N^* - N = \tfrac{1}{2}(m_s - m_t) + \tfrac{1}{4}p^2(1/\mu_{sq} - 1/\mu_{tq}) + v_{10} - v_8 - \bar{v}\,. \tag{13}$$

From Eqs. (7)–(12) we see that we get the Gell-Mann–Okubo octet mass formula[10]

$$\Lambda - N = \tfrac{1}{2}(\Xi - \Sigma) + \tfrac{1}{2}(\Xi - \Lambda)\,, \tag{14}$$

and the equal spacing rule for the decuplet

$$Y^* - N^* = \Xi^* - Y^* = \Omega - \Xi^*\,, \tag{15}$$

but nothing else. It is interesting, however, to see what happens when the $SU(3)$ symmetry-breaking potential goes to zero. This means we set v_{10} and v_8 equal to zero. Then, in addition to the Gell-Mann–Okubo mass formula and the decuplet equal-spacing rule, we obtain the following result:

$$Y^* - N^* = \Xi - \Sigma\,. \tag{16}$$

[10] This formula is usually written $3\Lambda + \Sigma = 2(N + \Xi)$. We have chosen to write it in terms of mass differences to emphasize that, in order to be conservative, the error should be compared to a mass difference rather than to a baryon mass.

Experimentally, the left-hand side of Eq. (16) is 147 MeV, while the right-hand side is 125 MeV, for a discrepancy of 22 MeV. This is somewhat worse than the discrepancy between the right and left sides of Eq. (14), which is 12 MeV.

Thus, we see that if we use $SU(6)$ wave functions in lowest-order perturbation theory and neglect the $SU(3)$-breaking potential we get a result which is too restrictive. One can conclude from this either that the $SU(3)$ symmetry-breaking interactions v_{10} and v_8 are present or that the $SU(6)$ wave functions are too restrictive. As a test of the latter assumption we can vary the baryon-octet wave functions from the values given by $SU(6)$ and see how the results are changed. According to $SU(6)$ there are equal amplitudes for a baryon to be made up of a sextet and quark and a triplet and quark. If we assume unequal amplitudes, calling them x and y respectively, with $x^2 + y^2 = 1$, we introduce one new parameter. The decuplet wave functions, being composed completely of the sextet and quark, are unchanged. We then get for the octet mass splittings the following expressions:

$$\Lambda - N = x^2\delta_s + \tfrac{1}{3}y^2(\delta_t + 2\delta_q)\,, \tag{17}$$

$$\Sigma - \Lambda = \tfrac{2}{3}[-x^2\delta_s + y^2\delta_t + (x^2 - y^2)\delta_q]\,, \tag{18}$$

$$\Xi - \Sigma = \tfrac{4}{3}x^2\delta_s + (y^2 - \tfrac{1}{3}x^2)\delta_q\,. \tag{19}$$

From Eqs. (17)–(19) we obtain only the Gell-Mann–Okubo mass formula, but not the extra unwanted equation, Eq. (16), which is in worse agreement with experiment.

We obtain the model adopted in LT if we set $x = 1$, $y = 0$, in Eqs. (17)–(19). We then obtain the formula

$$Y^* - N^* = \Xi - \Lambda\,.$$

Experimentally the right-hand side of this equation is 202 MeV, while, as stated previously, the left-hand side is 147 MeV. Thus, we again obtain a contradiction to experiment if the $SU(3)$ symmetry-breaking potential is neglected. On the other hand, if we set $y = 1$, $x = 0$, so that a member of the baryon octet is composed purely of a triplet and a quark, we do not obtain any relation between the octet and decuplet masses.

In the model we have taken δ_q, δ_t, and δ_s to be three different parameters. But if the sextet and triplet are really composed of two quarks, there may be some relation between the parameters δ_q, δ_t, and δ_s. Let us consider this possibility by forming a diquark as a bound state of two quarks in a manner similar to the way we formed a baryon as a bound state of a quark and a diquark. Proceeding as before, we introduce the projection operators $P(I_1I_2)$ for states of two quarks with isospin I_1 and I_2 and projection operators P_s and P_t for two quarks in a sextet state of spin one and a triplet state of spin zero, respectively. We also introduce the $SU(6)$-invariant potential V_{12}, the $SU(6)$ symmetry-breaking potential v_t' acting only in the triplet diquark state, and the $SU(3)$-breaking potentials Yv_{sy} and Yv_{ty}. Then the Hamiltonian H_{12} between two quarks can be written

$$\begin{aligned}H_{12}=&T(m_q,m_q)P(\tfrac{1}{2}\,\tfrac{1}{2})+T(m_q+\delta_q,\,m_q)P(\tfrac{1}{2}\,0)\\&+T(m_q,\,m_q+\delta_q)P(0\,\tfrac{1}{2})+T(m_q+\delta_q,m_q+\delta_q)P(0\,0)\\&+(V_{12}+Yv_{sy})P_s+(V_{12}+v_t'+Yv_{ty})P_t.\end{aligned}\tag{20}$$

If we neglect terms of order $p^2\delta_q$ and calculate the masses of the sextet and triplet in perturbation theory, we obtain

$$m_s=2m_q+\tfrac{1}{2}p^2/\mu_{qq}+V_{12}+\tfrac{2}{3}v_{sy},\quad \delta_s=\delta_q-v_{sy},\tag{21}$$

$$m_t=m_s+\tfrac{2}{3}(v_{ty}-v_{sy})+v_t',\qquad \delta_t=\delta_q-v_{ty}.\tag{22}$$

We see that these expressions for m_s, m_t, δ_s, and δ_t contain too many unknown parameters to be useful. However, if we assume that the $SU(3)$-breaking interaction can be neglected ($v_{sy}=v_{ty}=0$), we obtain $\delta_s=\delta_t=\delta_q$, which is far too restrictive a result. A somewhat less restrictive assumption is that $v_{sy}=v_{ty}\neq 0$. Then we have $\delta_s=\delta_t\neq\delta_q$. We see from Eq. (11) that this assumption is incompatible with using $SU(6)$ wave functions, since we obtain $\Sigma-\Lambda=0$. This is independent of whether we neglect the $SU(3)$-breaking interactions v_8 and v_{10}. Furthermore, even if the parameter x is kept free, if we neglect v_8 and v_{10} we obtain the sum rule,

$$\Lambda-N+\tfrac{1}{2}(\Sigma-\Lambda)=Y^*-N^*,$$

which is in clear disagreement with experiment.

Thus, our perturbation theory result is that if we set $\delta_s=\delta_t$, we must not neglect the $SU(3)$ symmetry-breaking interactions v_8 and v_{10}, and we must use baryon wave functions which differ from those required by $SU(6)$. If, however, we regard δ_s, δ_q, and δ_t as free parameters, we obtain only the Gell-Mann–Okubo formula and decuplet equal-spacing rule, independently of whether we neglect v_8 and v_{10}.

3. A HIGHER-ORDER APPROXIMATION

We can go beyond lowest-order perturbation theory by assuming that the baryon wave functions are not given by the $SU(6)$-invariant quantities shown in the Appendix, but that they have a more general form. We shall write the generalized baryon wave functions in terms of quark-diquark wave functions which are eigenstates of spin and isospin. We let $\eta_{2J}(I_sI_qI)$ be the spin-isospin wave function of a sextet of isospin I_s and a quark of isospin I_q combined to form a state of total isospin I and total spin J. Likewise we let $\eta(I_tI_qI)$ be the wave function of a triplet of isospin I_t combined with a quark of isospin I_q to form a state of total isospin I and spin $J=\frac{1}{2}$. We also let the space wave functions of a baryon B be ψ_{Bi}, where the subscript $i=1$ is omitted for N^* and Ω, the subscript i takes on the values $i=1, 2$ for Y^*, Ξ^*, and N, and $i=1, 2, 3$ for Λ, Σ, and Ξ. Then the baryon wave functions Ψ_B can be written

$$\begin{aligned}&\Psi_{N^*}=\eta_3(1\,\tfrac{1}{2}\,\tfrac{3}{2})\psi_{N^*},\\&\Psi_{Y^*}=\eta_3(1\,0\,1)\psi_{Y^*1}+\eta_3(\tfrac{1}{2}\,\tfrac{1}{2}\,1)\psi_{Y^*2},\\&\Psi_{\Xi^*}=\eta_3(\tfrac{1}{2}\,0\,\tfrac{1}{2})\psi_{\Xi^*1}+\eta_3(0\,\tfrac{1}{2}\,\tfrac{1}{2})\psi_{\Xi^*2},\\&\Psi_\Omega=\eta_3(0\,0\,0)\psi_\Omega,\\&\Psi_N=\eta_1(1\,\tfrac{1}{2}\,\tfrac{1}{2})\psi_{N1}+\eta(0\,\tfrac{1}{2}\,\tfrac{1}{2})\psi_{N2},\\&\Psi_\Lambda=\eta_1(\tfrac{1}{2}\,\tfrac{1}{2}\,0)\psi_{\Lambda1}+\eta(\tfrac{1}{2}\,\tfrac{1}{2}\,0)\psi_{\Lambda2}+\eta(0\,0\,0)\psi_{\Lambda3},\\&\Psi_\Sigma=\eta_1(1\,0\,1)\psi_{\Sigma1}+\eta_1(\tfrac{1}{2}\,\tfrac{1}{2}\,1)\psi_{\Sigma2}+\eta(\tfrac{1}{2}\,\tfrac{1}{2}\,1)\psi_{\Sigma3},\\&\Psi_\Xi=\eta_1(\tfrac{1}{2}\,0\,\tfrac{1}{2})\psi_{\Xi1}+\eta_1(0\,\tfrac{1}{2}\,\tfrac{1}{2})\psi_{\Xi2}+\eta(\tfrac{1}{2}\,0\,\tfrac{1}{2})\psi_{\Xi3}.\end{aligned}\tag{23}$$

The baryon wave functions which follow from $SU(6)$ can also be written in terms of $\eta_{2J}(I_sI_qI)$ and $\eta(I_tI_qI)$. These wave functions are given in the Appendix.

We can now obtain the masses of the baryons as solutions to the eigenvalue problem

$$(H_0+V)\Psi_B=B\Psi_B,\tag{24}$$

where H_0 and V are given by Eqs. (1) and (3). The states $\eta_{2J}(I_sI_qI)$ and $\eta(I_tI_qI)$ are eigenstates of the projection operators in H_0, but are not necessarily eigenstates of the projection operators in V. However, certain linear combinations of η_{2J} and η are eigenstates of the projectors P_{10} and P_8 or P_s and P_t in V. These linear combinations are given in the Appendix. We operate with H_0 on Ψ_B, then express the η_{2J} and η in terms of the eigenstates of the projectors P_{10}, P_8, P_s and P_t, operate with V, and finally reexpress the wave functions in terms of the η_{2J} and η. After taking the scalar products of the resulting expressions with η_{2J} and η, we obtain the following equations: (We suppress the subscripts B on the ψ_{Bi}, since it is clear from the equations which baryon is referred to.)

$$(T_1+V_{56}+v_{10}-N^*)\psi=0,\tag{25}$$

$$\begin{aligned}&[T_4+\tfrac{1}{3}(V_{56}+2V_{70})-Y^*]\psi_1+\tfrac{1}{3}\sqrt{2}V_0\psi_2=0,\\&\tfrac{1}{3}\sqrt{2}V_0\psi_1+[T_2+\tfrac{1}{3}(2V_{56}+V_{70})-Y^*]\psi_2=0,\end{aligned}\tag{26}$$

$$\begin{aligned}&[T_5+\tfrac{1}{3}(2V_{56}+V_{70}-2v_{10})-\Xi^*]\psi_1\\&\qquad+\tfrac{1}{3}\sqrt{2}(V_0-v_{10})\psi_2=0,\\&\tfrac{1}{3}\sqrt{2}(V_0-v_{10})\psi_1\\&\qquad+[T_3+\tfrac{1}{3}(V_{56}+2V_{70}-v_{10})-\Xi^*]\psi_2=0,\end{aligned}\tag{27}$$

$$(T_6+V_{56}-2v_{10}-\Omega)\psi=0,\tag{28}$$

$$\begin{aligned}&[T_1+\tfrac{1}{2}(V_{56}+V_{70}+v_8)+v_s-N]\psi_1\\&\qquad+\tfrac{1}{2}(V_0+v_8)\psi_2=0,\\&\tfrac{1}{2}(V_0+v_8)\psi_1+[T_7+\tfrac{1}{2}(V_{56}+V_{70}+v_8)\\&\qquad+v_t-N]\psi_2=0,\end{aligned}\tag{29}$$

$$\begin{aligned}&[T_2+\tfrac{1}{2}(V_{56}+V_{70})+v_s-\Lambda]\psi_1\\&\qquad+V_0\psi_2/\sqrt{12}+V_0\psi_3/\sqrt{6}=0,\\&V_0\psi_1/(\sqrt{12})+[T_8+\tfrac{1}{6}(V_{56}+5V_{70}+2v_t)\\&\qquad-\Lambda]\psi_2+[V_0/(\sqrt{18})+\tfrac{1}{3}(\sqrt{2}v_t)]\psi_3=0,\end{aligned}\tag{30}$$

$$V_0\psi_1/\sqrt{6}+[V_0/(\sqrt{18})+\tfrac{1}{3}(\sqrt{2}v_t)]\psi_2 +[T_9+\tfrac{1}{3}(V_{56}+2V_{70}+2v_t)-\Lambda]\psi_3=0,$$

$$[T_4+\tfrac{1}{3}(V_{56}+2V_{70}+2v_s)-\Sigma]\psi_1 -[V_0/(\sqrt{18})+\tfrac{1}{3}(\sqrt{2}v_s)]\psi_2+V_0\psi_3/\sqrt{6}=0,$$

$$-[V_0/(\sqrt{18})+\tfrac{1}{3}(\sqrt{2}v_s)]\psi_1 +[T_2+\tfrac{1}{6}(V_{56}+5V_{70}+2v_s)-\Sigma]\psi_2 -V_0\psi_3/(\sqrt{12})=0, \tag{31}$$

$$V_0\psi_1/\sqrt{6}-V_0\psi_2/(\sqrt{12})+[T_8+\tfrac{1}{2}(V_{56}+V_{70}) +v_t-\Sigma]\psi_3=0,$$

$$[T_5+\tfrac{1}{6}(V_{56}+5V_{70}-v_8+2v_s)-\Xi]\psi_1 -[(V_0-v_8)/\sqrt{18}+\tfrac{1}{3}(\sqrt{2}v_s)]\psi_2 +(V_0-v_8)\psi_3/(\sqrt{12})=0,$$

$$-[(V_0-v_8)/(\sqrt{18})+\tfrac{1}{3}(\sqrt{2}v_s)]\psi_1 +[T_3+\tfrac{1}{3}(V_{56}+2V_{70}-v_8+2v_s)-\Xi]\psi_2 -(V_0-v_8)\psi_3/(\sqrt{6})=0, \tag{32}$$

$$(V_0-v_8)\psi_1/(\sqrt{12})-(V_0-v_8)\psi_2/(\sqrt{6}) +[T_{10}+\tfrac{1}{2}(V_{56}+V_{70}-v_8)+v_t-\Xi]\psi_3=0,$$

where

$$V_0=V_{56}-V_{70}. \tag{33}$$

The decuplet equations are very similar to those in LT, except that here we are using a somewhat different interaction term from the one used in LT. However, when we turn to the octet equations, we find that for the nucleon we have two simultaneous equations to solve, whereas in LT we had only one. Likewise, for each of the Λ, Σ, and Ξ we now have three simultaneous equations to solve, whereas in LT we had only one for the Λ and two each for the Σ and Ξ.

In principle, once the potentials are specified, these sets of simultaneous differential equations can be solved to obtain the baryon masses as eigenvalues. However, following LT, we proceed in a simpler way, assuming that the potentials V_{56} and V_{70} are sufficiently deep square wells that the ψ_{Bi} vanish at the boundary. Then the momentum p appearing in the expressions for the kinetic energy is a constant rather than a differential operator, and the differential equations become algebraic equations.

These sets of simultaneous homogeneous algebraic equations have solutions only if their determinants vanish. We see that we are led to linear algebraic equations for the masses N^* and Ω, quadratic equations for the masses Y^*, Ξ^*, and N, and cubic equations for the masses Λ, Σ, and Ξ. In the case of the quadratic and cubic equations, the baryon masses are assumed to be the smallest roots of the equations. Quadratic and cubic algebraic equations can, of course, be solved in closed form, but the algebraic solution in terms of our parameters is quite complicated, especially so for the cubic equations. Furthermore, these solutions contain too many parameters to be useful. We shall spare the reader by not writing down these closed-form solutions.

It is more instructive to make use of our assumption that the parameters m_s, m_t, m_q, V_{56}, V_{70}, and V_0 are large, and the remaining parameters p, δ_s, δ_t, δ_q, v_{10}, v_8, v_s, v_t, and the difference m_s-m_t are small. If we expand the solutions to the simultaneous equations, keeping only linear terms in the small parameters, we obtain the perturbation-theory results of Sec. 2. The baryon mass differences, to this order, are independent of all the large parameters. The exact solutions for the mass differences, then, should be fairly insensitive to the values of the large parameters.

We can go beyond the lowest-order perturbation-theory results by expanding the exact solutions to second order in the small parameters. The mass differences among the members of the decuplet are then

$$Y^*-N^*=\tfrac{2}{3}\delta_s+\tfrac{1}{3}\delta_q-v_{10}+8V_0\gamma_s^2,$$
$$\Xi^*-Y^*=\tfrac{2}{3}\delta_s+\tfrac{1}{3}\delta_q-v_{10}, \tag{34}$$
$$\Omega-\Xi^*=\tfrac{2}{3}\delta_s+\tfrac{1}{3}\delta_q-v_{10}-8V_0\gamma_s^2,$$

where

$$\gamma_s=(\delta_s-\delta_q)/6V_0.$$

We see from these equations that to second order in δ_s and δ_q we no longer have the decuplet equal spacing rule, but just the sum rule

$$Y^*-N^*+\Omega-\Xi^*=2(\Xi^*-Y^*).$$

But since the decuplet equal spacing rule holds very well experimentally, we conclude that γ_s is small.

The octet mass differences are given by

$$\Lambda-N=\tfrac{1}{2}\delta_s+\tfrac{1}{6}\delta_t+\tfrac{1}{3}\delta_q-v_8 +V_0[2\zeta(3\gamma_s-\gamma_t)+9\gamma_s^2-6\gamma_s\gamma_t+5\gamma_t^2], \tag{35}$$
$$\Sigma-\Lambda=\tfrac{1}{3}(\delta_t-\delta_s)-4V_0[\zeta(\gamma_s+\gamma_t)+\gamma_s^2-\gamma_t^2],$$
$$\Xi-\Sigma=\tfrac{2}{3}\delta_s+\tfrac{1}{3}\delta_q-v_8+8V_0[\zeta\gamma_s+3\gamma_s^2-3\gamma_s\gamma_t],$$

where $\zeta=(m_s-m_t+v_s-v_t)/2V_0$ and $\gamma_t=(\delta_t-\delta_q)/6V_0$. We no longer have the Gell-Mann–Okubo mass formula, but rather the result

$$\Lambda-N-\tfrac{1}{2}(\Xi-\Sigma)-\tfrac{1}{2}(\Xi-\Lambda) =V_0(18\gamma_s\gamma_t-13\gamma_s^2+3\gamma_t^2). \tag{36}$$

Experimentally, the left-hand side of Eq. (36) is +12 MeV. Since V_0 is negative, we cannot satisfy Eq. (35) with $\gamma_s=0$. We can get best agreement with experiment in the following way. First, from Eq. (34) we write

$$\Xi^*-Y^*-(Y^*-N^*)=-8V_0\gamma_s^2. \tag{37}$$

Then we take the ratio R of Eq. (36) to Eq. (37), getting

$$R=\tfrac{1}{8}(13-18\gamma_t/\gamma_s-3\gamma_t^2/\gamma_s^2). \tag{38}$$

Since the deviation from the Gell-Mann–Okubo octet formula is much greater experimentally than the deviation from the equal spacing rule, we will obtain best agreement with experiment by maximizing R. We then obtain

$$\gamma_t/\gamma_s=-3.$$

Then from Eq. (36), using the experimental value 12 MeV, we obtain $V_0\gamma_s^2=-0.3$ MeV. With this value of $V_0\gamma_s^2$ we obtain from Eq. (37) the predictions

$$\Omega-\Xi^*-(\Xi^*-Y^*)=\Xi^*-Y^*-(Y^*-N^*)=2.4\text{ MeV}.$$

Unfortunately there is some ambiguity in comparing with the experimental values because of the finite widths of the resonances and the electromagnetic splittings. The most accurately known masses (in MeV) are[11] $N^{*++}=1236.0\pm0.6$, $Y^{*+}=1382.2\pm0.9$, $\Xi^{*0}=1528.9\pm1.1$, and $\Omega^-=1674\pm3$. Using these values, we obtain

$$\Omega^--\Xi^{*0}-(\Xi^{*0}-Y^{*+})=-1.6\pm3.8,$$
$$\Xi^{*0}-Y^{*+}-(Y^{*+}-N^{*++})=0.5\pm2.2.$$

However, if we assume that U-spin invariance holds, we should compare the negative members of each multiplet. But the masses of the negative members are not so well known, the errors being as large as 7 MeV.

4. DISCUSSION

The main objective of this paper was to introduce a two-particle model of baryons which is approximately invariant under $SU(6)$ and apply it to calculating the baryon mass differences. We have considered the mass splitting both in lowest-order perturbation theory, and to second order in the $SU(6)$- and $SU(3)$-breaking parameters.

We have found that we obtain best agreement with experiment in second order if

$$\gamma_t/\gamma_s=(\delta_t-\delta_q)/(\delta_s-\delta_q)=-3. \qquad (39)$$

But from Eq. (35) we see that the first-order correction to the $\Sigma-\Lambda$ mass difference is just $\frac{1}{3}(\delta_t-\delta_s)$. We can satisfy both Eq. (39) and the equation for $\Sigma-\Lambda$ if we take

$$\delta_t>\delta_q>\delta_s. \qquad (40)$$

In LT we constructed the baryons from a quark and a sextet diquark. We found that we could not break the Gell-Mann–Okubo octet formula without breaking the decuplet rule by a comparable amount. With the model of the present paper, which incorporates approximate $SU(6)$ invariance, we have additional flexibility. This is because the octet baryons contain a triplet, as well as a sextet, diquark. Therefore, here we are able to break the octet sum rule with only a small breaking of the decuplet formula.

However, because the parameters entering our model are not completely arbitrary, we have less flexibility than one might suppose. In particular, since V_0 must be negative, we cannot preserve the decuplet equal-spacing rule by setting $\gamma_s=0$, i.e., $\delta_s=\delta_q$. This is because, with V_0 negative and with $\gamma_s=0$, the Gell-Mann–Okubo formula is broken in the wrong direction. If we wish to set $\gamma_s=0$ so that the equal-spacing rule holds exactly, then we must weaken some of our other assumptions. For example, we could add an $SU(3)$ symmetry-breaking potential between the quark and diquark which depends on the isospin as well as the hypercharge. But for the present we do not wish to consider this, more complicated situation.

We can obtain an estimate for V_0 from our results. As a first approximation, we assume that the $\Sigma-\Lambda$ mass difference arises chiefly from the first-order term:

$$\Sigma-\Lambda\approx\tfrac{1}{3}(\delta_t-\delta_s)\approx80\text{ MeV}.$$

But

$$\tfrac{1}{3}(\delta_t-\delta_s)=2V_0(\gamma_t-\gamma_s)=-8V_0\gamma_s,$$

where we have used $\gamma_t/\gamma_s=-3$. Putting the result $\gamma_s\approx-10\text{ MeV}/V_0$ into $V_0\gamma_s^2=-0.3$ MeV, we obtain

$$V_0\approx-330\text{ MeV}.$$

Thus, if the deviation from the Gell-Mann–Okubo mass formula is to be explained as a second-order effect arising from the medium-strong mass splittings of the quark and diquark, baryonic states corresponding to the 70-dimensional representation of $SU(6)$ should occur in the vicinity of several hundred MeV above the baryon octet and decuplet. The exact position of these states will depend on details of the interaction, such as the $SU(3)$-breaking interactions in the 70, to which the lowest baryon octet and decuplet are relatively insensitive.

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APPENDIX

In this appendix, the symbol for a baryon denotes its wave function rather than its mass. We assume that the baryons are members of a 56-multiplet of $SU(6)$ and are constructed from a quark and a diquark. We use the symbols q_i, s_i, and t_i for the $SU(3)$ wave functions of the quark, sextet and triplet diquarks respectively. We denote the spin wave functions of a quark by α and β for spin up and spin down, respectively. Also we denote the spin wave functions of a sextet diquark of spin one by a, b, and c for z components 1, 0, and -1, respectively. The triplet diquark has spin zero, and we omit its wave function. We also define the spin wave functions ϕ and χ:

$$\phi=(\surd\tfrac{1}{3})a\beta+(\surd\tfrac{2}{3})b\alpha,\quad \chi=(\surd\tfrac{2}{3})a\beta-(\surd\tfrac{1}{3})b\alpha.$$

Then the wave functions of the members of the baryon decuplet with z component of the spin equal to one-half

[11] A. H. Rosenfeld, A. Barbaro-Galtieri, W. J. Podolsky, L. R. Price, P. Soding, C. G. Wohl, M. Roos, and W. J. Willis, Rev. Mod. Phys. 39, 1 (1967).

which follow from $SU(6)$ are

$$
\begin{aligned}
N^{*++}&=s_1q_1\phi\,,\\
N^{*+}&=[(\sqrt{\tfrac{1}{3}})s_1q_2+(\sqrt{\tfrac{2}{3}})s_2q_1]\phi\,,\\
N^{*0}&=[(\sqrt{\tfrac{2}{3}})s_2q_2+(\sqrt{\tfrac{1}{3}})s_3q_1]\phi\,,\\
N^{*-}&=s_3q_2\phi\,,\\
Y^{*+}&=[(\sqrt{\tfrac{1}{3}})s_1q_3+(\sqrt{\tfrac{2}{3}})s_4q_1]\phi\,,\\
Y^{*0}&=(\sqrt{\tfrac{1}{3}})(s_2q_3+s_4q_2+s_5q_1)\phi\,,\\
Y^{*-}&=[(\sqrt{\tfrac{1}{3}})s_3q_3+(\sqrt{\tfrac{2}{3}})s_5q_2]\phi\,,\\
\Xi^{*0}&=[(\sqrt{\tfrac{2}{3}})s_4q_3+(\sqrt{\tfrac{1}{3}})s_6q_1]\phi\,,\\
\Xi^{*-}&=[(\sqrt{\tfrac{2}{3}})s_5q_3+(\sqrt{\tfrac{1}{3}})s_6q_2]\phi\,,\\
\Omega&=s_6q_3\phi\,.
\end{aligned}
\tag{A1}
$$

Likewise, the wave functions of the members of the baryon octet with z component of the spin equal to one-half which follow from $SU(6)$ are

$$
\begin{aligned}
p&=[(\sqrt{\tfrac{1}{3}})s_1q_2-(\sqrt{\tfrac{1}{6}})s_2q_1]\chi+(\sqrt{\tfrac{1}{2}})t_1q_1\alpha\,,\\
n&=[(\sqrt{\tfrac{1}{6}})s_2q_2-(\sqrt{\tfrac{1}{3}})s_3q_1]\chi+(\sqrt{\tfrac{1}{2}})t_1q_2\alpha\,,\\
\Lambda&=\tfrac{1}{2}(s_4q_2-s_5q_1)\chi+(\sqrt{\tfrac{1}{3}})(\tfrac{1}{2}t_2q_2-\tfrac{1}{2}t_3q_1+t_1q_3)\alpha\,,\\
\Sigma^+&=[(\sqrt{\tfrac{1}{3}})s_1q_3-(\sqrt{\tfrac{1}{6}})s_4q_1]\chi+(\sqrt{\tfrac{1}{2}})t_2q_1\alpha\,,\\
\Sigma^0&=(\sqrt{\tfrac{1}{3}})(s_2q_3-\tfrac{1}{2}s_4q_2-\tfrac{1}{2}s_5q_1)\chi+\tfrac{1}{2}(t_2q_2+t_3q_1)\alpha\,,\\
\Sigma^-&=[(\sqrt{\tfrac{1}{3}})s_3q_3-(\sqrt{\tfrac{1}{6}})s_5q_2]\chi+(\sqrt{\tfrac{1}{2}})t_3q_2\alpha\,,\\
\Xi^0&=[(\sqrt{\tfrac{1}{6}})s_4q_3-(\sqrt{\tfrac{1}{3}})s_6q_1]\chi+(\sqrt{\tfrac{1}{2}})t_2q_3\alpha\,,\\
\Xi^-&=[(\sqrt{\tfrac{1}{6}})s_5q_3-(\sqrt{\tfrac{1}{3}})s_6q_2]\chi+(\sqrt{\tfrac{1}{2}})t_3q_3\alpha\,.
\end{aligned}
\tag{A2}
$$

The phases of the Clebsch-Gordan coefficients may differ from those in some other papers.

In terms of the spin-isospin wave functions $\eta_{2J}(I_sI_qI)$ and $\eta(I_tI_qI)$ defined in Sec. 3 of the text, these baryon wave functions can be written (suppressing charge indices)

$$
\begin{aligned}
N^*&=\eta_3(1\,\tfrac{1}{2}\,\tfrac{3}{2})\,,\\
Y^*&=(\sqrt{\tfrac{1}{3}})\eta_3(1\,0\,1)+(\sqrt{\tfrac{2}{3}})\eta_3(\tfrac{1}{2}\,\tfrac{1}{2}\,1)\,,\\
\Xi^*&=(\sqrt{\tfrac{2}{3}})\eta_3(\tfrac{1}{2}\,0\,\tfrac{1}{2})+(\sqrt{\tfrac{1}{3}})\eta_3(0\,\tfrac{1}{2}\,\tfrac{1}{2})\,,\\
\Omega&=\eta_3(0\,0\,0)\,,\\
N&=(\sqrt{\tfrac{1}{2}})\eta_1(1\,\tfrac{1}{2}\,\tfrac{1}{2})+(\sqrt{\tfrac{1}{2}})\eta(0\,\tfrac{1}{2}\,\tfrac{1}{2})\,,\\
\Lambda&=(\sqrt{\tfrac{1}{2}})\eta_1(\tfrac{1}{2}\,\tfrac{1}{2}\,0)+(\sqrt{\tfrac{1}{6}})\eta(\tfrac{1}{2}\,\tfrac{1}{2}\,0)\\
&\qquad+(\sqrt{\tfrac{1}{3}})\eta(0\,0\,0)\,,\\
\Sigma&=(\sqrt{\tfrac{1}{3}})\eta_1(1\,0\,1)-(\sqrt{\tfrac{1}{6}})\eta_1(\tfrac{1}{2}\,\tfrac{1}{2}\,1)\\
&\qquad+(\sqrt{\tfrac{1}{2}})\eta(\tfrac{1}{2}\,\tfrac{1}{2}\,1)\,,\\
\Xi&=(\sqrt{\tfrac{1}{6}})\eta_1(\tfrac{1}{2}\,0\,\tfrac{1}{2})-(\sqrt{\tfrac{1}{3}})\eta_1(0\,\tfrac{1}{2}\,\tfrac{1}{2})\\
&\qquad+(\sqrt{\tfrac{1}{2}})\eta(\tfrac{1}{2}\,0\,\tfrac{1}{2})\,.
\end{aligned}
\tag{A3}
$$

To obtain the equations of Sec. 3, it is necessary to have the eigenstates of the projectors P_{10}, P_8, P_s, and P_t. The baryon states defined in Eq. (A3) are eigenstates of P_{10} and P_8, while the states

$$\eta_1(1\,\tfrac{1}{2}\,\tfrac{1}{2})\,,\quad \eta(0\,\tfrac{1}{2}\,\tfrac{1}{2})\,,\quad \eta_1(\tfrac{1}{2}\,\tfrac{1}{2}\,0)\,,\quad \eta(\tfrac{1}{2}\,\tfrac{1}{2}\,1)\,,\ \text{and}\ \eta(\tfrac{1}{2}\,0\,\tfrac{1}{2})\,,$$

and the linear combinations

$$
\begin{aligned}
&(\sqrt{\tfrac{1}{3}})\eta(\tfrac{1}{2}\,\tfrac{1}{2}\,0)+(\sqrt{\tfrac{2}{3}})\eta(0\,0\,0)\,,\\
&(\sqrt{\tfrac{2}{3}})\eta_1(1\,0\,1)-(\sqrt{\tfrac{1}{3}})\eta_1(\tfrac{1}{2}\,\tfrac{1}{2}\,1)\,,\\
&(\sqrt{\tfrac{1}{3}})\eta_1(\tfrac{1}{2}\,0\,\tfrac{1}{2})-(\sqrt{\tfrac{2}{3}})\eta_1(0\,\tfrac{1}{2}\,\tfrac{1}{2})\,,
\end{aligned}
$$

are eigenstates of P_s and P_t. As stated in Sec. 3, the states $\eta_{2J}(I_sI_qI)$ and $\eta(I_tI_qI)$ are eigenstates of the projection operators in the free Hamiltonian.